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ALIASING EFFECTS AND SAMPLING THEOREMS OF SPHERICAL RANDOM FIELDS WHEN SAMPLED ON A FINITE GRID

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Abstract. Aliasing effects are investigated for spherical random fields sampled on a finite grid. Using the spherical harmonics expansion, it is shown that for a band-limited spherical random field its trend and spectrum can be uniquely reconstructed from the sampled field if the sampling points are judiciously designed. Analytical expressions are also obtained for aliasing errors in the trend and the spectrum when the field is not band-limited.

Key words and phrases: Gauss quadrature, Laplace series, sampling theorems, spectral analysis, spherical harmonics.

1. Introduction

Spherical random fields arise naturally in applications such as meteorology when the surface air temperature is of interest (e.g. Bourke (1988); North *et al.* (1992)). A spherical random field, denoted by $T(\theta, \phi)$, is a stochastic process defined on the unit sphere, with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ specifying the direction of a point on the sphere (Jones (1963); Roy (1973, 1976)). It may be assumed that $T(\theta, \phi)$ has finite second-order moments and can be decomposed as *trend plus noise*, so that

$$T(\theta, \phi) = \mu(\theta, \phi) + \epsilon(\theta, \phi), \tag{1.1}$$

where $\mu(\theta, \phi) := E\{T(\theta, \phi)\}$ represents the deterministic trend and $\epsilon(\theta, \phi)$ the random fluctuation with $E\{\epsilon(\theta, \phi)\} = 0$. Under suitable conditions (Section 4), $T(\theta, \phi)$ can be expressed in mean-square as a Laplace series

$$T(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} T_{mn} Y_n^m(\theta, \phi), \tag{1.2}$$

where the $Y_n^m(\theta, \phi)$ are the *spherical harmonics* defined by

$$Y_n^m(\theta, \phi) := (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi}, \tag{1.3}$$

with the $P_n^m(x)$ being the associated Legendre functions (Arfken (1970), Chapter 12). The Laplace coefficients T_{mn} in (1.2) can be obtained (in mean-square) from the integral

$$T_{mn} := \int_0^\pi \int_0^{2\pi} T(\theta, \phi) Y_n^{m*}(\theta, \phi) d\phi \sin\theta d\theta, \tag{1.4}$$

where the asterisk $*$ stands for complex conjugate.

Through the transformation that consists of (1.2) and (1.4), various aspects of the field $T(\theta, \phi)$, such as the global average, can be investigated on the basis of the Laplace coefficients T_{mn} (e.g. Bourke (1988); North *et al.* (1992)). In practice, however, measurements of $T(\theta, \phi)$ can only be obtained from a finite number of locations, so that the integral (1.4) has to be replaced with a finite sum that depends only on the discrete measurements (e.g. Bourke (1988)). This gives rise to the *aliasing errors* as they may occur in the Fourier analysis of univariate processes (Hannan (1970); Papoulis (1977)). In this article, we are concerned with the analytical expressions of the aliasing errors for some sampling schemes as well as the conditions under which the field $T(\theta, \phi)$ can be uniquely determined from its samples (Section 2). We are also interested in the aliasing errors associated with spectral approximations when the spectral analysis of $T(\theta, \phi)$ is restricted to the finite samples (Section 3). It is shown in particular that band-limited spherical random fields can be uniquely reconstructed from their discrete samples if the sampling points are judiciously designed.

2. Aliasing in Laplace coefficients

To study how the aliasing problem occurs in the Laplace coefficients when the field $T(\theta, \phi)$ is sampled at a finite number of locations, let $T(\theta_k, \phi_k)$, $k = 1, \ldots, K$, be the samples from $T(\theta, \phi)$ and $\tilde{T}_{mn}$ be the approximated Laplace coefficients obtained by replacing (1.4) with the finite sum

$$\tilde{T}_{mn} := \sum_{k=1}^{K} w_k T(\theta_k, \phi_k) Y_n^{m*}(\theta_k, \phi_k) \sin\theta_k, \tag{2.1}$$

where $w_k > 0$ is a weight function.

Assuming the Laplace series (1.2) converges at (θ_k, ϕ_k) to $T(\theta_k, \phi_k)$ in mean square (see Section 4), one can substitute $T(\theta_k, \phi_k)$ in (2.1) with the series in (1.2) so that $\tilde{T}_{mn}$ can be expressed (in mean square) as

$$\tilde{T}_{mn} = \sum_{v=0}^{\infty} \sum_{u=-v}^{v} d(m, n; u, v) T_{uv}, \tag{2.2}$$

where

$$d(m,n;u,v) := \sum_{k=1}^{K} w_k Y_v^u(\theta_k,\phi_k) Y_n^{m*}(\theta_k,\phi_k)\sin\theta_k. \tag{2.3}$$

Clearly, in order for the identity $\tilde{T}_{mn} = T_{mn}$ to hold for all (m,n) and for any $T(\theta,\phi)$, it is necessary and sufficient that $d(m,n;u,v)$ be a Kronecker delta, namely, $d(m,n;u,v) = \delta_{m-u}\delta_{n-v}$, where $\delta_0 = 1$ and $\delta_u = 0$ for $u \neq 0$. Since $d(m,n;u,v)$ is not a Kronecker delta in general, it is possible that the series (2.2) involve not only T_{mn} but also some other T_{uv}, and thus the aliasing errors occur, with those T_{uv} being aliases of T_{mn}. Note that the relationship between $\tilde{T}_{mn}$ and T_{uv} depends on the sampling points (θ_k,ϕ_k). In the following, a special sampling scheme is discussed with great detail.

