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DETECTION OF CHANGES IN LINEAR SEQUENCES

LAJOS HORVÁTH*

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, U.S.A.

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Abstract. We discuss the asymptotic properties of some tests to detect possible changes in the mean of linear processes.

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1. Introduction and results

Testing for a change in the mean of a sequence of observations is one of the most basic and important problems in change-point analysis. Assuming that the observations are independent normal random variables, Sen and Srivastava (1975*a*, 1975*b*), Worsley (1979, 1986) derived several methods to test for a shift in the normal mean. Yao and Davis (1986), Haccou *et al.* (1988), Gombay and Horváth (1990, 1994) and Horváth (1993*b*) derived the asymptotic distribution of the maximally selected likelihood ratio when changes in the parameters of the observations are tested for. Without assuming any parametric form of the distribution functions of the observations, Page (1954, 1955) and Csörgő and Horváth (1988) studied the detection of changes in the mean. Picard (1985), Kulpelger (1985), Giraitis and Leipus (1990, 1992), Tang and MacNeill (1993), Horváth (1993*a*) and Davis *et al.* (1995) considered tests when the observations are dependent. For the estimation of the time of change in the mean of dependent observations we refer to Picard (1985), Bai (1994) and Horváth and Kokoszka (1995).

We assume that the observations $\{X_i, 1 \le i \le n\}$ satisfy the model

$$X_i = \mu_i + e_i, \qquad 1 \le i \le n,$$

where the errors $\{e_i, 1 \le i \le n\}$ are given by the linear process

$$e_i = \sum_{0 \le j < \infty} a_j \epsilon_{i-j}. \tag{1.1}$$

We wish to test the null hypothesis

$$H_o : \mu_1 = \mu_2 = \cdots = \mu_n$$

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against the alternative H_A: there is an integer k^*, $1 \le k^* < n$, such that

$$\mu_1 = \cdots = \mu_k^* \ne \mu_{k^*+1} = \cdots = \mu_n.$$

The tests for H_o against H_A will be based on functionals of

$$Z_n^0(t) = Z_n(t) - tZ_n(1)$$

(cf. Theorems 1.2 and 1.3 for the distribution of weighted supremum functionals), where

$$Z_n(t) = \begin{cases} n^{-1/2} \sum\limits_{1 \le i \le (n+1)t} X_i, & \text{if} \quad 0 \le t < 1 \\ n^{-1/2} \sum\limits_{1 \le i \le n} X_i, & \text{if} \quad t = 1. \end{cases}$$

Throughout this paper we assume that

(1.2) $\{\epsilon_i, -\infty < i < \infty\}$ are independent, identically distributed random variables with $E\epsilon_i = 0$, $0 < \sigma^2 = E\epsilon_i^2 < \infty$ and $E|\epsilon_i|^\nu < \infty$ with some $\nu > 2$.

The random variables ϵ_i are smooth with density function f satisfying

$$\sup_{-\infty < s < \infty} \frac{1}{|s|} \int_{-\infty}^{\infty} |f(t+s) - f(t)| dt < \infty. \tag{1.3}$$

Also, the condition

$$a_k = O(k^{-\beta}), \quad \text{as } k \to \infty, \quad \text{with some } \beta > 3/2 \tag{1.4}$$

holds. Let

$$g(z) = \sum_{0 \le k < \infty} a_k z^k, \qquad z \in \mathbb{C}$$

and assume

$$g(z) \ne 0 \quad \text{for all} \quad |z| \le 1. \tag{1.5}$$

We note that conditions (1.2) and (1.4) imply that

$$Ee_i = 0 \quad \text{and} \quad \operatorname{var} e_i = \sigma^2 \sum_{0 \le k < \infty} a_k^2 < \infty.$$

First we obtain the necessary and sufficient condition for the weak convergence of Z_n^0 in weighted metrics. Using weight functions we can improve the power of the tests if the change occurs at the beginning or at the end of the observations (cf. Picard (1985) and Csörgő and Horváth (1988)). Let

$$\tau^2 = \sigma^2 \left(\sum_{0 \le j < \infty} a_j \right)^2 > 0 \tag{1.6}$$

and

$$Q = \Big\{ q : \inf_{\epsilon \le t \le 1-\epsilon} q(t) > 0, q \text{ is non-decreasing in a neighbourhood of zero and non-increasing in a neighbourhood of one} \Big\}. \tag{1.7}$$

