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SHARP ERROR BOUNDS FOR ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION FUNCTIONS FOR SCALE MIXTURES

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Abstract. Let Z be a random variable with the distribution function $G(x)$ and let s be a positive random variable independent of Z. The distribution function $F(x)$ of the scale mixture $X = sZ$ is expanded around $G(x)$ and the difference between $F(x)$ and its expansion is evaluated in terms of a quantity depending only on G and the moments of the powers of the variable of the form $s^{\delta/\rho} - 1$, where $\rho(> 0)$ and $\delta(= \pm 1)$ are parameters indicating the types of expansion. For $\delta = -1$, the bound is sharp under some extra conditions. Sharp bounds for errors of the approximations of the scale mixture of the standard normal and some gamma distributions are given either by analysis ($\delta = -1$) or by numerical computation ($\delta = 1$).

Key words and phrases: Asymptotic expansion, normal distribution, gamma distribution, scale mixture, sharp bound.

1. Introduction

Let Z be a random variable with the distribution function $G(x)$ and let s be a positive random variable independent of Z. The random variable $X = sZ$ is said to be a scale mixture of Z with the scale factor s, and its distribution function $F(x)$ is given by

$$F(x) = \Pr\{X \leq x\} = E_s[G(xs^{-1})]. \tag{1.1}$$

In many statistical situations, s depends on sample size n and approaches to a constant ($= 1$, if suitably normalized) as n tends to infinity. Studentized statistic Y/v, in which Y and v are independent, Y/σ follows the known distribution, and v^2 is an estimator of the squared scale factor σ^2, is a typical example of scale mixtures with $s = (v/\sigma)^{-1}$. Usually the distribution G is basic and well-known, whereas F varies according to the distribution of s, and becomes more complicated. Study on approximations of $F(x)$ and evaluation of its error bound has been made, assuming that G is either the standard normal or gamma distributions. They depend on the distribution of s only through its moments. For a review, see Fujikoshi and Shimizu (1990) and Fujikoshi (1993).

The purpose of the present article is to give a unified result in a more general setting and to obtain, by way of a simple analysis, improved or even sharp error bounds for two types of approximations. Numerical examples will be given for the scale mixtures of the standard normal and gamma distributions.

2. The main result

We tacitly assume that the scale factor s is close to one in some sense. To make our unified argument as simple as possible, we introduce the following transformation:

$$s = y^{\delta\cdot\rho} \quad \text{or} \quad y = s^{\delta/\rho},$$

where $\delta = \pm 1$ and ρ is a positive constant, which can be set to be 1, 1/2 or any other values depending on the situations. Suppose that the distribution G has either $\mathbf{R}^+ = (0, \infty)$ or $\mathbf{R} = (-\infty, \infty)$ as its support and is k times continuously differentiable there.

Let $g(x)$ be the probability density of G and for $j = 1, \ldots, k$, let $c_{\delta,j}(x)$ be defined by

$$(\partial^j/\partial y^j)G(xy^{-\delta\cdot\rho})\mid_{y=1} = -\delta^j \cdot c_{\delta,j}(x) \cdot g(x),$$

and write

$$\alpha_{\delta,j} \equiv \begin{cases} \max\{G(0), 1 - G(0)\}, & \text{if} \quad j = 0, \\ (1/j!) \sup\limits_{x} |c_{\delta,j}(x)| g(x), & \text{if} \quad j \geq 1. \end{cases}$$

First, we prove the basic

LEMMA 2.1. *For each $y > 0$, the distribution function $G(xy^{-\delta\cdot\rho})$ of the conditional distribution of $X = y^{\delta\cdot\rho}Z$ given y can be put in the form*

$$G(xy^{-\delta\cdot\rho}) = G_{\delta,k}(x, y) + \Delta_{\delta,k}(x, y), \tag{2.1}$$

where

$$G_{\delta,k}(x, y) = \begin{cases} G(x), & \text{if} \quad k = 1, \\ G(x) - \sum\limits_{j=1}^{k-1} \dfrac{\delta^j}{j!} c_{\delta,j}(x) g(x)(y-1)^j, & \text{if} \quad k \geq 2, \end{cases} \tag{2.2}$$

and

$$|\Delta_{\delta,k}(x, y)| \leq \alpha_{\delta,k}(y \vee y^{-1} - 1)^k. \tag{2.3}$$

The inequality cannot be improved. In other words, the constant $\alpha_{\delta,k}$ cannot be replaced by any smaller values.

