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REGRESSIONAL CHARACTERIZATION OF THE GENERALIZED INVERSE GAUSSIAN POPULATION

J. PUSZ

Mathematical Institute, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland

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Abstract. A characterization of the generalized inverse Gaussian population is obtained from a condition of constant regression of a suitable statistic on a linear one.

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1. Introduction

Let $X_1, X_2, \dots, X_n$ be a sample of size n (independently and identically distributed random variables (r.v's)) from a population with a distribution function $F(x)$. We write $\Lambda = \sum_{i=1}^{n} X_i$ for the sum of the observations and $S = S(X_1, X_2, \dots, X_n)$ for another statistic.

In many cases it is possible to find a statistic S which has constant regression on Λ. Moreover one can choose such S that this property determines $F(x)$. For the normal population and also for Poisson, binomial, negative binomial, gamma and hypergeometric such characterization was obtained by Laha and Lukacs (1960), Bolger and Harkness (1965), Gordon (1973). This characterization for the inverse Gaussian population was obtained by Khatri (1962), Roy and Wasan (1969) and Seshadri (1983).

The inverse Gaussian distribution with parameters α and β (i.e. the first-passage-time distribution of Browian motion with positive drift defined by Tweedie (1957)), is a particular case ($\mu = -1/2$) of the generalized inverse Gaussian distribution (considered by Barndorff-Nielsen and Halgreen (1977), see Seshadri ((1993), pp. 26–30) for an history of this distribution) determined by probability density function with parameters $\alpha > 0$, $\beta > 0$, $\mu \in (-\infty, \infty)$:

$$f_{\alpha,\beta,\mu}(x) = \frac{(\alpha/\beta)^{\mu/2}}{2K_\mu(2\sqrt{\alpha\beta})} x^{\mu-1} \exp(-\alpha x - \beta x^{-1}) \tag{1.1}$$

on $(0,\infty)$, where $K_\mu(z)$ is the modified Bessel function of the second kind

$$K_\mu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^\mu \int_0^\infty u^{-\mu-1}\exp\left(-u-\frac{z^2}{4u}\right)du$$

for complex z such that $\operatorname{Re} z^2 > 0$ (e.g. Watson (1958), p. 183).

We shall use the notation $X \sim GIG(\alpha,\beta,\mu)$ to denote that r.v. X has a generalized inverse Gaussian distribution with parameters α, β and μ.

In this note we give a characterization of the GIG population based on the constancy of regression of a suitable statistic S on Λ.

2. Statistics characterising GIG's populations

We state a moments characterization of $GIG(\alpha,\beta,\mu)$ distribution which is a basic result for our considerations.

LEMMA 2.1. *Let X be a positive r.v. and $m_n = EX^n$ its moments $(n = 0,1,\ldots)$ and $\alpha > 0$, $\beta > 0$, $\mu \in (-\infty,\infty)$. Then $X \sim GIG(\alpha,\beta,\mu)$ if, and only if,*

$$m_n = (\beta/\alpha)^{n/2}K_{\mu+n}(2\sqrt{\alpha\beta})K_\mu^{-1}(2\sqrt{\alpha\beta}), \qquad n = 0,1,\ldots.$$

Moreover, any $X \sim GIG(\alpha,\beta,\mu)$ has finite moments for all integer n and they are uniquely determined by the recurrence formulae

$$\alpha m_{n+2} = (\mu+n+1)m_{n+1} + \beta m_n \tag{2.1}$$

with $m_0 = 1$, $m_1 = (\beta/\alpha)^{1/2}K_{\mu+1}(2\sqrt{\alpha\beta})K_\mu^{-1}(2\sqrt{\alpha\beta})$.

PROOF. Assume that X has the $GIG(\alpha,\beta,\mu)$ distribution. Then using (1.1) we get

$$m_n = (\beta/\alpha)^{n/2}K_{\mu+n}(2\sqrt{\alpha\beta})K_\mu^{-1}(2\sqrt{\alpha\beta})$$

for any integer n. On application of identity

$$zK_{\nu-1}(z) - zK_{\nu+1}(z) = -2\nu K_\nu(z) \tag{2.2}$$

(see, Watson (1958), p. 79) leads to (2.1).

To prove the converse we use the Carleman criterion. We shall prove that if moments $(m_n)_{n=1,2,\ldots}$ satisfy (2.1) then the series $\sum_{n=1}^{\infty}(m_n)^{\;1/2n}$ is divergent.

