Ann. Inst. Statist. Math.
Vol. 49, No. 2, 211–229 (1997)

ESTIMATION IN A DISCRETE RELIABILITY GROWTH MODEL UNDER AN INVERSE SAMPLING SCHEME

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(Received March 16, 1995; revised March 4, 1996)

Abstract. This paper develops a discrete reliability growth (RG) model for an inverse sampling scheme, e.g., for destructive tests of expensive single-shot operations systems where design changes are made only and immediately after the occurrence of failures. For q_i, the probability of failure at the i-th stage, a specific parametric form is chosen which conforms to the concept of the Duane (1964, *IEEE Trans. Aerospace Electron. Systems*, **2**, 563–566) learning curve in the continuous-time RG setting. A *generalized linear model* approach is pursued which efficiently handles a certain non-standard situation arising in the study of large-sample properties of the maximum likelihood estimators (MLEs) of the parameters. Alternative closed-form estimators of the model parameters are proposed and compared with the MLEs through asymptotic efficiency as well as small and moderate sample size simulation studies.

Key words and phrases: Asymptotics, generalized linear model, maximum likelihood, nonhomogeneous geometric, reliability growth.

1. Introduction

As a system undergoes development, its reliability generally improves as testing exposes failure modes and appropriate design changes or corrective actions are subsequently implemented. At the end of a stage of testing on a given configuration of the system, the current reliability can be estimated from the test data available from that configuration alone. Alternatively, since the extent of that data can be limited, reliability growth (RG) models (in essence, regression models) can be pursued to attempt to utilize all of the available test data and obtain a more precise estimate of the system reliability.

The standard application of such an approach was initially popularized by Duane (1964). From an examination of the time between failure data of several industrial systems, he observed that the cumulative number of failures typically produced a linear relationship with the cumulative operating time t when plotted on a log-log scale. This phenomenon, subsequently referred to as the "learning-curve property", was given a concrete stochastic basis by Crow (1974) who assumed that the failures during the development stage of a new system follow a nonhomogeneous Poisson process (NHPP) with an intensity function of the Weibull form $\mu\delta t^{\delta-1}$. He provided a comprehensive treatment of the above model in the context of reliability growth and demonstrated its elegant inferential aspects for application purposes.

Concerning the reliability growth of one-shot devices, the test results are binary in nature as opposed to time-to-failure in a continuous-time framework. By establishing an analogy with the NHPP model, Crow (1983) and Finkelstein (1983) formulated a discrete reliability growth model, henceforth referred to as the C-F model, in which the failure probability q_i at the i-th system configuration (stage) decreases with i according to the functional form

$$q_i = \mu_1 [i^{\delta_1} - (i-1)^{\delta_1}], \quad 0 < \mu_1 < 1, \quad 0 < \delta_1 < 1. \tag{1.1}$$

Implicit in the derivation of (1.1) is the assumption that the functional form holds irrespective of the particular outcomes of the individual trials, as well as of the specific trial numbers at which design changes or corrective actions are implemented (Fries (1993)). Some estimation procedures for this model were suggested by Crow (1983) and Finkelstein (1983); asymptotic properties were studied by Bhattacharyya *et al.* (1989), Bhattacharyya and Ghosh (1991), and Johnson (1991).

If design changes are effected only and immediately upon failures, e.g., in the destructive testing of very expensive systems, the assumptions underlying the C-F model are unreasonable. In this article we develop a discrete RG model which incorporates the concept of a learning-curve in a simple manner, and provides a suitable description of system improvement in the operational setting where failure occurrences and design changes are synchronized. The model is presented in Section 2 along with a demonstration of its connection with the Duane postulate. In Section 3, we discuss maximum likelihood estimation which requires iterative solutions and also confronts a non-standard situation of asymptotics. A correspondence between the new model and the C-F model (1.1) is brought forth in Section 4, and a simple set of alternative closed-form estimators for the model parameters are constructed by exploiting this connection. Finally, in Section 5, the two sets of estimators are contrasted via examinations of asymptotic relative efficiency (ARE) and the results of small and moderate sample size simulation studies.

2. Formulation of the model

If design changes and corrective actions are made only when a failure occurs, the results for each configuration test would consist of a run of successes followed by a single failure. The number of trials (N_i) to the first failure under the i-th configuration, should therefore be modeled as a Geometric(q_i) random variable where q_i denotes the system failure probability at the i-th stage. These circumstances correspond, for instance, to repeated developmental testing for a given number of hardware copies—in which a success outcome in any trial permits the same system to be retrieved and retested in the next trial, but any system failure is catastrophic and eliminates the particular system for future testing. Examples in the military arena include : (1) "live fire testing" of the survivability performance of air, land, and sea vehicles against various threat munitions, (2) testing of the launch and landing capabilities of unmanned aerial vehicles, and (3) exercises of target homing characteristics of (unarmed) torpedos. For a parametric form of q_i, we refer to the Duane postulate of linearity of the cumulative number of failures and cumulative test time on a log-log scale. If we identify each trial to take a unit amount of time, then the cumulative number of trials until the i-th failure, $T_i = \sum_{j=1}^{i} N_j$, can be interpreted as the discrete analog of the cumulative test time up to the i-th failure in the continuous-time case. We consider the functional form

$$q_i = (\mu/\delta) i^{1-\delta}, \qquad 0 < \mu < \delta, \quad \delta > 1, \tag{2.1}$$

for the failure probability under the i-th configuration. With this choice, we have $E(T_i) = \sum_{j=1}^{i} q_j^{-1} = (\delta/\mu) \sum_{j=1}^{i} j^{\delta-1} \approx \mu^{-1} i^{\delta}$, (for moderately large i), thus conforming to the Duane curve.