2.1 *Sampling on finite grids*

To be more specific, consider sampling the field $T(\theta,\phi)$ on a finite grid of the form (θ_p,ϕ_q) for $p = 0,1,\ldots,N-1$ and $q = 0,1,\ldots,2M-1$. In this case, we can rewrite (2.1) as

$$\tilde{T}_{mn} = \sum_{p=0}^{N-1}\sum_{q=0}^{2M-1} w_p^\theta w_q^\phi T(\theta_p,\phi_q) Y_n^{m*}(\theta_p,\phi_q)\sin\theta_p, \tag{2.4}$$

where $w_p^\theta > 0$ and $w_q^\phi > 0$ are suitable weights. Similarly, we can rewrite (2.3) as

$$\begin{aligned} d(m,n;u,v) &= \sum_{p=0}^{N-1}\sum_{q=0}^{2M-1} w_p^\theta w_q^\phi Y_v^u(\theta_p,\phi_q) Y_n^{m*}(\theta_p,\phi_q)\sin\theta_p \\ &= C_{mn}C_{uv} I_{mn}^N(u,v) J_m^M(u), \end{aligned} \tag{2.5}$$

where

$$C_{mn} := (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}},$$

$$I_{mn}^N(u,v) := \sum_{p=0}^{N-1} w_p^\theta P_v^u(\cos\theta_p) P_n^m(\cos\theta_p)\sin\theta_p,$$

$$J_m^M(u) := \sum_{q=0}^{2M-1} w_q^\phi e^{i(u-m)\phi_q}.$$

In particular, let the sampling be *uniform* with respect to ϕ so that

$$\phi_q := q\pi/M \quad \text{and} \quad w_q^\phi := \pi/M. \tag{2.6}$$

Since $J_m^M(u) = 2\pi$ if $u = m + 2rM$ for some integer r and $J_m^M(u) = 0$ otherwise, it follows from (2.4) and (2.5) that

$$\tilde{T}_{mn} = \sum_{v=0}^{\infty}\sum_{r\in\mathbb{R}_m^M(v)} 2\pi C_{mn} C_{m+2rM,v} I_{mn}^N(m+2rM,v) T_{m+2rM,v}, \tag{2.7}$$

where, for fixed $v \geq 0$, $\mathbb{R}_m^M(v) := \{r : -(v+m)/(2M) \leq r \leq (v-m)/(2M)\}$ is the set of integers r for which $|m + 2rM| \leq v$. Further, since the parity relation $P_v^u(-x) = (-1)^{u+v} P_v^u(x)$ holds for any u and v (Arfken (1970), p. 561), the function $G(\theta) := P_v^u(\cos\theta) P_n^m(\cos\theta) \sin\theta$ is symmetric about $\pi/2$ in the sense that $G(\pi - \theta) = (-1)^{u+v+m+n} G(\theta)$. Therefore, if the sampling in θ is also symmetric about $\pi/2$ so that

$$\theta_p = \pi - \theta_{N-p-1} \qquad (p = 0, 1, \ldots, [N/2]), \tag{2.8}$$

it is easy to see that $I_{mn}^N(u, v) = 0$ whenever $u + v + m + n$ is odd; this implies, in particular, that $I_{mn}^N(m + 2rM, v) = 0$ for any $v = n + 2s + 1$ with s being an integer. (The results are not altered by including $\theta = 0$, the north pole, into the sampling points.) Combining this with (2.7) gives rise to the following theorem.

THEOREM 2.1. *Suppose the sampling points* (θ_p, ϕ_q) *are given by* (2.6) *and* (2.8). *If the Laplace series* (1.2) *converges in mean square to* $T(\theta_p, \phi_q)$ *at* (θ_p, ϕ_q), *then the approximated Laplace coefficients* $\tilde{T}_{mn}$ *defined by* (2.4) *can be written in mean square as*

$$\tilde{T}_{mn} = \sum_{(r,s) \in \mathbb{D}_{mn}^M} 2\pi C_{mn} C_{m+2rM, n+2s} I_{mn}^N(m + 2rM, n + 2s) T_{m+2rM, n+2s} \tag{2.9}$$

where $\mathbb{D}_{mn}^M := \{(r, s) : -n/2 \leq s < \infty, -(n + m + 2s)/(2M) \leq r \leq (n - m + 2s)/(2M)\}$.

As we can see from (2.9), with the sampling points satisfying (2.6) and (2.8), the aliases of T_{mn} in the expression of $\tilde{T}_{mn}$ consist of all Laplace coefficients of the form $T_{m+2rM,n+2s}$ with $(r, s) \in \mathbb{D}_{mn}^M \setminus (0, 0)$. Figure 1 depicts a subset of possible aliases in the case of $M = 3$, where identical symbols represent the locations of T_{mn} in the (m, n)-plane that are possible aliases of each other. For example, all T_{mn} located at "□" are possible aliases of each other in general and of T_{00} (the global average) in particular. Note that without further restrictions on θ_p the period of aliasing in coordinate n always equals 2 regardless of the sampling rate N in θ. For example, all Laplace coefficients $T_{2rM,2s}$ with $s = 1, 2, \ldots$ and $|r| \leq [s/M]$ (represented by □) may have contributions to the aliasing errors in $\tilde{T}_{00}$ no matter how large N is. This is true in particular when the sampling in θ is uniform with $\theta_p - \pi(p + 1)/(N + 1)$ for $p = 0, 1, \ldots, N \quad 1$.