PROOF. Put $Q_{\delta,0}(x) = G(x)$ and for $j - 1, 2, \ldots, k$, let $Q_{\delta,j}(x)$ be defined by the recurrence relation

$$Q_{\delta,j}(x) = -(j-1)Q_{\delta,j-1}(x) - \delta\rho x \frac{dQ_{\delta,j-1}(x)}{dx}.$$

It is easy to see, by mathematical induction, that we have

$$(\partial^j/\partial y^j)G(xy^{-\delta\cdot\rho}) = Q_{\delta,j}(xy^{-\delta\cdot\rho})y^{-j}.$$

In particular, $Q_{\delta,j}(x) = -\delta^j c_{\delta,j}(x)g(x)$, and applying Taylor's Theorem to $G(xy^{-\delta\cdot\rho})$, we obtain (2.1), (2.2) and

$$\Delta_{\delta,k}(x,y) = -(\delta^k/k!)c_{\delta,k}(u)g(u)y_0^{-k}(y-1)^k, \tag{2.4}$$

where $u = xy_0^{-\delta\cdot\rho}$ and where y_0 is a positive number lying between 1 and y. Inequality (2.3) itself and that it cannot be improved are simple consequences of (2.4). □

LEMMA 2.2. *We have for all x and $0 < y < 1$,*

$$|\Delta_{\delta,k}(x,y)| \le D_{\delta,k} \equiv \alpha_{\delta,0} + \alpha_{\delta,1} + \cdots + \alpha_{\delta,k-1}.$$

If $c_{-1,j}(x)$ has the same sign as x for all x and j, then we can take

$$D_{-1,k} = \alpha_{-1,0} \quad \textit{for} \quad k \ge 1.$$

PROOF. The case $k = 1$ is clear. In general, we have for $k \ge 2$,

$$\begin{aligned}|\Delta_{\delta,k}(x,y)| &= |G(xy^{-\delta\cdot\rho}) - G(x) + \sum_{j=1}^{k-1}(\delta^j/j!)c_{\delta,j}(x)g(x)(1-y)^j| \\ &\le |G(xy^{-\delta\cdot\rho}) - G(x)| + \sum_{j=1}^{k-1}(1/j!)|c_{\delta,j}(x)|g(x) \le D_{\delta,k}.\end{aligned}$$

To prove the second half, let $x > 0$. Then, $c_{-1,j}(x)g(x) \ge 0$, and

$$\begin{aligned}0 \le -\Delta_{-1,k}(x,y) &= (1/k!)c_{-1,k}(u)g(u)y_0^{-k}(1-y)^k \\ &= G_{-1,k}(x,y) - G(xy^\rho) \\ &= G(x) - \sum_{j=1}^{k-1}\frac{(1-y)^j}{j!}c_{-1,j}(x)g(x) - G(xy^\rho) \\ &\le G(x) - G(xy^\rho) \\ &\le \alpha_{-1,0}.\end{aligned}$$

Same argument can apply to the case $x < 0$. □

The lemmas make it possible to construct an approximating function to the distribution function $F(x)$ of the random variable $X = y^{\delta\cdot\rho}Z$ and to obtain its error bound: writing $G_{\delta,k}(x) = E_y[G_{\delta,k}(x,y)]$, we have from Lemma 2.1

$$|F(x) - G_{\delta,k}(x)| \le \alpha_{\delta,k}E[(y \vee y^{-1} - 1)^k].$$

Note that the error bound depends on the tail behavior of y^{-1}. However, in practical applications we often encounter the situations where the moments of y are computable, but that of y^{-k} is not, or even does not exist. We, then, need an error bound of the form

$$|F(x) - G_{\delta,k}(x)| \leq \beta_{\delta,k} E_y[|y-1|^k].$$

It is clear that we have only to prove the inequality

$$|G(xy^{-\delta\cdot\rho}) - G_{\delta,k}(x,y)| \leq \beta_{\delta,k}|y-1|^k, \qquad \text{for all } x \text{ and } y > 0, \tag{2.5}$$

or by introducing the nonnegative continuous function $\mathrm{J}(x,y)$ defined by

$$\mathrm{J}(x,y) \equiv \begin{cases} |G(xy^{-\delta\cdot\rho}) - G_{\delta,k}(x,y)|/|y-1|^k, & \text{for } \ y \neq 1, \\ |c_{\delta 1,k}(x)|g(x)/k!, & \text{for } \ y = 1, \end{cases} \tag{2.6}$$

to prove the inequality $\mathrm{J}(x,y) \leq \beta_{\delta,k}$.

