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# ON GEOMETRIC-STABLE LAWS, A RELATED PROPERTY OF STABLE PROCESSES, AND STABLE DENSITIES OF EXPONENT ONE*

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**Abstract.** Klebanov *et al.* (1985, *Theory Probab. Appl.*, **29**, 791–794) introduced a class of probability laws termed by them "geometrically-infinitely-divisible" laws, and studied in detail the sub-class of "geometrically-strictly-stable" laws. In Section 2 of the present paper, the larger sub-class of "geometric-stable" laws is (defined and) studied. In Section 3, a characterization of stable processes involving (stochastic integrals and) geometric-stable laws is presented. In Section 4, the asymptotic behaviour of stable densities of exponent one (and $|\beta| < 1$) is studied using only real analysis methods. In an Appendix, "geometric domains of attraction" to geometric-stable laws are investigated, motivated by the work of Mohan *et al.* (1993, *Sankhyā Ser. A*, **55**, 171–179).

*Key words and phrases*: Stable laws and processes, geometric-stable laws, geometric domains of attraction.

## 1. Introduction

Klebanov, Maniya and Melamed (1985)—referred to hereafter as KMM (1985)—considered a problem posed by V. M. Zolotarev, namely, the investigation of characteristic functions (ch.f.'s) (of probability distributions on the real line $\mathbb{R}$) satisfying the relations:

$$f(t) = g_p(t)\{p + (1-p)f(t)\}\ \forall t \in \mathbb{R} \tag{1.1}$$

for every $p \in (0,1)$, where $g_p$ is itself a ch.f. Such an $f$ may obviously be written in the form

$$f = \sum_{j=1}^{\infty} p(1-p)^{j-1} g_p^j = p \cdot g_p / \{1 - (1-p) g_p\} \tag{1.2}$$

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for every $p \in (0,1)$, and is necessarily infinitely divisible (inf. div.). KMM (1985) called such $f$ *geometric-infinitely-divisible* (GID) ch.f.'s, as the form (1.2) suggests. They established a de Finetti type result for such $f$ (analogous to that for inf. div. ch.f.'s), and, using that result, the following basic characterization:

PROPOSITION 1.1. (KMM, 1985) *$f$ is a GID ch.f. iff it is of the form $(1+\psi)^{-1}$, where $e^{-\psi}$ is an inf. div. ch.f. (i.e., iff $f$ is never zero and* $\exp(1-1/f)$ *is an inf. div. ch.f.).*

It may be worthwhile recording here an alternative proof of the above result: If $e^{-\psi}$ is an inf. div. ch.f., then $e^{-t\psi}$ is a ch.f. for every $t \geq 0$. Hence, for all $\alpha > 0$, $p > 0$, the "mixture"

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-pt\psi} t^{\alpha-1} e^{-t} dt = (1+p\psi)^{-\alpha}$$

is a ch.f. In particular, $(1+\psi)^{-1}$ is an inf. div. ch.f. and (1.2) holds for all $p \in (0,1)$ with $g_p =$ the (inf. div.) ch.f. $(1+p\psi)^{-1}$. Conversely, if $(1+\psi)^{-1}$ is a GID ch.f., then $g_p$, in (1.1) or (1.2), is equal to $(1+p\psi)^{-1}$, so that this is a ch.f. for every $p \in (0,1)$. Hence, for every $\alpha > 0$, $h_{\alpha,n} = (1+\alpha\psi/n)^{-n}$ is a ch.f. for every integer $n \geq \alpha$, and (the continuous function) $e^{-\alpha\psi} = \lim_{n\to\infty} h_{\alpha,n}$ is therefore a ch.f. Thus $e^{-\psi}$ is an inf. div. ch.f. Hence the proposition.

KMM (1985) studied the sub-class of GID laws of the form $(1+\psi)^{-1}$ where $e^{-\psi}$ is a "strictly stable" ch.f., calling such laws "geometric-strictly-stable" laws, and obtained a characterization thereof in terms of random sums of independent random variables. Other sub-classes such as of "geometric-semi-stable" laws, Mittag-Leffler laws, and the Linnik class, namely, of ch.f.'s of the form

$$f(t) = (1+\lambda|t|^\alpha)^{-1} \ \forall t \in \mathbb{R}; \qquad \lambda > 0, \ 0 < \alpha \leq 2 \tag{1.3}$$

were considered in some of the other papers listed as references below.

In the present paper, we consider properties of "geometric-stable" laws (Section 2), and a related property of stable processes (Section 3). Section 4 comprises a real analysis discussion of an asymptotic expansion for stable densities with exponent one. In an Appendix, we consider "geometric-stable" laws in relation to "geometric domains of attraction".

## 2. Properties of "geometric-stable" laws

Let $\alpha \in (0,2]$. The stable laws are given by $\exp(-\psi_\alpha)$, where (suppressing in our notation the dependence of the formula on the parameters $\lambda, \mu, \beta$)

$$\psi_\alpha(t) = \begin{cases} -i\mu t + \lambda|t|^\alpha\{1+i(\operatorname{sgn} t)\beta\tan(\pi\alpha/2)\} & \text{for} \quad \alpha \neq 1, \\ -i\mu t + \lambda|t|\{1+i(\operatorname{sgn} t)(2\beta/\pi)\log|t|\} & \text{for} \quad \alpha = 1, \end{cases} \tag{2.1}$$

where $\mu$ and $\beta$ are real numbers, with $|\beta| \leq 1$, and $\lambda > 0$. For detailed discussions of stable laws, we refer the reader to Gnedenko and Kolmogorov (1954), Ibragimov and Linnik (1971), Feller (1971), and Lukacs (1970), and to Ramachandran and Lau (1991) for some aspects thereof. By Proposition 1.1,

$$f_\alpha := (1+\psi_\alpha)^{-1} \tag{2.2}$$

is a GID ch.f. We refer to such an $f_\alpha$ as a "geometric-stable" ch.f. with exponent $\alpha$.