2.2 *Gaussian sampling theorem*

To further reduce the aliasing effects, consider the following *Gaussian sampling approach* (e.g. Bourke (1988)). The Gaussian sampling approach in this article should not be confused with the sampling approach using random sampling points that obey a Gaussian distribution. The word "Gaussian" refers to the Gaussian quadrature for numerical integration (e.g. Stoer and Bulirsch (1980)). Let x_p, for $p = 0, 1, \ldots, N - 1$, be the roots of the N-th Legendre polynomial $P_N(x) := P_N^0(x)$. Due to the parity relation $P_N(-x) = (-1)^N P_N(x)$, the roots

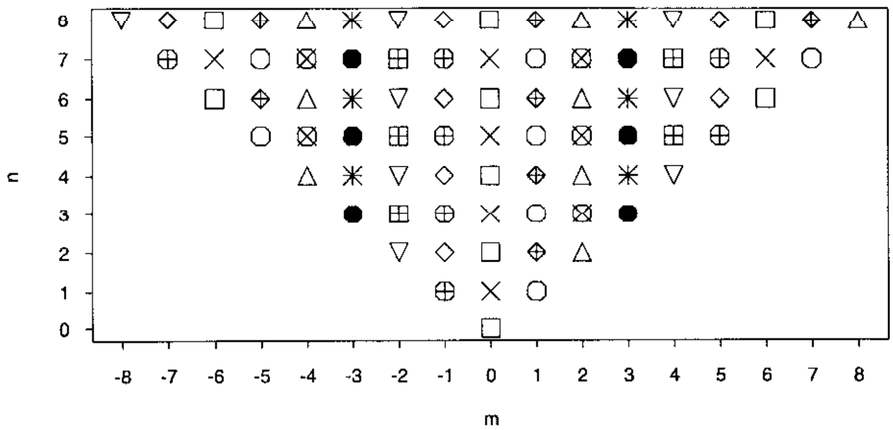

Fig. 1. Aliasing effects in $\tilde{T}_{mn}$ when $M = 3$ (N arbitrary): Coordinates of possible mutual aliases are represented with identical symbols.

can be arranged in symmetry so that $x_p = -x_{N-p-1}$. Let $T(\theta, \phi)$ be sampled with respect to θ using the sampling points

$$\theta_p := \arccos(x_p) \qquad (p = 0, 1, \dots, N-1). \tag{2.10}$$

The weights w_p^θ in Gaussian sampling are taken to be

$$w_p^\theta := W_p / \sin\theta_p \qquad (p = 0, 1, \dots, N-1), \tag{2.11}$$

where the W_p solve the (non-singular) system of linear equations

$$\sum_{p=0}^{N-1} P_k(x_p) W_p = 2\delta_k \qquad (k = 0, 1, \dots, N-1). \tag{2.12}$$

Note that the Gaussian sampling points θ_p in (2.10) satisfy the symmetry requirement (2.8). Table 1 gives some values for x_p, θ_p, and w_p^θ (Stoer and Bulirsch (1980)); for further values see Abramowitz and Stegun (1964).

With the choice of (2.10)–(2.12), a theorem due to Gauss (Stoer and Bulirsch (1980), Theorem 3.6.12) states that

(a) the weights W_p, and hence w_p^θ, are strictly positive;

(b) the identity

$$\sum_{p=0}^{N-1} w_p^\theta P(\cos\theta_p) \sin\theta_p = \sum_{p=0}^{N-1} W_p P(x_p) = \int_{-1}^{1} P(x) dx$$

holds for any polynomial $P(x)$ of degree strictly less than $2N$.

This theorem plays a crucial role in the evaluation of $I_{mn}^N(m+2rM, n+2s)$. In fact, since $P(x) := P_{n+2s}^{m+2rM}(x) P_n^m(x)$ is a polynomial of degree $2(n+s)$, therefore, with $-n/2 \le s < N - n$ and $0 \le n \le 2(N-1)$, the Gauss theorem leads to

$$I_{mn}^N(m + 2rM, n + 2s) = \int_{-1}^{1} P_{n+2s}^{m+2rM}(x) P_n^m(x) dx. \tag{2.13}$$

Table 1. Gaussian sampling points and weights.

N	w_p^θ	x_p	θ_p
1	$w_0 = 2$	$x_0 = 0$	$\theta_0 = \pi/2$
2	$w_0 = w_1 = 1.2247449$	$x_0 = -x_1 = 0.5773503$	$\theta_0 = 0.9553166$, $\theta_1 = 2.1862760$
3	$w_0 = w_2 = 0.8784105$	$x_0 = -x_2 = 0.7745967$	$\theta_0 = 0.6847192$, $\theta_2 = 2.4568735$
	$w_1 = 8/9$	$x_1 = 0$	$\theta_1 = \pi/2$
4	$w_0 = w_3 = 0.6842497$	$x_0 = -x_3 = 0.8611363$	$\theta_0 = 0.5332957$, $\theta_3 = 2.6082970$
	$w_1 = w_2 = 0.6934525$	$x_1 = -x_2 = 0.3399810$	$\theta_1 = 1.2238096$, $\theta_2 = 1.9176931$
5	$w_0 = w_4 = 0.5602532$	$x_0 = -x_4 = 0.9061798$	$\theta_0 = 0.4366349$, $\theta_4 = 2.7049577$
	$w_1 = w_3 = 0.5680074$	$x_1 = -x_3 = 0.5384693$	$\theta_1 = 1.0021768$, $\theta_3 = 2.1394158$
	$w_2 = 128/225$	$x_2 = 0$	$\theta_2 = \pi/2$

The orthogonality of $P_n^m(x)$ further implies that

$$I_{mn}^N(m, n+2s) = \frac{1}{2\pi C_{mn}^2}\delta_s \qquad (-n/2 \le s < N-n). \tag{2.14}$$

Therefore, for $0 \le n \le N$, the Gaussian sampling scheme annihilates all aliases of T_{mn} in (2.9) that correspond to $r = 0$ and $-n/2 \le s < N-n$ ($s \ne 0$); the remaining aliases $T_{m+2rM,n+2s}$, with $(r,s) \ne (0,0)$, are separated from T_{mn} by a distance of $D = 2\sqrt{r^2M^2+s^2}$. For $r \ne 0$, we have $D \ge 2M$; for $r = 0$, we have $D \ge \max\{2, 2(N-n)\}$, because possible aliases $T_{m,n+2s}$ occur only if $s \ge N-n$ and $s \ne 0$.