The condition for the weak convergence of Z_n^0 in weighted metrics will be given by the integral

$$I_{0,1}(q,c) = \int_0^1 \frac{1}{t(1-t)} \exp\left(-\frac{cq^2(t)}{t(1-t)} \right) dt. \tag{1.8}$$

THEOREM 1.1. *We assume that* H_o, (1.1)–(1.6) *hold and* $q \in Q$. *If* $I_{0,1}(q,c) < \infty$, *then we can define a sequence of Brownian bridges* $\{B_n(t), 0 \le t \le t\}$ *such that*

$$\sup_{0<t<1} |Z_n^0(t) - \tau B_n(t)|/q(t) = o_P(1). \tag{1.9}$$

If $\limsup_{t\to 0} t/q(t) < \infty$, $\limsup_{t\to 1}(1-t)/q(t) < \infty$ *and* (1.9) *holds with a sequence of Brownian bridges, then* $I_{0,1}(q,c) < \infty$ *for all* $c > 0$.

If we are interested only in the convergence in distribution of weighted supremum functional of Z_n^0, it can be established under weaker conditions than in Theorem 1.1. Let $\{B(t), 0 \le t \le\}$ denote a Brownian bridge.

THEOREM 1.2. *We assume that* H_o, (1.1)–(1.6) *hold and* $q \in Q$. *Then*

$$\sup_{0<t<1} |Z_n^0(t)|/q(t) \xrightarrow{\mathcal{D}} \tau \sup_{0<t<1} |B(t)|/q(t) \tag{1.10}$$

and

$$\sup_{0<t<1} Z_n^0(t)/q(t) \xrightarrow{\mathcal{D}} \tau \sup_{0<t<1} B(t)/q(t) \tag{1.11}$$

if and only if $I_{0,1}(q,c) < \infty$ *for some* $c > 0$.

The variance of $Z_n^0(t)$ is proportional to $\tau^2 t(1-t)$. However, $I_{0,1}((t(1-t))^{1/2}, c) = \infty$ for all $c > 0$, so the results in Theorems 1.1 and 1.2 cannot be used to get the asymptotic distribution of the standardized statistic

$$T_n = \frac{1}{\tau} \sup_{0<t<1} |Z_n^0(t)|/(t(1-t))^{1/2}. \tag{1.12}$$

Bai (1994) pointed out that T_n is related to the generalized likelihood ratio, if $\{\epsilon_i, -\infty < i < \infty\}$ are independent, identically distributed normal random variables. Let $a(t) = (2\log t)^{1/2}$ and $b(t) = 2\log t + \frac{1}{2}\log\log t - \frac{1}{2}\log \pi$.

THEOREM 1.3. *If H_0 and (1.1)–(1.6) hold, then*

$$\lim_{n\to\infty} P\{a(\log n)T_n \le t + b(\log n)\} = \exp(-2e^{-t})$$

for all t.

Theorems 1.1–1.2 will follow from a general invariance principle for tied down partial sums of stationary sequences. The Appendix contains the results and the proofs on weighted convergence of tied down sums of stationary sequence. The proofs of Theorems 1.1–1.3 are given in the next section.

2. Proofs of Theorems 1.1–1.3

The first lemma will show that $\{e_i = \sum_{0\le j<\infty} a_j\epsilon_{i-j}, -\infty < i < \infty\}$ is a stationary strongly mixing sequence. For the definition and properties of strongly mixing sequences we refer to Ibragimov and Linnik (1971) and Philipp and Stout (1975).

LEMMA 2.1. *If (1.1)–(1.6) hold, then $\{e_i - \infty < i < \infty\}$ is a stationary strongly mixing sequence with mixing coefficient $\rho(k) = k^{-\alpha}$ with some $\alpha > 0$.*

PROOF. The result follows from Gorodetskii (1977) and Withers (1981).

Combining Lemma 2.1 and the strong approximation of partial sums of strongly mixing random variables, we get the next lemma.