The following de Finetti type result holds for such $f$ (cf. Theorem 2.2 of Mohan *et al.* (1993) on "geometric-strictly-stable" ch.f.'s):

PROPOSITION 2.1. *A ch.f. $f$ is "geometric stable" iff it is (non-vanishing on $\mathbb{R}$ and) of the form*

$$f(t) = \lim_{n\to\infty} \{1 + n(1 - h(t/a_n)) + i\mu_n t\}^{-1} \tag{2.3}$$

*for some real sequences $\{a_n\}$ and $\{\mu_n\}$ with $a_n > 0$ $\forall n$ and some non-degenerate ch.f. $h$.*

PROOF. If (2.3) holds, then $f = (1+\psi)^{-1}$, where

$$\psi(t) = \lim_{n\to\infty} [n\{1 - h(t/a_n)\} + i\mu_n t].$$

Write $⨍$ to denote integration over $\mathbb{R}\setminus\{0\}$, and let $H$ denote the d.f. corresponding to the ch.f. $h$. Then, for an appropriate sequence $\{\mu_n'\}$ of real constants, we have for all $t\in\mathbb{R}$, as $n\to\infty$,

$$n ⨍ \left(e^{itx/a_n} - 1 - \frac{itx/a_n}{1+x^2}\right) dH(x) + i\mu_n' t \to -\psi(t),$$

i.e.,

$$n\tilde{\psi}(t/a_n) + i\mu_n' t \to -\psi(t),$$

$\tilde{\psi}$ being the log ch.f. of an inf. div. law with the Lévy representation $L(0,0,M,N)$, where $M = H$ on $(-\infty,0)$ and $N = H - 1$ on $(0,\infty)$. If $\tilde{f}$ is the ch.f. $\exp(\tilde{\psi})$, then we have from the above that

$$\{\tilde{f}(t/a_n)\}^n \exp(i\mu_n' t) \to \exp(-\psi(t)).$$

It follows immediately (from the theory of normed sums of i.i.d.r.v.'s) that $\exp(-\psi)$ is a stable ch.f., so $f$ is a geometric stable ch.f.

Conversely, if $f = (1+\psi_\alpha)^{-1}$, where $\exp(-\psi_\alpha)$ is a stable ch.f., then $\psi_\alpha(t) = n\psi_\alpha(t/n^{1/\alpha}) + i\mu_n t$ $\forall n \in \mathbb{N}$, for suitable real $\mu_n$, hence $= \lim_{n\to\infty}[n\{1 - h_\alpha(t/n^{1/\alpha})\} + i\mu_n t]$, with $h_\alpha = \exp(-\psi_\alpha)$ itself if $\alpha < 2$. If $\alpha = 2$, so that $\psi_\alpha(t)$ is of the form: $-i\mu t + \lambda t^2$, we may take $h_\alpha(t) = \exp(i\tilde{\mu}t - \tilde{\lambda}t^2)$, where $0 < \tilde{\lambda} < \lambda$ and $\tilde{\lambda} + \frac{1}{2}\tilde{\mu}^2 = \lambda$, $\mu_n = \tilde{\mu}\sqrt{n} - \mu$. Hence the proposition.

*Remark.* In the Appendix, in the course of the proofs of Theorems A.2 and A.3, it is shown that a ch.f. $1/(1+\psi)$ is GS (also) if and only if (A.5) holds. Thus (A.5) and (2.3) are equivalent conditions.

The following is the main result of this paper, and relates to some properties of such laws:

THEOREM 2.1. (i) *$f_\alpha$ belongs to the "domain of normal attraction" of the stable law* $\exp(-\psi_\alpha)$, $0 < \alpha \le 2$.

(ii) *The corresponding d.f. $F_\alpha$ has absolute moments of all orders $< \alpha$ but not of orders $\ge \alpha$ if $0 < \alpha < 2$. If $\alpha = 2$, $F_\alpha$ has m.g.f., and, in particular, has moments of all orders.*

(iii) *$F_\alpha$ is absolutely continuous (w.r. to Lebesgue measure).*

(iv) *The following $F_\alpha$ are all "self-decomposable" (hence "unimodal");* (a) *if $\alpha = 2$, without restriction on $\lambda$, $\mu$;* (b) *if $0 < \alpha < 1$ or if $1 < \alpha < 2$, for $\mu = 0$; if $\alpha = 1$, for $\mu = \beta = 0$.*

(v) *For $1 < \alpha < 2$, $\beta = 0$, $\mu \ne 0$, $F_\alpha$ is not self-decomposable.*

PROOF. (i) For the case $\alpha = 2$, we define $c^2 = 1 + (\mu^2/2\lambda)$. We then take:

$$a_n = \begin{cases} n^{1/\alpha} & \text{for} \quad \alpha \ne 2, \\ cn^{1/2} & \text{for} \quad \alpha = 2, \end{cases}$$

and

$$\mu_n = \begin{cases} \mu(1 - n^{1-1/\alpha}) & \text{for} \quad \alpha \ne 1, 2, \\ (2\lambda\beta/\pi)\log n & \text{for} \quad \alpha = 1, \\ \mu\left(1 - \dfrac{\sqrt{n}}{c}\right) & \text{for} \quad \alpha = 2. \end{cases}$$

Straightforward computations show that, as $n \to \infty$, for all $\alpha \in (0, 2]$,

$$\{f_\alpha(t/a_n)\}^n e^{i\mu_n t} \to e^{-\psi_\alpha(t)} \ \forall t \in \mathbb{R}. \tag{2.4}$$

The "norming constant" $(1/a_n)$ being of the form: const. $n^{-1/\alpha}$ in all cases, (2.4) means that $f_\alpha$ belongs, by definition, to the "domain of normal attraction" of the stable law $\exp(-\psi_\alpha)$. For relevant definitions and results, we refer the reader to (Section 35 of) Gnedenko and Kolmogorov (1954) or Ibragimov and Linnik (1971).