As an example, consider the case where $m = n = 0$ and $N = M$. It is readily shown from (2.9) and (2.14) that

$$\tilde{T}_{00} = T_{00} + \sum_{s=M}^{\infty} \sum_{r=-[s/M]}^{[s/M]} 2\pi C_{00} C_{2rM,2s} I_{00}^M(2rM, 2s) T_{2rM,2s}.$$

Clearly, if $T_{mn} = 0$ for any (m,n) with $n > 2M$, then the Gaussian sampling scheme leads to $\tilde{T}_{00} = T_{00}$. In this case, the global average T_{00} can be obtained without aliasing error from the discrete (Gaussian) samples $T(\theta_p, \phi_q)$. However, if the T_{mn} do not vanish for large wave numbers, then, in general, one *cannot* obtain aliasing-free T_{00} with the Gaussian sampling, regardless of the sampling rate N.

For a given pair of integers $N_0 \ge M_0 > 0$, a random field $T(\theta,\phi)$ is said to be *band-limited* with bandwidth (M_0, N_0) if $T(\theta,\phi)$ can be written as (1.2) with $T_{mn} = 0$ almost surely for any (m,n) such that $|m| > M_0$ and/or $n > N_0$. In practice, band-limited fields can be regarded as approximations to the fields whose Laplace coefficients T_{mn} decay sufficiently fast as the wave number n grows (e.g. North *et al.* (1992)). The following theorem states that a band-limited field is alias-free in $\tilde{T}_{mn}$ and can be perfectly reconstructed from its Gaussian samples if the sampling rate is sufficiently high.

THEOREM 2.2. *Suppose a random field $T(\theta,\phi)$ is band-limited with bandwidth (M_0, N_0) and the $\tilde{T}_{mn}$ in (2.4) are obtained from $T(\theta_p,\phi_q)$ by Gaussian sampling with $M > M_0$ and $N > \max(N_0, M-1)$. Then, (i) the identity $\tilde{T}_{mn} = T_{mn}$ holds almost surely for all $(m,n) \in \Omega_{MN} := \{(m,n) : 0 \le n \le N, |m| \le \min(M,n)\}$; (ii) for any (M', N') satisfying $M \ge M' \ge M_0$ and $N \ge N' \ge \max(N_0, M')$, the random field $T(\theta,\phi)$ can be reconstructed almost surely from the samples $T(\theta_p,\phi_q)$ according to*

$$T(\theta,\phi) = \sum_{p=0}^{N-1}\sum_{q=0}^{2M-1} w_p^\theta w_q^\phi T(\theta_p,\phi_q) K_{M'N'}(\theta,\phi;\theta_p,\phi_q)\sin\theta_p, \tag{2.15}$$

where $K_{M'N'}(\theta,\phi;\theta',\phi')$ is the reconstruction kernel defined by

$$K_{M'N'}(\theta,\phi;\theta',\phi') := \sum_{(m,n)\in\Omega_{M'N'}} Y_n^m(\theta,\phi) Y_n^{m*}(\theta',\phi'). \tag{2.16}$$

In the special case of $M' = N'$, the kernel can also be expressed as

$$K_{N'}(\theta,\phi;\theta',\phi') := \sum_{n=0}^{N'} \frac{2n+1}{4\pi} P_n(\cos\gamma), \tag{2.17}$$

where $\gamma \in [0,\pi]$ is the angle that separates the directions (θ,ϕ) and (θ',ϕ') in the spherical coordinate system and can be obtained from $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

PROOF. Since $T_{uv} = 0$ for any $|u| \ge M > M_0$, it follows that $T_{m+2rM,n+2s} = 0$ for any $r \ne 0$ whenever $|m| \le M$; further, since $T_{uv} = 0$ for any $v \ge N > N_0$, one obtains $T_{m,n+2s} = 0$ for any $s \ge N - n$ whenever $n \le N$. Combining these results with (2.9) and (2.14) leads to $\tilde{T}_{mn} = T_{mn}$ for any $(m,n) \in \Omega_{MN}$. To prove (2.15), it suffices to note that (1.2) can be rewritten as

$$T(\theta,\phi) = \sum_{(m,n)\in\Omega_{M'N'}} \tilde{T}_{mn} Y_n^m(\theta,\phi).$$

With $\tilde{T}_{mn}$ in this expression replaced by (2.4), one obtains (2.15). Finally, according to the addition theorem of spherical harmonics (Arfken (1970), p. 581),

$$\begin{aligned} P_n(\cos\gamma) &= \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(\theta,\phi)Y_n^{m*}(\theta',\phi') \\ &= \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^{m*}(\theta,\phi)Y_n^m(\theta',\phi'). \end{aligned} \tag{2.18}$$

This, combined with the fact that $\Omega_{N'N'} = \{(m,n) : 0 \le n \le N', |m| \le n\}$, implies that $K_{N'N'}(\theta,\phi;\theta',\phi')$ reduces to $K_{N'}(\theta,\phi;\theta',\phi')$ as defined by (2.17). □

With the Gaussian sampling scheme, it is easy to show that

$$\sum_{p=0}^{N-1} \sum_{q=0}^{2M-1} w_p^\theta w_q^\phi Y_v^u(\theta_p, \phi_q) Y_n^{m*}(\theta_p, \phi_q) = \delta_{u-n}\delta_{v-m} \tag{2.19}$$

for any $(m, n), (u, v) \in \Omega_{MN}$ because the sum can be written as $C_{mn}C_{uv}I_{mn}^N(u, v)J_m^M(u)$ which in turn equals $\delta_{u-n}\delta_{v-m}$ for $(m, n), (u, v) \in \Omega_{MN}$.