LEMMA 2.2. *If (1.1)–(1.6) hold, then we can define a Wiener process $\{W(t), 0 \le t < \infty\}$ such that*

$$\left|\sum_{1\le i\le k} e_i - \tau W(k)\right| \overset{a.s.}{=} o(k^{1/2-\beta}) \tag{2.1}$$

with some $\beta > 0$.

PROOF. It follows from Lemma 2.1 and Theorem 4 of Kuelbs and Philipp (1980).

Now it is very easy to prove Theorems 1.1 and 1.2.

PROOFS OF THEOREMS 1.1 and 1.2. It follows from Lemma 2.1 that

$$n^{-1/2} \sum_{1\le i\le (n+1)t} e_i \overset{\mathcal{D}[0,1]}{\longrightarrow} \tau W(t) \tag{2.2}$$

and by the modulus of continuity of Wiener processes (cf. Csörgő and Révész (1981)) we have

$$\max_{1\le x\le n}\left|\sum_{1\le i\le x} e_i - \tau W(x)\right| / x^{1/2} = O_P(1). \tag{2.3}$$

Hence by Lemma 2.2 condition C.2 in the Appendix is also satisfied and therefore Theorems 1.1 and 1.2 follow from Theorems 3.1 and 3.2.

Proof of Theorem 1.3. Using (2.1) we get that

$$|Z_n(1)| = O_P(1), \tag{2.4}$$

and therefore the strong approximation in Lemma 2.2 and Darling and Erdős (1956) yield

$$(2\tau^2 \log\log n)^{-1/2} \sup_{0<t\le 1/2} |Z_n^0(t)|/(t(1-t))^{1/2} \xrightarrow{\mathcal{D}} 1 \tag{2.5}$$

$$(2\tau^2 \log\log n)^{-1/2} \sup_{0<t<1/\log n} |Z_n^0(t)|/(t(1-t))^{1/2} \xrightarrow{\mathcal{D}} 1 \tag{2.6}$$

and

$$\sup_{1/\log n\le t\le 1/2} |Z_n^0(t)|/(t(1-t))^{1/2} = O_P((\log\log\log n)^{1/2}). \tag{2.7}$$

By the stationarity of $\{e_i, 1\le i<\infty\}$ we get from (2.5)–(2.7) that

$$(2\tau^2 \log\log n)^{-1/2} \sup_{1/2\le t<1} |Z_n^0(t)|/(t(1-t))^{1/2} \xrightarrow{\mathcal{D}} 1, \tag{2.8}$$

$$(2\tau^2 \log\log n)^{-1/2} \sup_{1-1/\log n\le t<1} |Z_n^0(t)|/(t(1-t))^{1/2} \xrightarrow{\mathcal{D}} 1 \tag{2.9}$$

and

$$\sup_{1/2\le t\le 1-1/\log n} |Z_n^0(t)|/(t(1-t))^{1/2} = O_P((\log\log\log n)^{1/2}). \tag{2.10}$$

It follows from (2.4), (2.6) and (2.9) that

$$\begin{aligned}\sup_{0<t\le 1/\log n} &|Z_n^0(t)|/(t(1-t))^{1/2}\\ &= \max_{1\le k\le n/\log n} k^{-1/2}\left|\sum_{1\le i\le k} e_i\right| + O_P((\log\log n)^{1/2}/\log n)\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\sup_{1-1/\log n<t<1} &|Z_n^0(t)|/(t(1-t))^{1/2}\\ &= \max_{n-n/\log n\le k<n} \frac{1}{(n-k)^{1/2}}\left|\sum_{k<i\le n} e_i\right|\\ &\quad + O_P((\log\log n)^{1/2}/\log n).\end{aligned} \tag{2.12}$$

In light of (2.5)–(2.12), Theorem 1.3 is proven if we show that

$$(2.13)\quad \lim_{n\to\infty} P\left\{ a(\log n)\frac{1}{\tau}\max_{1\le k\le n/\log n}\frac{1}{k^{1/2}}\left|\sum_{1\le i\le k} e_i\right| \le t+b(\log n),\right.$$
$$\left. a(\log n)\frac{1}{\tau}\max_{n-n/\log n\le k<n}\frac{1}{(n-k)^{1/2}}\left|\sum_{k<i\le n} e_i\right| \le s+b(\log n)\right\}$$
$$= \exp(-(e^{-t}+e^{-s})).$$