*Remark.* A result due to B. V. Gnedenko (see, for instance, Gnedenko and Kolmogorov (1954) or Ibragimov and Linnik (1971)) asserts that, in the cases $0 < \alpha < 2$, we must then have

$$\begin{aligned} F_\alpha(x) &= \lambda_1|x|^{-\alpha}\{1 + o(1)\} \quad \text{as} \quad x \to -\infty, \\ 1 - F_\alpha(x) &= \lambda_2 x^{-\alpha}\{1 + o(1)\} \quad \text{as} \quad x \to \infty, \end{aligned} \tag{2.5}$$

where $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, with $\lambda_1 + \lambda_2 = \lambda$, the $\lambda_j$ being the same constants as appear in the Lévy representation $L(\mu, 0, M, N)$ for $-\psi_\alpha$, namely, $M(u) = \lambda_1/|u|^\alpha$ for $u < 0$, $N(u) = -\lambda_2/u^\alpha$ for $u > 0$, and $\lambda$ being the same as in (2.1).

(ii) This assertion follows at once from (2.5) in the cases $0 < \alpha < 2$. It also follows immediately from the form of the ch.f. $f_\alpha$ since, as $t \to 0$, $f_\alpha(t) =$

$1 - \psi_\alpha(t) + o(|t|^\alpha)$, from the general results in Ramachandran (1969), also partly reproduced in Section 1.3 of Ramachandran and Lau (1991). If $\alpha = 2$, $f_\alpha$ is an "analytic ch.f."—equivalently $F_\alpha$ has m.g.f., and, in particular, moments of all orders exist.

(iii) We begin by noting an auxiliary result of some independent interest.

LEMMA 2.1. (a) *If $\alpha \neq 1$, $F_\alpha$ is the d.f. of the random variable (r.v.)*

$$XZ^{1/\alpha} + \mu(Z - Z^{1/\alpha}),$$

*where $X$ is a r.v. with the stable ch.f.* $\exp(-\psi_\alpha)$ *and $Z$ is a r.v. independent of $X$ and having a standard exponential d.f.*

(b) *If $\alpha = 1$, $F_\alpha$ is the d.f. of the r.v.*

$$XZ + (2\beta\lambda/\pi)Z\log Z,$$

*where $X$ and $Z$ are as described in* (a).

PROOF. (a) For $\alpha \neq 1$,

$$\begin{aligned} &E\{\exp[iu\{XZ^{1/\alpha} + \mu(Z - Z^{1/\alpha})\}]\} \\ &\quad = EE\{\exp[\cdots] \mid Z\} \\ &\quad = \int_0^\infty E\exp[iu\{t^{1/\alpha}X + \mu(t - t^{1/\alpha})\}]\cdot e^{-t}dt \\ &\quad = \int_0^\infty \exp\{-t\psi_\alpha(u) - t\}dt = \{1 + \psi_\alpha(t)\}^{-1} = f_\alpha(u). \end{aligned}$$

Hence (a). A similar computation leads to conclusion (b). Hence the lemma.

Continuing with the proof of (iii), we see that

$$\begin{aligned} F_\alpha(x) &= P[XZ^{1/\alpha} + \mu(Z - Z^{1/\alpha}) \le x] \\ &= \int_0^\infty G_\alpha\left(\frac{x}{t^{1/\alpha}} - \mu t^{1-1/\alpha} + \mu\right)e^{-t}dt, \qquad \text{if} \quad \alpha \neq 1 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} F_\alpha(x) &= P[XZ + (2\beta\lambda/\pi)Z\log Z \le x] \\ &= \int_0^\infty G_\alpha\left(\frac{x}{t} - (2\beta\lambda/\pi)\log t\right)e^{-t}dt, \qquad \text{if} \quad \alpha = 1 \end{aligned} \tag{2.7}$$

where, in all cases $G_\alpha$ is the stable d.f. with $\exp(-\psi_\alpha)$ as ch.f.

For $x > 0$, we may recast (2.6) and (2.7) respectively in the forms:

$$F_\alpha(x) = \alpha x^\alpha \int_0^\infty G_\alpha\left\{v - \mu\left(\frac{x}{v}\right)^{\alpha-1} + \mu\right\}\exp\{-(x/v)^\alpha\}v^{-\alpha-1}dv, \qquad \text{if} \quad \alpha \neq 1, \tag{2.8}$$

and

$$(2.9)\quad F_\alpha(x) = x\int_0^\infty G_\alpha\{v - (2\beta\lambda/\pi)(\log x - \log v)\}\exp\{-(x/v)\}v^{-2}dv, \qquad \text{if}\quad \alpha = 1.$$

For $x < 0$, dual forms hold. $G_\alpha$ and $p_\alpha = G'_\alpha$, the p.d.f. of $G_\alpha$, both being bounded and continuous on $\mathbb{R}$, a straightforward application of the dominated convergence theorem justifies differentiating the integrals appearing in the RHS of (2.8) and (2.9) w.r. to $x$ under the integral sign, leading to the conclusion that $P_\alpha = F'_\alpha$ exists and is continuous on $(0,\infty)$—and, by a dual argument, on $(-\infty, 0)$ as well. Hence (iii).

(iv) We need to show that, in both the cases (a) and (b) here, $f_\alpha/f_\alpha(c\cdot)$ is a ch.f. for every $c \in (0,1)$.