In principle, the best possible value for $\beta_{\delta,k}$ in (2.5) can be computed as soon as the functional form of G is given, by numerically maximizing the $\mathrm{J}(x,y)$, with respect to x and $y > 0$. In this article, we will first derive it for $\delta = -1$, under a suitable condition, and an upper bound for $\beta_{\delta,k}$, under less restrictive situations, by way of a simple argument. In the later section, we will present the best possible values for $\beta_{1,k}$ using numerical computations. Now we prove,

THEOREM 2.1. *Put*

$$\beta_{\delta,k} = (\alpha_{\delta,k}^{1/k} + D_{\delta,k}^{1/k})^k.$$

Then the inequality

$$|G(xy^{-\delta\cdot\rho}) - G_{\delta,k}(x,y)| \leq \beta_{\delta,k}|y-1|^k,$$

holds for all x and $y > 0$.

PROOF. Let c be any given constant between 0 and 1. From (2.4) it follows that for $y \geq c$,

$$|\Delta_{\delta,k}(x,y)| \leq c^{-k}\alpha_{\delta,k}|y-1|^k.$$

On the other hand, if $0 < y < c$, then we have

$$\begin{aligned} |\Delta_{\delta,k}(x,y)| &= |y-1|^{-k}|\Delta_{\delta,k}(x,y)||y-1|^k \\ &\leq (1-c)^{-k} D_{\delta,k}|y-1|^k. \end{aligned}$$

Equating the right-hand sides of the two inequality, we find that $c = \{1 + (D_{\delta,k}/\alpha_{\delta,k})^{1/k}\}^{-1}$ is the best choice, and this completes the proof. □

The following theorem is useful to obtain the sharp bound. Examples will be given in the later sections.

THEOREM 2.2. *Suppose that G is analytic on its support and that $c_{-1,j}(x)$ has the same sign as x for all x and j. We also assume that $c_{-1,j}(x)g(x)$ goes to 0 as $|x| \to \infty$. Then,*

$$|G(xy^\rho) - G_{-1,k}(x,y)| \leq \beta_{-1,k}|y-1|^k,$$

where $\beta_{-1,k} = \alpha_{-1,0}$. The positive constant $\beta_{-1,k}$ cannot be replaced by a smaller one.

PROOF. In view of Lemma 2.1, we have only to prove

$$\sup_x \sup_{0<y\leq 1} \mathrm{J}(x,y) - \alpha_{-1,0} \geq \alpha_{-1,k}. \tag{2.7}$$

Let $0 < c < 1$ be an arbitrary number. If $0 < y \leq c$, then keeping the second half of Lemma 2.2 in mind,

$$\mathrm{J}(x,y) \leq (1-c)^{-k}|\Delta_{-1,k}(x,y)| \leq (1-c)^{-k}\alpha_{-1,0}.$$

Suppose next $c < y \leq 1$. For a given $x \neq 0$, $G(xy^\rho)$, as a function of y, can be developed in a series around $y = 1$:

$$\begin{aligned} G(xy^\rho) &= G(x) - \sum_{j=1}^{\infty} \frac{(1-y)^j}{j!} c_{-1,j}(x)g(x) \\ &- G_{-1,k}(x,y) - \sum_{j=k}^{\infty} \frac{(1-y)^j}{j!} c_{-1,j}(x)g(x). \end{aligned}$$

If $x > 0$, then each term of the summand is non-negative and

$$\begin{aligned} \mathrm{J}(x,y) &= (1-y)^{-k} \sum_{j=k}^{\infty} \frac{(1-y)^j}{j!} c_{-1,j}(x)g(x) \\ &\leq \sum_{j=k}^{\infty} \frac{(1-c)^{j-k}}{j!} c_{-1,j}(x)g(x) = (1-c)^{-k}|\Delta_{-1,k}(x,c)| \\ &\leq (1-c)^{-k}\alpha_{-1,0}. \end{aligned}$$