From (2.1) we see that for $n \ge n_0 \ge \max[1,\alpha-\mu]$, $m_{n+1} > m_n$ and

$$m_{n_0+k} < \frac{x(x+1)\cdots(x+k-1)}{\alpha^k}m_{n_0}, \qquad k = 1,2,\ldots,$$

where $x = n_0+\mu+\beta > 0$. From this

$$(m_{n_0+k})^{-1/2(n_0+k)} > \left(\frac{\alpha^k}{m_{n_0}}\right)^{1/2(n_0+k)} \frac{1}{(x(x+1)\cdots(x+k-1))^{1/2(n_0+k)}}.$$

Since $\lim_{k\to\infty}(\alpha^k/m_{n_o})^{1/2(n_0+k)} = \sqrt{\alpha} > 0$ and

$$\frac{1}{(\sqrt{x}\sqrt{x+1}\cdots\sqrt{x+k-1})^{1/(n_0+k)}} > \frac{n_0+k}{\sqrt{x}+\sqrt{x+1}+\cdots+\sqrt{x+k-1}+n_0} > \frac{1}{2\sqrt{x+k}}$$

then for large k and some positive $c > 0$,

$$(m_{n_0+k})^{-1/2(n_0+k)} > \frac{c}{\sqrt{x+k}}.$$

Clearly $\sum_{k=1}^{\infty} 1/\sqrt{x+k} = \infty$ and this implies that $\sum_{n=1}^{\infty}(m_n)^{-1/2n} = \infty$. □

THEOREM 2.1. *Let $X_1, X_2, \ldots, X_n$ be a sample of size n (i.e. $X_1, X_2, \ldots, X_n$ are independent and identically distributed) taken from a positive population with finite moments $m_{-2} = EX_1^{-2}$, $m_{-1} = EX_1^{-1}$, $m_1 = EX_1$ and*

$$\Lambda = X_1 + X_2 + \cdots + X_n,$$
$$S = \Lambda\left(p\sum_{k=1}^{n} X_k^{-1} + q\sum_{k=1}^{n} X_k^{-2}\right) - nq\sum_{k=1}^{n} X_k^{-1},$$

where $q > 0$ and p is real such that

$$pm_{-1} + qm_{-2} > 0. \tag{2.3}$$

Then

$$E(S \mid \Lambda) = c \tag{2.4}$$

holds almost everywhere if, and only if, the following two conditions are satisfied:

(i) *$X_1 \sim GIG(\alpha, \beta, \mu)$ for some $\alpha > 0$, $\beta > 0$ and $\mu \in (-\infty, \infty)$,*

(ii) *there exists $\gamma > 0$ such that $p = (\mu - 1)\gamma$, $q = \beta\gamma$, $c = n(n\mu - 1)\gamma$.*

PROOF. To prove the necessity of conditions (i) and (ii) we assume that (2.4) is satisfied. By Lemma 1.1.1 of Kagan, Linnik and Rao (1973), S has constant regression on Λ if, and only if,

$$E\left(\sum_{j=1}^{n} X_j\left(p\sum_{k=1}^{n} X_k^{-1} + q\sum_{k=1}^{n} X_k^{-2}\right) - nq\sum_{k=1}^{n} X_k^{-1}\right)\exp\left(it\sum_{j=1}^{n} X_j\right)$$
$$= cE\exp\left(it\sum_{j=1}^{n} X_j\right).$$

Let $\varphi(t) = E\exp(itX_1)$ be the characteristic function of X_1 and $f(t) = EX_1^{-2}\exp(itX_1)$. Then this formula is equivalent to

$$(c-np)(-f''(t))^n - (n-1)nqif'(t)(-f''(t))^{n-1}$$
$$= (n-1)npf'(t)f'''(t)(-f''(t))^{n-2} + (n-1)nqif(t)f'''(t)(-f''(t))^{n-2}.$$

Since $f''(t) = -\varphi(t)$ then there exist a neighbourhood V of the origin such that for $t \in V$, $f''(t) \neq 0$. The above equation is equivalent to

$$i\frac{c - np}{(n-1)n}(f''(t))^2 - qf'(t)f''(t) = (ipf'(t) - qf(t))f'''(t) \tag{2.5}$$

for $t \in V$. From (2.5) for $t = 0$ we get

$$\frac{c - np}{(n-1)n} = p + \gamma, \qquad \gamma = p(m_{-1}m_1 - 1) + q(m_{-2}m_1 - m_{-1}), \tag{2.6}$$

and using (2.3) we see that

$$\gamma > -q\frac{m_{-2}}{m_{-1}}(m_{-1}m_1 - 1) + q(m_{-2}m_1 - m_{-1}) = q\frac{m_{-2} - m_{-1}^2}{m_{-1}} > 0.$$

Let

$$h(t) = \frac{ipf'(t) - qf(t)}{f''(t)}, \qquad t \in V. \tag{2.7}$$

The differential equation (2.5) reduces in view of (2.7) to

$$h'(t) + i\gamma = 0$$

and $h(t) = a - i\gamma t$, $t \in V$ for some constant a. From (2.7)

$$ipf'(t) - qf(t) = (a - i\gamma t)f''(t) \quad \text{for} \quad t \in V. \tag{2.8}$$