3. Aliasing in spectra of homogeneous fields

Since the field $T(\theta, \phi)$ has finite second-order moments, the correlation structure of $T(\theta, \phi)$ can be described by its *covariance function* defined as

$$\begin{aligned} R(\theta, \phi; \theta', \phi') &:= E\{(T(\theta, \phi) - \mu(\theta, \phi))(T(\theta', \phi') - \mu(\theta', \phi'))\} \\ &= E\{\epsilon(\theta, \phi)\epsilon(\theta', \phi')\}. \end{aligned}$$

When the covariance function depends only on the angle $\gamma \in [0, \pi]$ that separates the directions (θ, ϕ) and (θ', ϕ') in the spherical coordinate system, the field $T(\theta, \phi)$ is said to be *homogeneous* (isotropic) *in covariance*, or simply *homogeneous*; in other words, for a homogeneous field $T(\theta, \phi)$, the covariance function $R(\theta, \phi; \theta', \phi')$ becomes

$$R(\theta, \phi; \theta', \phi') = \sigma^2 \rho(\cos\gamma),$$

where $\sigma^2 := R(\theta, \phi; \theta, \phi)$ is the variance of $T(\theta, \phi)$ and $\rho(x)$ is called the *correlation function* of $T(\theta, \phi)$. (Note that $|\rho(x)| \leq \rho(1) = 1$.) For a homogeneous field, its correlation structure is invariant under any rotations of the sphere (Hannan (1970), p. 99).

3.1 *Spectral representations*

Since $|\rho(x)| \leq 1$ for all $x \in [-1, 1]$, one can always define the *power* (or *variance*) *spectrum* of the field $T(\theta, \phi)$ as

$$\rho_n := \int_{-1}^{1} \rho(x) P_n(x) dx \qquad (n = 0, 1, \ldots), \tag{3.1}$$

upon assuming that $\rho(x)$ is *Lebesgue measurable* (see Section 4). It is not difficult to show, by appropriate variable substitutions (Hobson (1955), p. 342), that

$$\begin{aligned} \rho_n &= \int_0^\pi \rho(\cos\gamma) P_n(\cos\gamma) \sin\gamma d\gamma \\ &= \frac{1}{8\pi^2} \int_0^\pi \int_0^{2\pi} \left\{ \int_0^\pi \int_0^{2\pi} \rho(\cos\gamma) P_n(\cos\gamma) d\phi \sin\theta d\theta \right\} d\phi' \sin\theta' d\theta'. \end{aligned}$$

This, combined with (1.4) and (2.18), yields $\rho_n = \mathrm{Var}\{T_{mn}\}/(2\pi\sigma^2) \geq 0$ for any (m, n), where $\mathrm{Var}\{\cdot\}$ represents the variance of a random variable.

Assuming the sequence $\{n\rho_n\}$ is *summable* (Roy (1976)), one can also expand $\rho(x)$ for $x \in [-1, 1]$ as a Legendre series

$$\rho(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \rho_n P_n(x), \tag{3.2}$$

whose uniform convergence is ensured by the fact that $|P_n(x)| \leq 1$ for all $x \in [-1, 1]$. According to the addition theorem (2.18), one can further write

$$\begin{aligned} \rho(\cos\gamma) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 2\pi\rho_n Y_n^m(\theta, \phi) Y_n^{m*}(\theta', \phi') \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 2\pi\rho_n Y_n^{m*}(\theta, \phi) Y_n^m(\theta', \phi'). \end{aligned} \tag{3.3}$$

Both (3.2) and (3.3) may be called *spectral representation* of $\rho(\cos\gamma)$.

Let μ_{mn} be the Laplace coefficients of $\mu(\theta, \phi)$, as defined by (1.4) with $\mu(\theta, \phi)$ in place of $T(\theta, \phi)$. It is evident that $\mu_{mn} = E\{T_{mn}\}$; further, using (3.3) and the orthogonality of the spherical harmonics (Arfken (1970), p. 572), it can be shown that

$$\begin{aligned} \mathrm{Cov}\{T_{mn}, T_{uv}\} &:= E\{(T_{mn} - \mu_{mn})(T_{uv}^* - \mu_{uv}^*)\} \\ &= 2\pi\sigma^2 \rho_n \delta_{n-v} \delta_{m-u}. \end{aligned} \tag{3.4}$$

Therefore, for a homogeneous field $T(\theta, \phi)$ with $\{n\rho_n\}$ being summable, the Laplace coefficients T_{mn} are *uncorrelated* with mean μ_{mn} and variance $2\pi\sigma^2\rho_n$ (Jones (1963); Hannan (1970); Roy (1973, 1976)). In addition, if $T(\theta, \phi)$ is Gaussian, then the T_{mn} are Gaussian and mutually independent (Roy (1973)). The Laplace series (1.2), if converges in mean square, can be regarded as a decomposition of $T(\theta, \phi)$ into a sequence of uncorrelated random variables that retain the spectral characteristics of $T(\theta, \phi)$ in their variances by (3.4); thus (1.2) can be called a *spectral representation* of the random field $T(\theta, \phi)$.