Since Lemma 2.2 and Darling and Erdős (1956) yield that

$$(2.14)\quad \lim_{n\to\infty} P\left\{ a(\log n)\frac{1}{\tau}\max_{1\le k\le n/\log n}\frac{1}{k^{1/2}}\left|\sum_{1\le i\le k} e_i\right| \le t+b(\log n)\right\}$$
$$= \exp(-e^{-t})$$

and

$$(2.15)\quad \lim_{n\to\infty} P\left\{ a(\log n)\frac{1}{\tau}\max_{n-n/\log n\le k<n}\frac{1}{(n-k)^{1/2}}\left|\sum_{k<i\le n} e_i\right| \le s+b(\log n)\right\}$$
$$= \exp(-e^{-s}),$$

(2.13) is established if we can prove asymptotic independence.

Let

$$X_{i,n} = \sum_{0\le j\le n-3n/\log n} a_j\epsilon_{i-j}, \qquad n-n/\log n\le i\le n.$$

It is easy to see that $\{X_i, 1\le i\le n/\log n\}$ and $\{X_{i,n}, n-n/\log n\le i\le n\}$ are independent random vectors. Next we note that

$$X_i - X_{i,n} = \sum_{0\le j<\infty}\tilde a_j\epsilon_{i-j}, \qquad n-n/\log n\le i\le n$$

where

$$\tilde a_j = \begin{cases} 0, & \text{if } \ 0\le j\le n-3n/\log n \\ a_j, & \text{if } \ n-3n/\log n<j<\infty. \end{cases}$$

Using the proof of Proposition 1 of Bai (1994) we get

$$(2.16)\qquad P\left\{\max_{n-n/\log n\le k<n}\frac{1}{(n-k)^{1/2}}\left|\sum_{k<i\le n}(X_i - X_{i,n})\right| > x\right\}$$
$$\le \frac{9\sigma^2}{x^2}\left(\sum_{0\le j<\infty}\tilde a_j\right)^2\left(1+\sum_{1\le j\le n/\log n}\frac{1}{j}\right)$$
$$+\frac{9\sigma^2}{x^2}\sum_{0\le j<\infty}\left(\sum_{j<i<\infty}\tilde a_j\right)^2\left(\frac{\log n}{n}+\sum_{1\le j\le n}\frac{1}{j}\right).$$

Applying (1.4) we can find two constants C_1 and C_2 depending on β only, such that

$$\left(\sum_{0\le j<\infty}\tilde{a}_j\right)^2 \le C_1 n^{2-2\beta} \tag{2.17}$$

and

$$\sum_{0\le j<\infty}\left(\sum_{j<i<\infty}\tilde{a}_j\right)^2 \le C_2 n^{3-2\beta}. \tag{2.18}$$

Choosing $x = 1/\log n$ in (2.16), we get from (1.4), (2.17) and (2.18) that

$$\max_{n-n/\log n\le k<n}\frac{1}{(n-k)^{1/2}}\left|\sum_{k<i\le n}X_i\right|$$
$$=\max_{n-n/\log n\le k<n}\frac{1}{(n-k)^{1/2}}\left|\sum_{k<i\le n}X_{i,n}\right| + O_P(1/\log n),$$

which gives the asymptotic independence in (2.13). □

Appendix

Let $\{\xi_k, -\infty < k < \infty\}$ be a sequence of stationary random variables and define

$$S_n(t) = \begin{cases} n^{1/2}\displaystyle\sum_{1\le i\le(n+1)t}\xi_i, & 0\le t<1\\ n^{-1/2}\displaystyle\sum_{1\le i\le n}\xi_i, & \text{if}\quad t=1.\end{cases}$$