(a) If $\alpha = 2$, then $f_\alpha(t) = (1 - i\mu t + \lambda t^2)^{-1}$, $\mu$ real, $\lambda > 0$. For $0 < c < 1$, we have

$$\begin{aligned} f_\alpha(t)/f_\alpha(ct) &= c^2 + (1-c^2)\{1 - i\mu ct/(1+c)\}f_\alpha(t) \\ &= c^2 + \frac{1}{2}(1-c^2)\{(1-k_c)(1-i\theta_1 t)^{-1} + (1+k_c)(1-i\theta_2 t)^{-1}\}, \end{aligned}$$

where $k_c = (1-c)\mu/\{(1+c)\sqrt{\mu^2+4\lambda}\}$; $\theta_1 = \frac{1}{2}(\mu - \sqrt{\mu^2+4\lambda})$, and $\theta_2 = \frac{1}{2}(\mu + \sqrt{\mu^2+4\lambda})$: we note that $|k_c| < 1$ and $\theta_1$, $\theta_2$ are real. Thus $f_\alpha/f_\alpha(c\cdot)$ is a convex linear combination of three ch.f.'s and hence is itself a ch.f.

(b) If $0 < \alpha < 1$ or $1 < \alpha < 2$ and $\mu = 0$, or if $\alpha = 1$ and $\mu = \beta = 0$, then, for $0 < c < 1$,

$$f_\alpha(\cdot)/f_\alpha(c\cdot) = c^\alpha + (1-c^\alpha)f_\alpha$$

and, as a convex linear combination of two ch.f.'s, is itself a ch.f.

*Remark.* The families (a) and (b) include:

(A) The Linnik class of ch.f.'s defined by (1.3). Their self-decomposability has been pointed out in Lin (1994), who also noted that their absolute continuity is an immediate consequence of their self-decomposability as well as proved that property independently. Self-decomposability also implies "unimodality" (Yamazato (1977)). Hence, a Linnik-type ch.f. pertains to a unimodal d.f.—a fact derived using a different, *ad hoc* complex analysis argument by R. G. Laha (see Lukacs (1970)).

(B) The Mittag-Leffler laws, supported on $[0,\infty)$ and with $(1+t^\alpha)^{-1}$, $t \geq 0$, for $0 < \alpha < 1$, as their Laplace-Stieltjes transform, and their conjugate d.f.'s. They correspond to $0 < \alpha < 1$, $\beta = \pm 1$, in (2.1); their self-decomposability was pointed out by Pillai (1985).

(v) Let $1 < \alpha < 2$, $\beta = 0$, $\mu \neq 0$ in (2.1). Then

$$f_\alpha(t) = (1 - i\mu t + \lambda|t|^\alpha)^{-1}.$$

For $0 < c < 1$,

$$f_\alpha(t)/f_\alpha(ct) = c^\alpha + (1-c^\alpha)(1 - i\mu k_c t)f_\alpha(t),$$

where $k_c = (c - c^\alpha)/(1 - c^\alpha)$. We check below that, for $\gamma \neq 0$, $(1 - i\gamma t)f_\alpha(t)$ is the Fourier-Stieltjes transform of a function of bounded variation, which is not everywhere increasing, and so cannot be a d.f. It follows that $f_\alpha/f_\alpha(c\cdot)$ is not a ch.f., and so $f_\alpha$ is not self-decomposable.

We note that, for $\alpha > 1$,

$$P_\alpha(x) = F'_\alpha(x) = \int_0^\infty p_\alpha\left(\frac{x}{t^{1/\alpha}} - \mu t^{1-1/\alpha} + \mu\right) e^{-t} t^{-(1/\alpha)} dt$$

is bounded (and continuous) on $\mathbb{R}$, since $p_\alpha$ is bounded (and continuous) on $\mathbb{R}$ and since $\int_0^\infty e^{-t} t^{-(1/\alpha)} dt < \infty$ for such $\alpha$. Also,

$$P'_\alpha(x) = \int_0^\infty p'_\alpha\left(\frac{x}{t^{1/\alpha}} - \mu t^{1-1/\alpha} + \mu\right) e^{-t} t^{-(2/\alpha)} dt$$

is integrable over $\mathbb{R}$, since

$$\begin{aligned} \int_{-\infty}^{\infty} |P'_\alpha(x)| dx &\leq \int_0^\infty e^{-t} t^{-(2/\alpha)} \left\{ \int_{-\infty}^{\infty} \left| p'_\alpha\left(\frac{x}{t^{1/\alpha}} - \mu t^{1-1/\alpha} + \mu\right)\right| dx \right\} dt \\ &= \left(\int_0^\infty e^{-t} t^{-1/\alpha} dt\right) \int_{-\infty}^{\infty} |p'_\alpha(v)| dv < \infty, \end{aligned}$$

since $p'_\alpha(v)$ is bounded on $\mathbb{R}$ and is $O(|v|^{-\alpha-2})$ as $|v| \to \infty$.