Taking $t = 0$ in (2.8) we get

$$a = pm_{-1} + qm_{-2}.$$

Moreover from(2.8) we get that f have derivatives of any order on V and

$$(a - i\gamma t)f^{(n+2)}(t) = i(p + n\gamma)f^{(n+1)}(t) - qf^{(n)}(t), \qquad n = 0, 1, \ldots. \tag{2.9}$$

Then (2.9) implies ($m_n = EX_1^n$)

$$am_n = (p + n\gamma)m_{n-1} + qm_{n-2}, \qquad n \geq 0.$$

By Lemma 2.1 we conclude that $X_1 \sim GIG(\alpha, \beta, \gamma)$ for $\beta = q/\gamma$, $\mu = p/\gamma + 1$, $\alpha = (\mu - 1)m_{-1} + \beta m_{-2}$. From (2.6) we obtain $c = n(n\mu - 1)\gamma$.

Now to prove the sufficiency of (2.4) we assume that the density function $f_{\alpha,\beta,\mu}(x)$ of the population is given by (1.1) and that p, q and c satisfy (ii). In this case we compute the right side of (2.7) ($t \in V = R$)

$$h_{\alpha,\beta,\mu}(t) = (\alpha - it)\frac{p\frac{2}{z(t)}K_{\mu-1}(z(t)) + q\beta^{-1}K_{\mu-2}(z(t))}{K_\mu(z(t))},$$

where $z(t) = 2\sqrt{\beta(\alpha - it)}$.

If $\mu = 1$ then from (ii) $p = 0$ and $h_{\alpha,\beta,1}(t) = \gamma(\alpha - it)$ since $K_1(z) = K_{-1}(z)$.

If $\mu \neq 1$ then by application of identity (2.2)

$$h_{\alpha,\beta,\mu}(t) = \frac{p}{\mu - 1}(\alpha - it) + \frac{1}{\beta}(\alpha - it)\left(q - \frac{p\beta}{\mu - 1}\right)\frac{K_{\mu-2}(z(t))}{K_\mu(z(t))}$$

and from (ii) $h_{\alpha,\beta,\mu}(t) = \gamma(\alpha - it)$.

In both cases

$$ipf'(t) - qf(t) = \gamma(\alpha - it)f''(t)$$

and by simple computation we see that f satisfies (2.5) for any real t. Since (2.5) is equivalent to (2.4) this ends the proof. □

Similarly as in Theorem 2.1 we can prove the following more general.

THEOREM 2.2. *Let $X_1, X_2, \ldots, X_n$ be a sample of size n taken from a positive population with finite moments $m_{-2} = EX_1^{-2}$, $m_{-1} = EX_1^{-1}$, $m_1 = EX_1$ and $q_1, \ldots, q_n$, $p_1, \ldots, p_n$ be real numbers such that $q = \sum_{k=1}^n q_k > 0$, $p = \sum_{k=1}^n p_k$ and $pm_{-1} + qm_{-2} > 0$. Let*

$$\Lambda = X_1 + X_2 + \cdots + X_n,$$
$$S = \Lambda\left(\sum_{k=1}^n p_k X_k^{-1} + \sum_{k=1}^n q_k X_k^{-2}\right) - n\sum_{k=1}^n q_k X_k^{-1}.$$

Then the regression of S on Λ is constant if, and only if, the population is $GIG(\alpha, \beta, \mu)$ with parameters satisfying condition (ii) *of Theorem* 2.1.

REFERENCES

Barndorff-Nielsen, O. and Halgreen, Ch. (1977). Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions, *Zeit. Wahrscheinlichkeitsth.*, **38**, 309–311.

Bolger, E. M. and Harkness, W. L. (1965). Characterization of some distributions by conditional moments, *Ann. Math. Statist.*, **36**, 703–705.

Gordon, F. S. (1973). Characterizations of populations using regression properties, *Ann. Statist.*, 1(1), 114–126.

Kagan, A. M., Linnik, Y. V. and Rao, C. R. (1973). *Characterization Problems in Mathematical Statistics*, Wiley, New York.

Khatri, C. G. (1962). A characterization of the inverse Gaussian distribution, *Ann. Math. Statist.*, **33**, 800–803.

Laha, R. G. and Lukacs, E. (1960). On some characterization problems connected with quadratic regression, *Biometrika*, **47**, 335–343.

Roy, L. K. and Wasan, M. T. (1969). A characterization of the inverse Gaussian distribution, *Sankhyā Ser. A*, **31**, 217–218.

Seshadri, V. (1983). The inverse Gaussian distribution: some properties and characterizations, *Canad. J. Statist.*, **11**(2), 131–136.

Seshadri, V. (1993). The inverse Gaussian distribution, Clarendon Press, Oxford.

Tweedie, M. C. K. (1957). Statistical properties of inverse Gaussian distributions. I, *Ann. Math. Statist.*, **28**, 362–377.

Watson, G. N. (1958). *A Treatise of the Theory of Bessel Functions*, University Press, Cambridge.