We assume that

C.1 there is $\sigma > 0$ such that

$$S_n(t)\overset{\mathcal{D}[0,1]}{\longrightarrow}\sigma W(t),$$

where $\{W(t), 0\le t\le 1\}$ is a Wiener process and

C.2 there are two sequences of Wiener processes $\{W_n^*(t), 0\le t<\infty\}$, $\{W_n^{**}(t), 0\le t<\infty\}$ and a sequence of positive numbers σ_n such that

$$\max_{1\le x\le n}\left|\sum_{1\le i\le x}\xi_i - \sigma_n W_n^*(x)\right|/x^{1/2} = O_P(1),$$

$$\max_{1\le x\le n}\left|\sum_{-x\le i\le -1}\xi_i - \sigma_n W_n^{**}(x)\right|/x^{1/2} = O_P(1).$$

and

$$0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < \infty.$$

We are interested in the weighted approximations of the tied-down partial sums process

$$S_n^0(t) = S_n(t) - tS_n(1), \qquad 0 \le t \le 1.$$

We find the necessary and sufficient condition for the weak convergence of $S_n^0(t)$ in weighted metrics, if conditions C.1 and C.2 are satisfied. We recall that the set Q and the integral $I_{0,1}(q,c)$ are defined in (1.7) and (1.8), respectively.

THEOREM A.1. *We assume that* C.1, C.2 *hold and* $q \in Q$. *If* $I_{0,1}(q,c) < \infty$ *for all* $c > 0$, *then we can define a sequence of Brownian bridges* $\{B_n(t), 0 \le t \le 1\}$ *such that*

$$\text{(A.1)} \qquad \sup_{0<t<1} |S_n^0(t) - \sigma B_n(t)|/q(t) = o_P(1).$$

If $\limsup_{t\to 0} t/q(t) < \infty$, $\limsup_{t\to 1}(1-t)/q(t) < \infty$ *and* (3.1) *holds with a sequence of Brownian bridges, then* $I_{0,1}(q,c) < \infty$ *for all* $c > 0$.

PROOF. It follows from C.1 that there is a sequence of Wiener processes $\{W_n(t), 0 \le t \le 1\}$ such that

$$\sup_{0\le t\le 1} |S_n(t) - \sigma W_n(t)| = o_P(1)$$

(cf. Remark 2 in Shorack and Wellner ((1986), p. 49)). Hence

$$\text{(A.2)} \qquad \sup_{0\le t\le 1} |S_n^0(t) - \sigma B_n(t)| = o_P(1),$$

where

$$\text{(A.3)} \qquad B_n(t) - W_n(t) - tW_n(1).$$

We assume that $I_{0,1}(q,c) < \infty$. Then we have

$$\text{(A.4)} \qquad \lim_{t\to 0} q(t)/t^{1/2} = \infty \quad \text{and} \quad \lim_{t\to 1} q(t)/(1-t)^{1/2} = \infty$$

(cf. Csörgő and Horváth (1993), p. 180). If $\{B(t), 0 \le t \le 1\}$ stands for Brownian bridge, then

$$\text{(A.5)} \qquad \lim_{\epsilon\to 0} \sup_{0<t\le\epsilon} |B(t)|/q(t) = 0 \quad \text{a.s.}$$

(cf. Csörgő and Horváth (1993), p. 189). Next we show that

$$\text{(A.6)} \qquad \lim_{\epsilon\to 0} \limsup_{n\to\infty} P\left\{ \sup_{0\le t\le\epsilon} |S_n^0(t)|/q(t) > \delta \right\} = 0$$

for all $\delta > 0$. It follows from the definition of $S_n^0(t)$ that

$$\sup_{0<t\le\epsilon} |S_n^0(t)|/q(t) \le \sup_{0<t\le\epsilon} |S_n(t)|/q(t) + \sup_{0<t\le\epsilon} |S_n(1)|t/q(t)$$

and

$$\sup_{0<t\le\epsilon} |S_n(1)t|/q(t) \le \epsilon^{1/2}|S_n(1)| \sup_{0<t\le\epsilon} t^{1/2}/q(t).$$

Using C.1 and (A.4) we obtain that

$$\text{(A.7)} \qquad \lim_{\epsilon\to 0} \limsup_{n\to\infty} P\left\{\sup_{0<t\le\epsilon} |S_n(1)|t/q(t) > \delta\right\} = 0$$

for all $\delta > 0$. Next we note that

$$\begin{aligned}\text{(A.8)} \quad & \sup_{0<t\le\epsilon} |S_n(t)|/q(t) \\ & = n^{-1/2} \sup_{1/(n+1)\le t\le\epsilon} \left|\sum_{1\le i\le(n+1)t} \xi_i\right| /q(t) \\ & \le n^{-1/2} \sup_{1/(n+1)\le t\le\epsilon} \left|\sum_{1\le i\le(n+1)t} \xi_i - \sigma_n W_n^*((n+1)t)\right| /q(t) \\ & \quad + n^{-1/2} \sup_{1/(n+1)\le t\le\epsilon} \sigma_n |W_n^*((n+1)t)|/q(t).\end{aligned}$$