For $\gamma \neq 0$, we then consider the integrable function $P_\alpha + \gamma P'_\alpha$, and set

$$V_{\alpha,\gamma}(x) = \int_{-\infty}^{x} \{P_\alpha(t) + \gamma P'_\alpha(t)\} dt.$$

A straightforward computation shows that $V_{\alpha,\gamma} * E_\gamma = F_\alpha$, where $E_\gamma$ is an exponential/conjugate exponential d.f. with parameter $\gamma$, having $\{1/(1 - it\gamma)\}$ as its ch.f. Hence

$$(1 - it\gamma) f_\alpha(t) = \int e^{itx} dV_{\alpha,\gamma}(x) \ \forall t \in \mathbb{R}.$$

Bug $V_{\alpha,\gamma}$, though a function of bounded variation, is not non-decreasing, since $P_\alpha + \gamma P'_\alpha$ takes negative values near the origin, as we check below: while $P_\alpha$ is bounded, we have, for $x > 0$,

$$P'_\alpha(x) = \int_0^\infty p'_\alpha(v - \mu(x/v)^{\alpha-1} + \mu) e^{-(x/v)^\alpha} \alpha x^{\alpha-2} v^{1-\alpha} dv$$

so that, as $x \to 0+$,

$$P'_\alpha(x)/\{\alpha x^{\alpha-2}\} \to \int_0^\infty p'_\alpha(v + \mu) v^{1-\alpha} dv$$

which is finite and negative since $p'_\alpha(v + \mu) \leq 0$ for $v \geq 0$, $\mu$ being the (unique) mode of the stable law with the particular $\exp(-\psi_\alpha)$ as its ch.f. to which we have restricted our attention. Hence $P_\alpha(x) + \gamma P'_\alpha(x) \to -\infty$ as $x \to 0+$, if $\gamma > 0$. A dual relation obtains as $x \to 0-$ if $\gamma < 0$. (Since the total 'algebraic' variation of $V_{\alpha,\gamma}$ on $\mathbb{R}$ is $= 1$, being the value of its Fourier-Stieltjes transform at the origin, $P_\alpha + \gamma P'_\alpha$ has to take positive values as well on $\mathbb{R}$.) Hence $V_{\alpha,\gamma}$ cannot be a d.f., and assertion (v) is proved.

## 3. A property of stable processes involving geometric-stable laws

Let $\{X(t), t \geq 0\}$ be a homogeneous stochastic process, continuous in probability, with independent increments and with $X(0) \equiv 0$. Then, in particular, $E\exp\{iuX(t)\} = \exp(-t\psi(u))$ where $e^{-\psi}$ is the ch.f. of $X(1)$. One may define the stochastic integral (in the sense of convergence in probability) of a real-valued continuous function $g$ defined on an interval $[a, b] \subset [0, \infty)$, denoted by $\int_\alpha^b g(t)dX(t)$, w.r.t. the process $\{X(t)\}$. For definitions and details, we refer the reader to Lukacs (1975) as also to Section 6.1 of Ramachandran and Lau (1991) for additional references, especially for the result cited in the next paragraph.

It was proved in Ramachandran (1994)—also see Ramachandran and Lau (1991)—that, for $\{X(t)\}$ to be a stable process (i.e., for $X(1)$ to have a stable ch.f.), it is necessary that, for every $y > 0$, and sufficient that, for some $y > 0$, the stochastic integral $\int_0^y tdX(t)$ have the same distribution as $X(s_y) + t_y$ for some $s_y > 0$ and real $t_y$.

In our present context, we can establish the following result suggested by the one just cited.

THEOREM 3.1. *A necessary and sufficient condition for $\{X(t)\}$ to be a stable process is that, $T$ being a r.v. independent of the process $\{X(t)\}$ and with standard exponential d.f., the r.v.*

$$\left(\int_0^T tdX(t)\right) \Big/ T$$

*have the same distribution as $X(\gamma T) + \delta T$, for some $\gamma > 0$ and real $\delta \cdot \gamma \in [1/3, 1]$ necessarily then. The common d.f. of these two r.v.'s is then a geometric-stable law with the same exponent as the process (the precise relationship being given by relation (3.2) below).*

PROOF. Let $e^{-\psi}$ be the ch.f. of $X(1)$. Then

$$(3.1)\qquad \begin{aligned} Ee^{iu(\int_0^T tdX(t)/T)} &= \int_0^\infty Ee^{iu(\int_0^t vdX(v)/t)}e^{-t}dt \\ &= \int_0^\infty \exp\left\{-\int_0^t \psi(uv/t)dv\right\}e^{-t}dt \\ &= \int_0^\infty \exp\left\{-t\int_0^1 \psi(uv)dv - t\right\}dt \\ &= \left\{1 + \int_0^1 \psi(uv)dv\right\}^{-1}. \end{aligned}$$

Also,

$$(3.2)\qquad \begin{aligned} Ee^{iu\{X(\gamma T)+\delta T\}} &= \int_0^\infty Ee^{iu\{X(\gamma t)+\delta t\}}e^{-t}dt \\ &= \int_0^\infty e^{-\gamma t\psi(u)+iu\delta t-t}dt = \{1 - i\delta u + \gamma\psi(u)\}^{-1}. \end{aligned}$$

Our identical distribution assumption is therefore equivalent to

$$\int_0^1 \psi(uv)dv = \gamma\psi(u) \quad i\delta u \tag{3.3}$$

where $e^{-\psi}$ is a ch.f. It is immediate from Ramachandran (1994)—also see Ramachandran and Lau (1991)—that the stable ch.f.'satisfy (3.3) and further are the only ch.f.'s which do so, with the exponent $\alpha$ being given by $\int_0^1 v^\alpha dv = \gamma$, i.e., $\alpha = (1/\gamma) - 1$. Hence the first assertion of the theorem. It is then immediate from (3.2) that the common d.f. of the two r.v.'s under discussion is a geometric-stable law with the same exponent as the process itself, the precise relationship between the ch.f.'s of these r.v.'s and ($e^{-\psi} =$) the ch.f. of $X(1)$ being given by (3.2).