3.2 *Aliasing effects in spectral approximation*

Suppose the homogeneous random field $T(\theta, \phi)$ is sampled at (θ_p, ϕ_q), $(p = 0, 1, \ldots, N-1; q = 0, 1, \ldots, 2M-1)$, and that the integral (1.4) is replaced by the finite sum (2.4), where the ϕ_q and w_q^ϕ are given by (2.6) and the θ_p satisfy (2.8). What we are concerned with in the following is how the aliasing errors occur when the average

$$\tilde{\rho}_n := \frac{1}{2n+1} \sum_{m=-n}^{n} \frac{1}{2\pi\sigma^2} \mathrm{Var}\{\tilde{T}_{mn}\} \tag{3.5}$$

is employed to approximate the spectrum $\rho_n = \mathrm{Var}\{T_{mn}\}/(2\pi\sigma^2)$.

To investigate the aliasing errors in $\tilde{\rho}_n$, it suffices to quote Theorem 2.1. In fact, according to Theorem 2.1, equation (2.9) holds in mean square when the

Laplace series (1.2) converges and when the sampling is on a finite grid defined by (2.6) and (2.8). Since the random variables $T_{m+2rM,n+2s}$ are uncorrelated with $\text{Var}\{T_{m+2rM,n+2s}\} = 2\pi\sigma^2\rho_{n+2s}$, as shown in (3.4), it follows that

$$\text{Var}\{\tilde{T}_{mn}\} = 2\pi\sigma^2 \sum_{s=-[n/2]}^{\infty} Q_{mn}^{MN}(n+2s)\rho_{n+2s}, \tag{3.6}$$

where

$$Q_{mn}^{MN}(v) := \sum_{r\in\mathbb{R}_m^M(v)} 4\pi^2 C_{mn}^2 C_{m+2rM,v}^2 \{I_{mn}^N(m+2rM,v)\}^2. \tag{3.7}$$

The next theorem is a direct consequence of (3.6) concerning the aliasing errors in $\tilde{\rho}_n$.

THEOREM 3.1. *Let the random field $T(\theta,\phi)$ be homogeneous and the sequence $\{n\rho_n\}$ be summable. If (1.2) holds in mean square, then, with the sampling points (θ_p,ϕ_q) satisfying (2.6) and (2.8), the approximated spectrum $\tilde{\rho}_n$ in (3.5) can be written as*

$$\tilde{\rho}_n = \sum_{s=-[n/2]}^{\infty} A_n^{MN}(n+2s)\rho_{n+2s}, \tag{3.8}$$

where $A_n^{MN}(v)$ is defined by

$$A_n^{MN}(v) := \frac{1}{2n+1}\sum_{m=-n}^{n} Q_{mn}^{MN}(v) \tag{3.9}$$

with $Q_{mn}^{MN}(v)$ given by (3.7).

As we can see from (3.8), the aliasing period in $\tilde{\rho}_n$ equals 2 regardless of M and N because the aliasing coefficients $A_n^{MN}(n+2s)$, $(s\neq 0)$, do not vanish in general without further restrictions on the sampling points θ_p. For example, with $M=N$ and $n=0$, equation (3.8) becomes

$$\tilde{\rho}_0 = \rho_0 + \sum_{s=1}^{\infty} A_0^N(2s)\rho_{2s} \tag{3.10}$$

where $A_0^N(2s) := A_0^{NN}(2s)$ can be expressed as

$$A_0^N(2s) = \sum_{r=-[s/N]}^{[s/N]} 4\pi^2 C_{00}^2 C_{2rN,2s}^2 \{I_{00}^N(2rN,2s)\}^2.$$

The second term in (3.10) defines the total aliasing error in $\tilde{\rho}_0$. To investigate the contribution of different wave numbers to the total aliasing error in (3.10), the

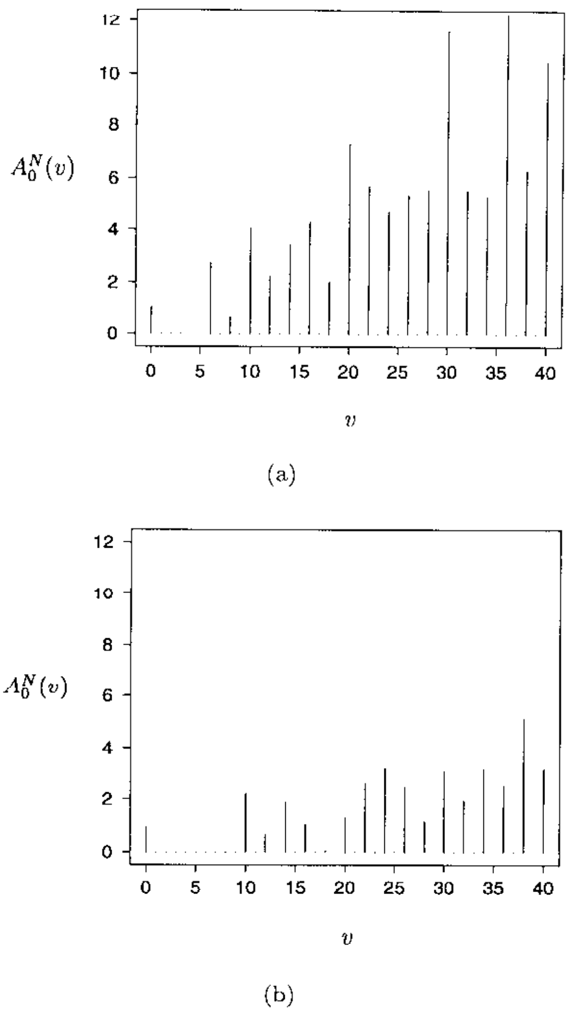

Fig. 2. Aliasing coefficients $A_0^N(v)$ in the approximated spectrum $\tilde{\rho}_0$ using the Gaussian sampling scheme: (a) $M = N = 3$; (b) $M = N = 5$.