By C.2 we have

$$\begin{aligned} & n^{-1/2} \sup_{1/(n+1)\le t\le\epsilon} \left|\sum_{1\le i\le(n+1)t} \xi_i - \sigma_n W_n^*((n+1)t)\right| /q(t) \\ & \le \sup_{1\le k\le n} \left|\sum_{1\le i\le k} \xi_i - \sigma_n W_n^*(k)\right| /k^{1/2} \sup_{0<t\le\epsilon} \frac{((n+1)t)^{1/2} n^{-1/2}}{q(t)} \\ & = O_P(1) \sup_{0<t\le\epsilon} \frac{t^{1/2}}{q(t)}\end{aligned}$$

and therefore by (A.4) we have

$$\begin{aligned}\text{(A.9)} \quad \lim_{\epsilon\to 0} \limsup_{n\to\infty} P\Bigg\{ n^{-1/2} & \sup_{1/(n+1)\le t\le\epsilon} \Bigg| \sum_{1\le i\le(n+1)t} \xi_i \\ & - \sigma_n W_n^*((n+1)t)\Bigg| /q(t) > \delta \Bigg\} = 0\end{aligned}$$

for all $\delta > 0$. Observing that

$$n^{-1/2} \sup_{1/(n+1)\le t\le \epsilon} |W_n^*((n+1)t)|/q(t) \stackrel{\mathcal{D}}{=} \left(\frac{n}{n+1}\right)^{1/2} \sigma_n \sup_{1/(n+1)\le t\le \epsilon} |W(t)|/q(t), \tag{A.10}$$

we get (cf. Csörgő and Horváth (1983), p. 189) that

$$\lim_{\epsilon\to 0} \sup_{0<t\le \epsilon} |W(t)|/q(t) = 0 \quad \text{a.s.} \tag{A.11}$$

Putting together (A.7)–(A.11) we obtain (A.6). By the stationarity of $\{\xi_i, -\infty < i < \infty\}$ and C.2 we get similarly to (A.6) that

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} P\left\{\sup_{1-\epsilon\le t<1} |S_n^0(t)|/q(t) > \delta\right\} = 0 \tag{A.12}$$

for all $\delta > 0$. Similarly to (A.5) we have

$$\lim_{\epsilon\to 0} \sup_{1-\epsilon<t<1} |B(t)|/q(t) = 0 \quad \text{a.s.} \tag{A.13}$$

Now (A.2), (A.5), (A.6), (A.12) and (A.13) yield the first part of Theorem A.1.

Next we assume that (A.1) holds. First we show that $\lim_{t\to 0} t/q(t) = 0$. By definition, $S_n(t) = 0$, if $0 \le t < 1/(n+1)$, and therefore by C.1 we have

$$\sup_{0<t<1/(n+1)} |S_n^0(t)|/q(t) = |S_n(1)| \sup_{0<t<1/(n+1)} t/q(t) = O_P(1) \tag{A.14}$$

and

$$\sup_{n/(n+1)<t<1} |S_n^0(t)|/q(t) = O_P(1). \tag{A.15}$$

Hence (A.1) yields

$$\sup_{0<t<1/(n+1)} |B_n(t)|/q(t) = O_P(1) \tag{A.16}$$

and

$$\sup_{n/(n+1)<t<1} |B_n(t)|/q(t) = O_P(1). \tag{A.17}$$

The distribution of B_n does not depend on n, and therefore $I_{0,1}(q,c) < \infty$ for some $c > 0$ (cf. Csörgő and Horváth (1993), p. 189). Since $I_{0,1}(q,c) < \infty$ for some $c > 0$, we also have (A.4), and therefore (A.14) and (A.15) can be replaced by

$$\sup_{0<t<1/(n+1)} |S_n^0(t)|/q(t) = o_P(1) \tag{A.18}$$

and

(A.19)
$$\sup_{n/(n+1)<t<1} |S_n^0(t)|/q(t) = o_P(1).$$

Thus (A.1) yields

(A.20)
$$\sup_{0<t<1/(n+1)} |B_n(t)|/q(t) = o_P(1)$$

and

(A.21)
$$\sup_{n/(n+1)<t<1} |B_n(t)|/q(t) = o_P(1).$$

The distribution of B_n does not depend on n, and therefore by Csörgő and Horváth ((1993), p. 189) we conclude that $I_{0,1}(q,c) < \infty$ for all $c > 0$.