The same argument can apply to the case $x < 0$ to obtain the same inequality. (The only difference is that the terms of the summand are all non-positive in this case, so that $c_{-1,j}(x)$ should be replaced by $-c_{-1,j}(x)$.) As c is arbitrary, we have proved the inequality $\mathrm{J}(x,y) \leq \alpha_{-1,0}$ which is to hold for all x and $0 < y \leq 1$. On the other hand, we have, for any fixed x,

$$\sup_{0<y\leq 1} \mathrm{J}(x,y) \geq \mathrm{J}(x,1) \equiv \frac{1}{k!}|c_{-1,k}(x)|g(x).$$

In particular, the definition of the $\alpha_{-1,k}$ leads to

$$\sup_x \sup_{0<y\le 1} \mathrm{J}(x,y) \ge \sup_x \frac{1}{k!}|c_{-1,k}(x)|g(x) = \alpha_{-1,k}.$$

As we have already proved the inequality $\sup_x \sup_{0<y\le 1} \mathrm{J}(x,y) \le \alpha_{-1,0}$, we obtain $\alpha_{-1,0} \ge \alpha_{-1,k}$. Finally, we shall prove the equality in (2.7). In fact, we have from (2.1) and (2.2),

$$\begin{aligned}\lim_{x\to\infty}\lim_{y\to 0} \mathrm{J}(x,y) &= \left| G(0) - \lim_{x\to\infty}\left\{G(x) - \sum_{j=1}^{k-1}(1/j!)c_{-1,j}(x)g(x)\right\}\right| \\ &= 1 - G(0),\end{aligned}$$

and similarly,

$$\lim_{x\to -\infty}\lim_{y\to 0} \mathrm{J}(x,y) = G(0).$$

This means that

$$\sup_x \sup_{y>0} \mathrm{J}(x,y) = \alpha_{-1,0},$$

as was to be proved. □

3. The normal distribution

Scale mixture of the standard normal distribution is of special interest and has many applications. Suppose that Z follows the standard normal distribution with the cumulative distribution function Φ and the density ϕ. Thus, we write Φ and ϕ instead of G and g. As most applications are related to a certain studentized statistic, it is natural to take $\rho = 1/2$ in this case. Therefore, we consider the transform $s \to y = s^{2\delta}$.

Noting that

$$(\partial^j/\partial y^j)\Phi(xy^{-1/2})\,|_{y=1} = -(1/2)^j a_{1,j}(x)\phi(x),$$

and

$$(\partial^j/\partial y^j)\Phi(xy^{1/2})\,|_{y=1} = -(-1/2)^j a_{-1,j}(x)\phi(x),$$

where

$$a_{1,j}(x) = H_{2j-1}(x) \text{ (Hermite polynomial)} \quad \text{and,}$$

$$a_{-1,j}(x) = \sum_{i=0}^{j-1}(2i-1)!!\binom{j-1}{i}x^{2j-2i-1},$$

and where $(2i-1)!! = (2i-1)\cdot(2i-3)\cdots 3\cdot 1$, with the convention $(-1)!! = 1$, we obtain from (2.2)

$$\Phi_{\delta,k}(x,y) = \begin{cases} \Phi(x), & \text{if } k = 1, \\ \Phi(x) - \displaystyle\sum_{j=1}^{k-1}\frac{\delta^j}{2^j\cdot j!}a_{\delta,j}(x)(y-1)^j\phi(x), & \text{if } k \ge 2. \end{cases} \tag{3.1}$$

Table 1. The polynomials $a_{\delta,j}(x)$.

j	$a_{1,j}(x)$	$a_{-1,j}(x)$
1	x	x
2	$x^3 - 3x$	$x^3 + x$
3	$x^5 - 10x^3 + 15x$	$x^5 + 2x^3 + 3x$
4	$x^7 - 21x^5 + 105x^3 - 105x$	$x^7 + 3x^5 + 9x^3 + 15x$
5	$x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x$	$x^9 + 4x^7 + 18x^5 + 60x^3 + 105x$
6	$x^{11} - 55x^9 + 990x^7 - 6930x^5$ $+17325x^3 - 10395x$	$x^{11} + 5x^9 + 30x^7 + 150x^5 + 525x^3 + 945x$

Table 2. The numerical values of $\alpha_{\delta,j}$.