aliasing coefficients $A_0^N(v)$ are plotted in Fig. 2 against the wave number v for the Gaussian sampling scheme. Note that $A_0^N(0) = 1$ and $A_0^N(2s-1) = 0$ for all $s \geq 1$. With Gaussian sampling, we also have $A_0^N(2s) = 0$ for $s = 1, \ldots, N-1$. Figure 2 also reveals that the nonzero aliasing coefficients tend to increase in magnitude as the wave number grows and decrease in magnitude as the sampling rate N increases.

3.3 *Gaussian sampling theorem*

A homogeneous random field $T(\theta, \phi)$ is said to have a *band-limited spectrum* with bandwidth N_ρ if $\rho_n = 0$ for any $n > N_\rho$. For such a field, the next theorem guarantees the elimination of aliasing errors in $\tilde{\rho}_n$ when the Gaussian sampling scheme is employed.

THEOREM 3.2. *Suppose the random field $T(\theta,\phi)$ is homogeneous and has a band-limited spectrum with bandwidth N_ρ. Let the samples $T(\theta_p,\phi_q)$ be obtained by Gaussian sampling with $N \geq M > N_\rho$. If the Laplace series (1.2) converges in mean square, then the identity $\mathrm{Var}\{\tilde{T}_{mn}\} = \mathrm{Var}\{T_{mn}\} = 2\pi\sigma^2\rho_n$ holds for any $(m,n) \in \Omega_{MN}$ so that $\tilde{\rho}_n = \rho_n$ for $n = 0,1,\ldots,M$. Further, if $\tilde{\rho}_n$ in (3.5) is replaced by*

$$\tilde{\rho}_n^M := \frac{1}{2(M\wedge n)+1} \sum_{m=-(M\wedge n)}^{M\wedge n} \frac{1}{2\pi\sigma^2}\,\mathrm{Var}\{\tilde{T}_{mn}\},$$

where $M \wedge n := \min(M,n)$, then the identity $\tilde{\rho}_n^M = \rho_n$ holds for $n = 0,1,\ldots,N$. In both cases, the correlation function $\rho(x)$ can be perfectly reconstructed from $\tilde{\rho}_n$ or $\tilde{\rho}_n^M$ using equation (3.2).

PROOF. First, the band-limited property of spectrum ensures that $\rho_{n+2s} = 0$ for $s \geq (M-n)/2$. Further, with $(m,n) \in \Omega_{MN}$ and $-n/2 \leq s < (M-n)/2$, it is easy to show that $\mathbb{R}_m^M(n+2s) = \{0\}$. This, combined with (2.14) and (3.7), leads to $Q_{mn}^{MN}(n+2s) = \delta_s$ for $-n/2 \leq s < (M-n)/2$ and $(m,n) \in \Omega_{MN}$. The assertions follow immediately from (3.6) and (3.8). ⊔

Note that Theorem 3.2 does not require $\mu(\theta,\phi) = E\{T(\theta,\phi)\}$ to be band-limited since the spectrum is the only consideration there. However, if the field $T(\theta,\phi)$ itself needs to be reconstructed, a band-limited $\mu(\theta,\phi)$ is desirable. This gives rise to the next theorem.

THEOREM 3.3. *Suppose that $T(\theta,\phi)$ is a homogeneous field of the form (1.1), where $\mu(\theta,\phi)$ is band-limited with bandwidth (M_μ, N_μ) and $\epsilon(\theta,\phi)$ has a band-limited spectrum with bandwidth N_ρ. If the samples $T(\theta_p,\phi_q)$ are obtained from Gaussian sampling with $N \geq M > \max(M_\mu, N_\mu, N_\rho)$, then, for any (M',N') satisfying $M \geq M' \geq \max(M_\mu, N_\rho)$ and $N \geq N' \geq \max(N_\mu, M')$, the reconstruction equation (2.15) holds with probability one.*

PROOF. Let the right-hand side of (2.15) be denoted by $\tilde{T}(\theta,\phi) = \tilde{\mu}(\theta,\phi) + \tilde{\epsilon}(\theta,\phi)$, where $\tilde{\mu}(\theta,\phi)$ and $\tilde{\epsilon}(\theta,\phi)$ are the resulting fields when $T(\theta_p,\phi_q)$ is replaced with $\mu(\theta_p,\phi_q)$ and $\epsilon(\theta_p,\phi_q)$, respectively, on the right-hand side of (2.15). Applying Theorem 2.2 to $\mu(\theta,\phi)$ yields $\mu(\theta,\phi) = \tilde{\mu}(\theta,\phi)$ and hence $T(\theta,\phi) - \tilde{T}(\theta,\phi) = \epsilon(\theta,\phi) - \tilde{\epsilon}(\theta,\phi)$. Therefore, it suffices to show that $E\{(\epsilon(\theta,\phi) - \tilde{\epsilon}(\theta,\phi))^2\} = 0$. To this end, one can combine the expression of $\tilde{\epsilon}(\theta,\phi)$ with (2.16), (3.3), and the band-limited property of ρ_n to obtain

$$\begin{aligned} E\{\epsilon(\theta,\phi)\tilde{\epsilon}(\theta,\phi)\} = & \sum_{(m,n)\in\Omega_{M'N'}} \sum_{(u,v)\in\Omega_{M'N'}} 2\pi\sigma^2\rho_n Y_v^u(\theta,\phi) Y_n^{m*}(\theta,\phi) \\ & \times \sum_{p=0}^{N-1}\sum_{q=0}^{2M-1} w_p^\theta w_q^\phi Y_v^u(\theta_p,\phi_q) Y_n^{m*}(\theta_p,\phi_q)\sin\theta_p. \end{aligned}$$