The next theorem shows that the weighted supremum functionals of S_n^0 converges in distribution under weaker conditions than needed for weak convergence in weighted metrics. Let $\{B(t), 0 \le t \le 1\}$ be a Brownian bridge.

THEOREM A.2. *We assume that* C.1, C.2 *hold and* $q \in Q$. $I_{0,1}(q,c) < \infty$ *for some* $c > 0$ *if and only if*

(A.22)
$$\sup_{0<t<1} |S_n^0(t)|/q(t) \xrightarrow{\mathcal{D}} \sigma \sup_{0<t<1} |B(t)|/q(t)$$

and

(A.23)
$$\sup_{0<t<1} S_n^0(t)/q(t) \xrightarrow{\mathcal{D}} \sigma \sup_{0<t<1} B(t)/q(t).$$

PROOF. First we note, that if (A.22) and (A.23) hold true, then the limit distributions are almost surely finite and therefore $I_{0,1}(q,c) < \infty$ for some $c > 0$ (cf. Csörgő and Horváth (1993), p. 188).

Next we prove that if $I_{0,1}(q,c) < \infty$ for some $c > 0$, then (A.22) also holds. Similar arguments give (A.23). Given q, by Lemma 4.1.1 in Csörgő and Horváth (1993), there are constants $0 \le c_1 = c_1(q) < \infty$ and $0 \le c_2 = c_2(q) < \infty$ such that

(A.24)
$$\lim_{\epsilon\to 0} \sup_{0<t\le\epsilon} |B(t)|/q(t) = c_1 \quad \text{a.s.},$$

(A.25)
$$\lim_{\epsilon\to 0} \sup_{1-\epsilon\le t<1} |B(t)|/q(t) = c_2 \quad \text{a.s.}$$

and

(A.26)
$$\lim_{\epsilon\to 0} \sup_{0<t\le\epsilon} |W(t)|/q(t) = c_1 \quad \text{a.s.},$$

(A.27)
$$\lim_{\epsilon\to 0} \sup_{0<t\le\epsilon} |W(t)|/q(1-t) = c_2 \quad \text{a.s.},$$

where $\{W(t), 0 \le t < \infty\}$ denotes a Wiener process. We use the construction in Proof of Theorem A.1. It follows from (A.2) that

(A.28)
$$\sup_{\epsilon\le t\le 1-\epsilon} |S_n^0(t) - \sigma B_n(t)|/q(t) = o_P(1)$$

for all $0 < \epsilon < 1/2$. Next we show that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} P\left\{\left|\sup_{0<t\le\epsilon}|S_n^0(t)|/q(t) - c_1\right| > \delta\right\} = 0 \tag{A.29}$$

and

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} P\left\{\left|\sup_{1-\epsilon\le t<1}|S_n^0(t)|/q(t) - c_2\right| > \delta\right\} = 0 \tag{A.30}$$

for all $\delta > 0$. Following the proofs in (A.7) and (A.8) we get that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} P\left\{\left|\sup_{0<t\le\epsilon}|S_n(t)|/q(t) - n^{-1/2}\sigma_n \sup_{1/(n+1)\le t\le\epsilon}|W_n^*((n+1)t)|/q(t)\right| > \delta\right\} = 0 \tag{A.31}$$

for all $\delta > 0$, where W_n^* are the Wiener processes in C.2. Using (A.10) and (A.26) we get that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} P\left\{\left|n^{-1/2}\sigma_n \sup_{1/(n+1)\le t\le\epsilon}|W_n^*((n+1)t)|/q(t) - c_1\right| > \delta\right\} = 0 \tag{A.32}$$

for all $\delta > 0$. Now it is clear that (A.29) follows from (A.31) and (A.32).

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