j	0	1	2	3	4	5	6
$\alpha_{1,k}$	0.5	0.1210	0.0688	0.0481	0.0370	0.0300	0.0252
$\alpha_{-1,k}$	0.5	0.1210	0.0791	0.0608	0.0501	0.0431	0.0380

For the derivation of these formulas, see, e.g., Shimizu (1987), Fujikoshi and Shimizu (1990), Fujikoshi (1993). Polynomial $a_{\delta,j}(x)$ and the numerical values of

$$\alpha_{\delta,j} = \sup\{|a_{\delta,j}(x)|\phi(x)/(2^j \cdot j!)\}$$

for small js are listed in Tables 1 and 2, respectively.

By setting $c_{\delta,j}(x) = 2^{-j}a_{\delta,j}(x)$ we can use Theorems 2.1 and 2.2 to obtain,

$$|F(x) - \Phi_{\delta,k}(x)| \leq \beta_{\delta,k}E[|s^{2\delta} - 1|^k],$$

where

$$\Phi_{\delta,k}(x) = \begin{cases} \Phi(x), & \text{if} \quad k = 1, \\ \Phi(x) - \sum_{j=1}^{k-1} \frac{\delta^j}{2^j \cdot j!} a_{\delta,j}(x)\phi(x)E[(s^{2\delta} - 1)^j], & \text{if} \quad k \geq 2, \end{cases} \tag{3.2}$$

and where the constants $\beta_{\delta,k}$ are defined by

$$\beta_{1,k} = (\alpha_{1,k}^{1/k} + D_{1,k}^{1/k})^k, \qquad \text{and} \qquad \beta_{-1,k} = 1/2. \tag{3.3}$$

Table 3 gives some numerical values of β's.

We note that Table 3 shows that our result gives substantial improvements on the previously obtained upper bounds: $\beta_{1,2} = 1.94$ (Hall (1979)), $\beta_{1,2} = 2.48$, $\beta_{1,4} = 11.26$, $\beta_{1,6} = 51.24$ (Shimizu (1987)), $\beta_{-1,2} = 1.84$ (Fujikoshi (1987)). On the other hand, the method used here for $\delta = 1$ gives only very poor results as compared to the case $\delta = -1$. We shall give sharp bounds in the later section.

Table 3. The numerical values of $\beta_{\delta,k}$.

k	1	2	3	4	5	6
$\beta_{1,k}$	0.621	0.940	1.939	3.471	6.320	11.66
$\beta_{-1,k}$	0.5	0.5	0.5	0.5	0.5	0.5

Example. (*t-distribution*) Let χ^2_n be a chi-squared random variable with degree of freedom n, and let $s=\sqrt{n/\chi^2_n}$. Then the variable sZ follows the t-distribution with the degree of freedom n, with the distribution function $F_n(x)$, say, which is approximated by (3.1) with $\delta=-1$, and an upper error bound $0.5\cdot E(s^{-2}-1)^k$. It is easy to see that

$$E(s^{-2p})=\frac{n(n+2)(n+4)\cdots(n+2p-2)}{n^p},$$

which, in turn, implies that if k is even, then both $E(s^{-2}-1)^{k-1}$ and $E(s^{-2}-1)^k$ are of the order $n^{-k/2}$. In particular, we have $E(s^{-2}-1)=0$ and $E(s^{-2}-1)^2=2/n$. It follows that for $k=2$, $F_n(x)$ can be put in the form

$$F_n(x)=\Phi(x)+\theta/n,$$

where and in what follows θ denotes a quantity not greater than 1 in the absolute value. If k is an even integer greater than 2, then, by combining the last term of the summands on the right-hand side of (3.1) with the error term we see that $F_n(x)$ can be put in the form

$$F_n(x)=\Phi^*_{-1,k}(x)+\theta P_k(n)n^{-k+1}$$

where

$$\Phi^*_{-1,k}(x)=\Phi(x)-\left\{\sum_{j=1}^{k/2-1}\left(A_{2j-1,k}(x)n^{-2j+2}+A_{2j,k}(x)n^{-2j+1}\right)\right\}\phi(x)$$

with $A_{2j-1,k}(x)$ and $A_{2j,k}(x)$ being polynomials of degree at most $2k-5$ and where $P_k(x)$ is a polynomial of degree $k/2-1$. Thus, for example,

$$F_n(x)=\begin{cases}\Phi(x)-(x^3+x)\phi(x)/4n+\theta(7/n^2+24/n^3) \quad (k=4), \qquad \text{and}\\ \Phi(x)-\{(x^3+x)/4n+(3x^7+25x^5+59x^3+93x)/96n^2\\ \quad +(x^7+3x^5+9x^3+15x)/8n^3\}\phi(x)\\ \quad +\theta\{67/n^3+1057/n^4+1920/n^5\} \quad (k=6).\end{cases}$$