## 4. Asymptotic behaviour of stable densities with exponent one

The following limited asymptotic expansion for stable densities was developed in the context of a possible extension of Theorem 2.1 (v) to the case $\alpha = 1$. It is given here as likely to be of independent interest, since it uses only real variable arguments—asymptotic expansions involving an appeal to Cauchy's theorem were provided long ago by Skorokhod (1954). Feller (1971) gives such for the case $\alpha \neq 1$, and the result below could supplement the discussion there. For the cases $\alpha = 1$, $\beta = \pm 1$ in (2.1), we still have to depend on Skorokhod (1954) for asymptotic expansions: these are also reproduced in Ibragimov and Linnik (1971).

PROPOSITION 4.1. *Let $p(x, 1, \beta)$ denote the density function of the stable law $\exp(-\psi_\alpha)$, with $\alpha = 1$, $\mu = 0$, $\lambda = 1$ in (2.1). Then,*

$$p(x, 1, \beta) = \left(\frac{1 \pm \beta}{\pi}\right)\frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \quad \textit{as} \quad x \to \pm\infty \ (\textit{respectively}).$$

PROOF. We have

$$\begin{aligned}
\pi p(x, 1, \beta) &= \operatorname{Re}\int_0^\infty e^{-itx - t(1+i\theta\log t)}dt, \qquad \theta = 2\beta/\pi, \qquad (4.1)\\
&= \int_0^\infty e^{-t}\cos(tx + \theta t\log t)dt\\
&= \int_0^\infty \cos tx \cdot e^{-t}\cos(\theta t\log t)dt\\
&\quad - \int_0^\infty \sin tx \cdot e^{-t}\sin(\theta t\log t)dt\\
&= I_1(x) + I_2(x), \qquad \text{say.}
\end{aligned}$$

$$\begin{aligned}
I_1(x) &= \frac{\sin tx}{x}e^{-t}\cos(\theta t\log t)\Big]_0^\infty\\
&\quad + \frac{1}{x}\int_0^\infty \frac{\sin tx}{x}e^{-t}\{\cos(\theta t\log t) + \theta(1+\log t)\sin(\theta t\log t)\}dt
\end{aligned}$$

on integrating by parts. The first term on the RHS above is zero. Integrating by parts again in the second term, we have

$$(4.2)\quad I_1(x) = \frac{1}{x^2} - \frac{1}{x^2}\int_0^\infty \cos tx e^{-t}\Big\{\cos(\theta t\log t)[\theta^2(1+\log t)^2+1] + \sin(\theta t\log t)\left[\frac{1}{t} - 2\theta(1+\log t)\right]\Big\}dt$$

$$= \frac{1}{x^2} + \frac{1}{x^2}\int_0^\infty \cos tx\phi_1(t)dt, \qquad \text{where} \quad \int_0^\infty |\phi_1(t)|dt < \infty,$$

noting that $\sin(\theta t\log t)/t \sim \theta\log t$ as $t \to 0+$ and $\int_0^1 |\log t|dt < \infty$.

$$I_2(x) = \frac{\cos tx}{x}e^{-t}\sin(\theta t\log t)]_0^\infty - \frac{1}{x}\int_0^\infty \cos tx e^{-t}\{\theta\cos(\theta t\log t)(1+\log t) - \sin(\theta t\log t)\}dt$$

on integrating by parts. The first term reduces to zero. Integrating by parts again in the second term, we have

$$(4.3)\quad I_2(x) = \frac{\theta}{x^2}\int_0^\infty e^{-t}\frac{\sin tx}{t}\cos(\theta t\log t)dt + \frac{1}{x^2}\int_0^\infty e^{-t}\sin tx\{\sin(\theta t\log t)[1-\theta^2(1+\log t)^2] - 2\theta\cos(\theta t\log t)(1+\log t)\}dt$$

$$= I_3(x) + I_4(x), \qquad \text{say,}$$

where

$$(4.4)\qquad x^2 I_4(x) = \int_0^\infty \sin tx\phi_2(t)dt, \qquad \text{with} \quad \int_0^\infty |\phi_2(t)|dt < \infty.$$

We check below that

$$(4.5)\qquad x^2 I_3(x) \to \pm\theta\frac{\pi}{2} = \pm\beta \quad \text{as} \quad x \to \pm\infty \text{ (respectively).}$$

For definiteness, let $x > 0$. Given $\epsilon > 0$, choose and fix $R$ to satisfy both the conditions

$$(4.6)\qquad \text{(a)} \quad \left|\frac{\pi}{2} - \int_0^R \frac{\sin v}{v}dv\right| < \epsilon, \qquad \text{(b)} \quad \int_R^\infty \frac{e^{-t}}{t}dt < \epsilon.$$

We have by the dominated convergence theorem,

$$\int_0^{Rx} e^{-t}\frac{\sin tx}{t}\cos(\theta t\log t)dt = \int_0^R e^{-(v/x)}\frac{\sin v}{v}\cos\left(\theta\frac{v}{x}\log\left(\frac{v}{x}\right)\right)dv \to \int_0^R \frac{\sin v}{v}dv \quad \text{as} \quad x \to \infty,$$

so that, for some $X(\geq 1)$,

$$\left|\int_0^{Rx} e^{-t}\frac{\sin tx}{t}\cos(\theta t\log t)dt - \int_0^R \frac{\sin v}{v}dv\right| < \epsilon \quad \text{for} \quad x \geq X. \tag{4.7}$$

By (4.6b),

$$\left|\int_{Rx}^\infty e^{-t}\frac{\sin tx}{t}\cos(\theta t\log t)dt\right| \leq \int_{Rx}^\infty \frac{e^{-t}}{t}dt < \epsilon \quad \text{for} \quad x \geq X. \tag{4.8}$$

(4.6a), (4.7) and (4.8) imply that (4.5) holds as $x \to \infty$. $I_3(\cdot)$ being an odd function, (4.5) holds as $x \to -\infty$ also.