This, according to (2.19) and (3.3), reduces to

$$\sum_{(m,n)\in\Omega_{M'N'}} 2\pi\sigma^2\rho_n Y_n^m(\theta,\phi) Y_n^{m*}(\theta,\phi) = \sigma^2\rho(1) = \sigma^2.$$

Similarly, one obtains $E\{\tilde{\epsilon}^2(\theta,\phi)\} = \sigma^2$. The proof is complete upon noting that $E\{\epsilon^2(\theta,\phi)\} = \text{Var}\{T(\theta,\phi)\} = \sigma^2$. □

As a by-product of Theorem 3.3, the following corollary establishes the equivalence between having a band-limited spectrum and being band-limited.

COROLLARY 3.1. *A zero-mean homogeneous field $\epsilon(\theta,\phi)$ has a band-limited spectrum with bandwidth N_0 if and only if the field is band-limited with bandwidth (N_0, N_0).*

PROOF. The "if" is trivial since by definition $\epsilon(\theta,\phi)$ has a Laplace expansion of the form (1.2) with $\epsilon_{mn} = 0$ and hence $\text{Var}\{\epsilon_{mn}\} = 2\pi\rho_n = 0$ for $n > N_0$. The "only if" can be shown by applying Theorem 3.3 to $\epsilon(\theta,\phi)$. In fact, with $M' = N' = N_0$ and $\epsilon(\theta,\phi)$ in place of $T(\theta,\phi)$, one can use (2.16) to rewrite (2.15) as a series of the form (1.2), where the sum is limited to $(m,n) \in \{(m,n) : 0 \leq n \leq N_0, |m| \leq n\}$ and the coefficients ϵ_{mn} are defined by

$$\epsilon_{mn} := \sum_{p=0}^{N-1} \sum_{q=0}^{2M-1} w_p^\theta w_q^\phi \epsilon(\theta_p,\phi_q) Y_n^{m*}(\theta_p,\phi_q) \sin\theta_p$$

with $\epsilon(\theta_p,\phi_q)$ obtained by Gaussian sampling for some $N \geq M > N_0$. The assertion follows from the definition of band-limited fields. □

4. Convergence of Laplace series

The mean-square convergence of Laplace series (1.2) plays an important role in Theorems 2.1, 3.1, and 3.2. In order to ensure the convergence, it is necessary and sufficient that (i) the Laplace series with coefficients μ_{mn} converges to $\mu(\theta,\phi)$ and (ii) the Laplace series with coefficients $\epsilon_{mn} := T_{mn} - \mu_{mn}$ converges in mean square to $\epsilon(\theta,\phi)$.

To satisfy condition (i), it is sufficient that $\mu(\theta,\phi)$ be absolutely integrable on the sphere, continuous at (θ,ϕ), and of bounded variation in a certain sense (Hobson (1955), pp. 342–345); in other words, condition (i) requires $\mu(\theta,\phi)$ to be "sufficiently smooth." To study condition (ii), let

$$\epsilon_N(\theta,\phi) := \sum_{(m,n)\in\Omega_N} \epsilon_{mn} Y_n^m(\theta,\phi)$$

be the truncated Laplace series, where $\Omega_N := \Omega_{NN} = \{(m,n) : 0 \leq n \leq N, |m| \leq n\}$; then, it is straightforward to show that

$$E|\epsilon(\theta,\phi) - \epsilon_N(\theta,\phi)|^2 = R(\theta,\phi;\theta,\phi) - 2\int R(\theta,\phi;\theta',\phi') K_N(\theta,\phi;\theta',\phi')$$
$$+ \iint R(\theta',\phi';\theta'',\phi'') K_N(\theta,\phi;\theta',\phi') K_N(\theta,\phi;\theta'',\phi''),$$

where the integrals are defined on the unit sphere. Since $K_N(\theta,\phi;\theta',\phi')$ tends to a Dirac delta on the sphere as $N\to\infty$, it follows that $E|\epsilon(\theta,\phi)-\epsilon_N(\theta,\phi)|^2\to 0$ provided $R(\theta',\phi';\theta'',\phi'')$ is continuous in (θ',ϕ') and (θ'',ϕ'') at $(\theta,\phi;\theta,\phi)$. This is equivalent to requiring $\epsilon(\theta,\phi)$ to be continuous in mean square at (θ,ϕ) (Loève (1978), Chapter XI).

Based on these results, one can conclude that if $\mu(\theta,\phi)$ is "sufficiently smooth" and $\epsilon(\theta,\phi)$ is continuous in mean square the Laplace series (1.2) converges to $T(\theta,\phi)=\mu(\theta,\phi)+\epsilon(\theta,\phi)$ in mean square. Since the mean-square continuity of $\epsilon(\theta,\phi)$ also ensures (ordinary) continuity of $\rho(x)$, the spectrum ρ_n exists as defined by (3.1).

5. Concluding remarks

In this paper, we investigate the aliasing errors in the spherical harmonic analysis of spherical random fields when sampled on a finite grid. The Gaussian sampling approach is employed to eliminate the aliasing errors for band-limited fields. Future research should extend these results to other orthonormal expansions of spherical random fields with suitable sampling schemes.

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