Remark. Shimizu (1995) obtained somewhat stronger result which states

$$\left|\Pr\{X\in A\}-\int_A\phi_{1,k}(x)dx\right|\le\gamma_k E[|s^2-1|^k], \tag{3.4}$$

where A is an arbitrary Borel set in $\mathbf{R}$ and $\phi_{1,k}(x)$ is the derivative of $\Phi_{1,k}(x)$, i.e.

$$\phi_{1,k}(x) = \sum_{j=0}^{k-1} \frac{1}{2^j \cdot j!} H_{2j}(x)\phi(x)E[(s^2-1)^j],$$

and where $\gamma_1 = 2.0$, $\gamma_2 = 2.142$, $\gamma_3 = 11.353$, and $\gamma_4 = 23.678$. Of these the first two are known to be sharp. The sharp bounds for $k = 3$ and 4 *are conjectured to* be 2.227 and 2.287, respectively. In any case, by taking $A = (-\infty, x]$, inequality (3.3) becomes

$$|F(x) - \Phi_{1,k}(x)| \le \gamma_k E[|s^2 - 1|^k].$$

Compare the values of γs with those of βs in Table 3.

4. The gamma distribution

Scale mixtures of chi-square distributions are also basic, and appear in many applications. More generally, we can consider mixtures of gamma distribution with the distribution function $G(x;\lambda)$ and the density $g(x;\lambda) = \mathrm{e}^{-x}x^{\lambda-1}/\Gamma(\lambda)$ $(x > 0)$. The approximating function corresponding to (2.2) is given by

$$(4.1)\quad G_{\delta,k}(x, y : \lambda) \equiv \begin{cases} G(x;\lambda), & \text{if} \quad k = 1, \\ G(x;\lambda) - \displaystyle\sum_{j=1}^{k-1} \frac{\delta^j}{j!} c_{\delta,j}(x;\lambda) g(x;\lambda)(y-1)^j, & \text{if} \quad k \ge 2, \end{cases}$$

where

$$c_{1,j}(x;\lambda) = xL_{j-1}^{(\lambda)}(x), \qquad \text{and} \qquad c_{-1,j}(x;\lambda) = (-1)^j x L_{j-1}^{(\lambda-j)}(x),$$

and where $L_p^{(\lambda)}(x)$ is the Laguerre polynomial defined by $L_0^{(\lambda)}(x) = 1$ and

$$\begin{aligned} L_p^{(\lambda)}(x) &= (-1)^p x^{-\lambda} \mathrm{e}^x d^p(x^{p+\lambda}\mathrm{e}^{-x})/dx^p) \\ &= \sum_{\ell=0}^{p} (-1)^\ell (p+\lambda)^{(\ell)} \binom{p}{\ell} x^{p-\ell}. \end{aligned}$$

Derivation of the polynomials and the numerical values of

$$\alpha_{\delta,k}(\lambda) = \sup |c_{\delta,k}(x;\lambda) g(x;\lambda)/k!|$$

were given in Fujikoshi (1987, 1993), and Fujikoshi and Shimizu (1989, 1990). They are cited in Tables 4, 5 and 6 below.

Now we can use Theorems 2.1 and 2.2 to obtain

$$|F(x;\lambda) - G_{\delta,k}(x;\lambda)| \le \beta_{\delta,k}(\lambda) E[|s^\delta - 1|^k],$$

Table 4. The polynomials $c_{\delta,j}(x;\lambda)$.

j	$c_{1,j}(x;\lambda)$	$c_{-1,j}(x;\lambda)$
1	x	x
2	$x(x-\lambda-1)$	$x(x+1-\lambda)$
3	$x\{x^2-2(\lambda+2)x$	$x\{x^2+2(1-\lambda)x$
	$+(\lambda+1)(\lambda+2)\}$	$+(1-\lambda)(2-\lambda)\}$
4	$x\{x^3-3(\lambda+3)x^2$	$x\{x^3+3(1-\lambda)x^2$
	$+3(\lambda+2)(\lambda+3)x$	$+3(1-\lambda)(2-\lambda)x$
	$-(\lambda+1)(\lambda+2)(\lambda+3)\}$	$+(1-\lambda)(2-\lambda)(3-\lambda)\}$

Table 5. The numerical values of $\alpha_{1,k}(\lambda)$.