The proposition then follows from relations (4.1)–(4.5) and the Riemann-Lebesgue theorem applied to $\emptyset_1$ and $\emptyset_2$.

*Remark.* $p(x,1,1) \to 0$ as $x \to \infty$ more rapidly than any given negative power of $x$; dually for $p(x,1,-1)$ as $x \to -\infty$. As stated above, a precise estimate is given by Skorokhod (1954).

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## Appendix. GS laws and GDA's

In the classical theory of limit distributions for sums of independent random variables, the concepts of 'domain of attraction (DA)' and 'domain of partial attraction (DPA)' to (qualifying) limit laws are well-known—see, for instance, Gnedenko and Kolmogorov (1954), Sections 35–37.

Their 'geometric' analogues may be defined as follows, as in Mohan *et al.* (1993). $N_p$ will denote a r.v. with the geometric d.f.: $P[N_p = j] = p(1-p)^{j-1}$ for $j = 1, 2, \ldots$, and $\xrightarrow{d}$ will, as usual, denote convergence in distribution/in law.

The law $H$ will be said to belong to the *geometric domain of partial attraction* (GDPA) of the law $F$ if, for some $p_n$ with $p_n \to 0$, there exist (on some pr. space) i.i.d.r.v.'s $X_j$ with $H$ as d.f., and r.v.'s $N_{p_n}$, independent of the $X_j$, such that

$$\sum_{j=1}^{N_{p_n}}(X_j - c_n)/a_n \xrightarrow{d} F \tag{A.1}$$

for some sequences $\{a_n\}$, $\{c_n\}$ of real numbers, with $a_n > 0\ \forall n$. Here, we write $Z_n \xrightarrow{d} F$ to mean that the corresponding sequence of d.f.'s (of the $Z_n$) converges vaguely to $F$.

If $c_n = 0\ \forall n$, in the above set-up, $H$ will be said to be in the *strict-sense* GDPA of $F$.

If $p_n = 1/n\ \forall n$, $H$ will be said to be in the *geometric domain of attraction* (GDA) of $F$; if, in addition, $c_n = 0\ \forall n$ as well, to be in the *strict-sense* GDA of $F$.

Mohan *et al.* (1993) established (among other related results) the following:

(a) A law with non-empty GDPA is necessarily GID.

(b) If the strict-sense GDA of $F$ is non-empty, then $F$ is necessarily geometric-strictly-stable (GSS).

(c) If $h$ belongs to the GDPA of the (GID) law with ch.f. $1/(1+\psi)$, then $h$ belongs to the DPA of the (ID) law $e^{-\psi}$, and conversely.

(d) If $h$ belongs to the strict-sense GDA of the (GSS) law with ch.f. $1/(1+\psi)$, then $h$ belongs to the strict-sense DA of the strictly-stable law with ch.f. $e^{-\psi}$, and conversely.

(e) A GSS law belongs to its own strict-sense GDA.

Also, in the classical theory, it is well-known that a law with non-empty DA is necessarily stable, and that a stable law belongs to its own DA.

We examine below the corresponding problems for GS laws: cf. (c), (b) and (d) above, respectively.

THEOREM A.1. *A* GS *law belongs to its own* GDA.

PROOF. Let $f_\alpha = 1/(1+\psi_\alpha)$, $0 < \alpha \le 2$, where $\psi_\alpha$ is given by (2.1). We take:

$$a_n = n^{1/\alpha}, \quad c_n = \mu\left(\frac{1}{n^{1/\alpha}} - \frac{1}{n}\right), \qquad \text{for} \quad \alpha \neq 1 \text{ or } 2,$$

$$a_n = n^{1/\alpha}, \quad c_n = (2\beta\lambda/\pi)(\log n/n), \qquad \text{for} \quad \alpha = 1,$$

and

$$a_n = cn^{1/2}, \quad c_n = \mu\left(\frac{1}{c\cdot n^{1/2}} - \frac{1}{n}\right), \quad c = \left(1 + \frac{\mu^2}{2\lambda}\right)^{1/2}, \qquad \text{for} \quad \alpha = 2.$$

Straightforward computations show that, for the above choice of constants, as $n \to \infty$,

$$n\{e^{itc_n}(1+\psi_\alpha(t/a_n)) - 1\} \to \psi_\alpha(t),\ \forall t \in \mathbb{R},$$

equivalent to

$$\frac{u_n(t)}{u_n(t) + n\{1-u_n(t)\}} \to f_\alpha(t), \qquad t \in \mathbb{R},$$

where

$$u_n(t) = f_\alpha(t/a_n)\cdot e^{-itc_n}. \tag{A.2}$$

It readily follows (cf. Mohan *et al.* (1993)) that (A.1) is satisfied for the above choice of the constants, with $p_n = 1/n\ \forall n$. Hence the theorem.

THEOREM A.2. *A law with non-empty* GDA *is necessarily* GS.

PROOF. Proceeding as in the proof of assertion (a) above (Theorem 4.1 in Mohan *et al.* (1993)), we see that if $u_n$ is defined as in (A.2) above with $h$ in place of $f_\alpha$ ($h$ being in the GDA of $f$), we must have

$$f(t) = \lim_{n\to\infty} \frac{1}{1+n\{1-u_n(t)\}} = \frac{1}{1+\psi(t)}, \quad \text{say,}$$

so that

$$\begin{aligned} \text{(A.3)} \qquad e^{-\psi(t)} &= e^{-\lim n\{1-u_n(t)\}} \\ &= \lim\{u_n(t)\}^n \quad \text{since } 1-u_n(t) = O\left(\frac{\psi(t)}{n}\right) \\ &= \lim\left\{h\left(\frac{t}{a_n}\right)\right\}^n e^{i\mu_n t} \end{aligned}$$

for some real sequence $\{\mu_n\}$. As is well-known, (A.3) implies that $e^{-\psi}$ is a stable ch.f. Hence the theorem.