λ	k						
	0	1	2	3	4	5	6
0.5	1.0	0.2420	0.1377	0.0962	0.0739	0.0600	0.0505
1	1.0	0.3679	0.2306	0.1682	0.1325	0.1093	0.0930
1.5	1.0	0.4626	0.3135	0.2380	0.1920	0.1648	0.1576
2	1.0	0.5414	0.3919	0.3086	0.2549	0.2567	0.2507
2.5	1.0	0.6103	0.4675	0.3809	0.3558	0.3677	0.3663
3	1.0	0.6722	0.5414	0.4552	0.4708	0.4985	0.5062

Table 6. The numerical values of $\alpha_{-1,k}(\lambda)$.

λ	k						
	0	1	2	3	4	5	6
0.5	1.0	0.2420	0.1582	0.1215	0.1002	0.0861	0.0759
1	1.0	0.3679	0.2707	0.2240	0.1954	0.1755	0.1606
1.5	1.0	0.4626	0.3701	0.3243	0.2954	0.2749	0.2186
2	1.0	0.5414	0.4630	0.4250	0.4016	0.3853	0.3731
2.5	1.0	0.6103	0.5517	0.5271	0.5142	0.5067	0.5023
3	1.0	0.6722	0.6375	0.6333	0.6393	0.6393	0.6472

where

$$(4.2)\qquad \beta_{\delta,k}(\lambda)=\begin{cases}(\alpha_{\delta,k}(\lambda)^{1/k}+D_{\delta,k}(\lambda)^{1/k})^k, & \text{if either } \delta=1 \text{ or } \lambda>1\\ 1, & \text{if } \delta=-1 \text{ and } \lambda\le 1,\end{cases}$$

and where

$$D_{\delta,1}(\lambda)=D_{\delta,2}(\lambda),\qquad \text{and}$$
$$D_{1,k}(\lambda)=1+\alpha_{1,1}(\lambda)+\cdots+\alpha_{1,k-1}(\lambda)\qquad (k\ge 3).$$

Table 7. The numerical values of $\beta_{1,k}(\lambda)$.

λ	k					
	1	2	3	4	5	6
0.5	1.242	2.207	3.880	6.947	12.65	23.34
1	1.368	2.722	5.100	9.511	17.83	33.60
1.5	1.463	3.131	6.136	11.80	22.79	45.54
2	1.542	3.488	7.095	14.00	29.13	59.88
2.5	1.611	3.813	8.011	16.80	36.08	76.12
3	1.673	4.117	8.902	19.70	43.55	94.14

Table 8. The numerical values of $\beta_{-1,k}(\lambda)$.

λ	k					
	1	2	3	4	5	6
0.5	1.0	1.0	1.0	1.0	1.0	1.0
1	1.0	1.0	1.0	1.0	1.0	1.0
1.5	1.463	3.305	6.977	14.44	29.59	56.38
2	1.542	3.694	8.154	17.48	36.90	77.19
2.5	1.611	4.047	9.272	20.49	44.37	94.92
3	1.673	4.375	10.37	23.60	52.14	113.8

Values of βs are given in Tables 7 and 8. Compare these with

$$\beta_{1,2}(1) \leq 2.77 \text{ (Hall (1979))}, \qquad \text{and} \qquad \beta_{-1,2}(1) \leq 4.47 \text{ (Fujikoshi (1987))}.$$

5. Numerical results

As we saw in the previous sections, the method given in the general setting only gives poor results for $\delta = 1$. In this section, we use the numerical method applicable, in theory, to arbitrary cases, including the case $\delta = 1$. For a given $y (0 < y < 1)$, we maximize the absolute value of

$$\Delta_{1,k}(x,y) \equiv \Phi(xy^{-1/2}) - \left(\Phi(x) - \sum_{j=1}^{k-1} \frac{1}{2^j \cdot j!} H_{2j-1}(x)\phi(x)(y-1)^j \right).$$