Conversely, let $h \in \mathrm{DA}(e^{-\psi})$, where $e^{-\psi}$ is a stable ch.f. By definition, then, for some real sequences $\{a_n\}$, $\{\mu_n\}$ with $a_n > 0\ \forall n$,

$$\text{(A.4)} \qquad \{h(t/a_n)\}^n e^{i\mu_n t} \to e^{-\psi(t)}\ \forall t \in \mathbb{R}, \qquad \text{as} \quad n \to \infty.$$

We show below that, for some real sequence $\{\nu_n\}$,

$$\text{(A.5)} \qquad \psi(t) = \lim_{n\to\infty} n\{1-h(t/a_n)e^{i\nu_n t}\}\ \forall t \in \mathbb{R}.$$

A first-principles proof may be obtained as follows, proceeding as in the discussion of the central limit theorem in the context of "row-wise independent, uniformly asymptotically negligible summands" as, for instance, in Loève (1960), Section 22.

Let $H_n = H(a_n\cdot)$, where $H$ is the d.f. with $h$ as ch.f. For some arbitrarily fixed $\tau > 0$, let $\tau_n = \int_{|x|<\tau} x\,dH_n(x)$, $\tilde{H}_n = H_n(\cdot+\tau_n)$, $h_n$ and $\tilde{h}_n$ be the ch.f.'s of $H_n$ and $\tilde{H}_n$ respectively, so that $h_n(t) = h(t/a_n)$, $\tilde{h}_n(t) = h_n(t)e^{-it\tau_n}$. Then, for a fixed $b > 0$, we have the estimate (*op. cit.*, p. 304):

$$\text{(A.6)} \qquad \max_{|t|\le b} n|\tilde{h}_n(t)-1| \le c\cdot n\int_0^b |\log|h_n(t)||dt \to c\cdot\int_0^b \mathrm{Re}\,\psi(t)dt$$

for a constant $c = c(\tau, b) > 0$, taking (A.4) into account for the last assertion. Hence (for large $n$ for any fixed $t$)

$$n|\log\tilde{h}_n(t) - \tilde{h}_n(t) + 1| \le n|\tilde{h}_n(t)-1|^2 \to 0 \qquad \text{as} \quad n\to\infty,$$

and, therefore, for a real sequence $\{\lambda_n\}$,

$$\text{(A.7)} \qquad n(\tilde{h}_n(t)-1) + i\lambda_n t \to -\psi(t) \qquad \text{as} \quad n\to\infty.$$

(A.6) and (A.7) imply that $\{\lambda_n\}$ is a bounded sequence, so that

$$\rho_n := \lambda_n/n = O(1/n) \text{ and } \to 0 \quad \text{as} \quad n \to \infty. \tag{A.8}$$

We now note that

$$\begin{aligned} &|n(\tilde{h}_n(t) - 1 + i\rho_n t) - n(\tilde{h}_n(t)e^{i\rho_n t} - 1)| \\ &\quad = |n(\tilde{h}_n(t) - 1)(1 - e^{i\rho_n t}) + n(1 + i\rho_n t - e^{i\rho_n t})| \\ &\quad \leq n|\tilde{h}_n(t) - 1||\rho_n||t| + n|\rho_n|^2 t^2 \to 0 \quad \text{as} \quad n \to \infty \ \forall t \in \mathbb{R}, \end{aligned}$$

in view of (A.6) and (A.8). It follows that

$$n(\tilde{h}_n(t)e^{i\rho_n t} - 1) \to -\psi(t)$$

which is the same as (A.5). Hence the converse part of our assertion below (Theorem A.3) follows.

A quick proof of the converse part, appealing to Gnedenko's "transfer theorem" as in Mohan *et al.* (1993), would be:

$$\text{With} \quad p_n - 1/n, \quad N_{(1/n)}/n \xrightarrow{d} E$$

where $E$ is the standard exponential d.f., since the ch.f. of the r.v. on the left $= e^{it/n}/\{n - (n-1)e^{it/n}\} \to 1/(1 - it)$ as $n \to \infty \ \forall t \in \mathbb{R}$.

Let the $X_j$ be i.i.d. r.v.'s with d.f. $H$ in the domain of attraction of the stable ch.f. $e^{-\psi}$, so that, for suitable $\{a_n\}$, $\{c_n\}$,

$$\sum_{j=1}^{n}(X_j - c_n)/a_n \xrightarrow{d} F, \quad \text{the stable d.f. with} \quad e^{-\psi} \quad \text{as ch.f.}$$

Then, by the cited theorem due to Gnedenko,

$$\sum_{j=1}^{N_{(1/n)}} (X_j - c_n)/a_n \xrightarrow{d} G,$$

where $g$, the ch.f. of $G$, is given by $\int_0^\infty \{f(t)\}^x dE(x) = g(t)$, so that $g = 1/(1+\psi)$, and $H$ belongs to the GDA of the GS law with $1/(1 + \psi)$ as ch.f.

The following is immediate from the above.

THEOREM A.3. *$h \in \mathrm{GDA}(f)$ if and only if $h \in \mathrm{DA}(e^{-\psi})$, where $f = 1/(1 + \psi)$ is a GS law (and $e^{-\psi}$ is a stable law).*

It follows that the necessary and sufficient conditions for $H$ to belong to the domain of attraction of the stable law with $e^{-\psi_\alpha}$ as ch.f., as given by Theorems 1 and 2 of Section 35 of Gnedenko and Kolmogorov (1954), hold *verbatim* for $H$ to belong to the GDA of the GS law with $1/(1 + \psi_\alpha)$ as ch.f.

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