To this end we search for the zeros of

$$\begin{aligned} f_k(x;y) &\equiv \sqrt{2\pi} \frac{\partial \Delta_{1,k}(x,y)}{\partial x} \\ &= y^{-1/2} \exp(-x^2/2y) - \sum_{j=0}^{k-1} \frac{1}{2^j \cdot j!} H_{2j}(x)(y-1)^j \cdot \exp(-x^2/2). \end{aligned}$$

Letting $u = x^2/2$, the zeros of $f_k(x; y)$ are found by locating zeros of

$$\mathrm{I}(u) = y^{-1/2}\mathrm{e}^{Au} - \sum_{j=0}^{k-1} \frac{1}{2^j \cdot j!} H_{2j}(\sqrt{2u})(y-1)^j, \tag{5.1}$$

where $A = 1 - 1/y$. Differentiating,

$$\mathrm{I}^{(j)}(u) = \frac{\partial^j \mathrm{I}(u)}{\partial u^j} = y^{-1/2} A^j \mathrm{e}^{Au} + \text{polynomial of degree } k - j - 1.$$

In particular, we have

$$\mathrm{I}^{(k-1)}(u) = y^{-1/2} A^{k-1} \mathrm{e}^{Au} - (y-1)^{k-1},$$

which has the unique positive zero

$$u_0 = -\frac{(2k-1)y}{2(y-1)} \log y.$$

Starting from this, we can find at most two positive zeros of $\mathrm{I}^{(k-2)}(u)$, and then at most three positive zeros of $\mathrm{I}^{(k-3)}(u)$. Continuing this process, we arrive at k positive zeros $u_1, u_2, \ldots, u_k$ of $\mathrm{I}(u)$. Thus, for each given y, the maximum of $|\Delta_{1,k}(x, y)/(y-1)^k|$ will be attained at one of $x_j = \sqrt{2u_j}$, $j = 1, 2, \ldots, k$. In this way, we have obtained numerically that for $k \le 6$, the ratio $\mathrm{J}(y) = \sup_x |\Delta_{1,k}(x,y)|/|(y-1)^k|$ is monotone decreasing and $\beta_{1,k}$ is obtained as $\lim_{y\to+0} \mathrm{J}(y)$. In fact, $\beta_{1,k} = 0.5$, $k \le 6$ for the normal distribution. A similar argument can apply to the scale mixture of the exponential distribution and we obtain, $\beta_{1,k}(1) = 1.0$, for $k \le 6$.

Compare the results with Tables 3 and 7.

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References

Fujikoshi, Y. (1987). Error bounds for asymptotic expansions of scale mixtures of distributions, *Hiroshima Math. J.*, **17**, 309–324.

Fujikoshi, Y. (1993). Error bounds for asymptotic approximations of some distribution functions, *Multivariate Analysis: Future Directions* (ed. C. R. Rao), 181–208, North-Holland, Amsterdam.

Fujikoshi, Y. and Shimizu, R. (1989). Asymptotic expansions of some mixtures of univariate and multivariate distributions, *J. Multivariate Anal.*, **30**, 279–291.

Fujikoshi, Y. and Shimizu, R. (1990). Asymptotic expansions of some distributions and their error bounds—the distributions of sums of independent random variables and scale mixtures, *Sugaku Expositions*, **3**, 75–96.

Hall, P. (1979). On measures of the distance of a mixture from its parent distribution, *Stochastic Process. Appl.*, **8**, 357–365.

Heyde, C. C. and Leslie, J. R. (1976). On moment measures of departure from the normal and exponential laws, *Stochastic Process. Appl.*, **4**, 317–328.

Shimizu, R. (1987). Error bounds for asymptotic expansion of the scale mixtures of the normal distribution, *Ann. Inst. Statist. Math.*, **30**, 611–622.

Shimizu, R. (1989). Expansion of the scale mixtures of the gamma distribution, *J. Statist. Plann. Inference*, **21**, 305–314.

Shimizu, R. (1995). Expansion of the scale mixtures of the multivariate normal distribution with error bound evaluated in L_1-norm, *J. Multivariate Anal.*, **53**, 126–138.