Incidentally, Dubman and Sherman (1969) studied a discrete RG model under the same inverse sampling scheme, but they assumed q_i to be proportional to δ^i, which amounts to a "geometric decay" in the failure probability q_i. Our model (2.1), represents an exponential decay in q_i. Henceforth, we shall refer to (2.1) as a *Nonhomogeneous Geometric* (NHG) model for reliability growth. Note that with the aforementioned correspondence of a trial with a unit time, the NHG model (2.1) is precisely a discrete version of the PEXP model discussed in Sen and Bhattacharyya (1993).

Fries (1993) proposed a discrete RG model by taking q_i to be of the form

$$q_i = \mu[i^{\delta} - (i-1)^{\delta}]^{-1}, \qquad 0 < \mu < 1, \quad \delta > 1 \tag{2.2}$$

and applied it to reliability test data for certain missile systems undergoing developmental testing program. Since the two forms of q_i in (2.1) and (2.2) are close even for moderately large i's, the asymptotic results derived in this paper hold for both models.

To keep the exposition simple, we confine our attention to the inference procedures for a single unit undergoing reliability growth under the proposed inverse sampling setting. The properties of the estimators we develop for a single unit easily generalize to testing a fixed number $k > 1$ identical prototypes provided that all the units are observed until the same number (n) of failures. While the treatment of multiple units do not induce any additional analytical complexity, the only modification appears in the expression for the asymptotic variance-covariance matrix of the estimators which absorb the number of units tested as a scaling constant.

3. Maximum likelihood estimation

Under the sampling scheme of observing the failure sequence of a system upto its n-th configuration, the data consist of N_i, the number of trials until the first failure under the i-th configuration, $i = 1, \ldots, n$. The N_i's are assumed to be independent geometric random variables with means $1/q_i = (\delta/\mu)i^{\delta-1}$. For the sake of brevity, we write the mean in the log-linear form $\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)$, where

$$(3.1) \qquad \boldsymbol{\beta}' = (\beta_1, \beta_2), \qquad \beta_1 = \log(\delta/\mu), \qquad \beta_2 = \delta - 1 \qquad \text{and} \qquad \boldsymbol{x}_i' = (1, \log i).$$

The log-likelihood becomes

$$\log L = -\boldsymbol{\beta}'\sum_{i=1}^{n}\boldsymbol{x}_i - \sum_{i=1}^{n}(N_i - 1)\log[1 - \exp(-\boldsymbol{\beta}'\boldsymbol{x}_i)], \qquad 0 < \beta_1 < \infty, \quad 0 < \beta_2 < \infty,$$

and we have

$$(3.2) \qquad \boldsymbol{l}_n(\boldsymbol{\beta}) \equiv \frac{\partial \log L}{\partial \boldsymbol{\beta}} = -\sum_{i=1}^{n}\boldsymbol{x}_i + \sum_{i=1}^{n}(N_i - 1)[\exp(\boldsymbol{\beta}'\boldsymbol{x}_i) - 1]^{-1}\boldsymbol{x}_i$$

$$(3.3) \qquad \boldsymbol{A}_n(\boldsymbol{\beta}) \equiv -\frac{\partial^2 \log L}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'} = \sum_{i=1}^{n}(N_i - 1)[\exp(\boldsymbol{\beta}'\boldsymbol{x}_i) - 1]^{-2}\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)\boldsymbol{x}_i\boldsymbol{x}_i'.$$

As for the existence and uniqueness of the MLE, we observe from expression (3.3) that on the complement of the set

$$S_n^* = [N_i = 1 \text{ for at least } (n-1) \text{ of the } i\text{'s}, \ i = 1, \ldots, n],$$

$\boldsymbol{A}_n(\boldsymbol{\beta})$ is positive definite and so the log-likelihood is strictly concave ensuring a unique maximum of the likelihood function. Unless the maximum is attained outside the parameter space, it corresponds to the unique solution of $\boldsymbol{l}_n(\boldsymbol{\beta}) = \mathbf{0}$. In the sequel, we take the MLE $\hat{\boldsymbol{\beta}}_n$ to be this unique solution and investigate its large sample properties. Note that $P(S_n^*)$ tends to zero as $n \to \infty$, and so the existence of the MLE is ensured in an asymptotic sense.

To study the asymptotic properties of the MLEs, we first note that our model with the parameterization as in (3.1) can be recast in the framework of a *generalized linear model*. A univariate generalized linear model (see the book by McCullagh and Nelder (1983) for a detailed description and analyses of a generalized linear model) is formulated by modeling the independent data $\{y_i\}_{i=1,\ldots,n}$ through the family of pdf's $\{f(y_i, \theta_i)\}_{i=1,\ldots,n}$ assuming the exponential class structure

$$f(y_i, \theta_i) \propto \exp(\theta_i y_i - b(\theta_i)), \qquad i = 1, 2, \ldots, n,$$

$b(\cdot)$ being an arbitrary differentiable function related to the mean of y_i through the equation $\partial b(\theta_i)/\partial\theta_i = E_{\theta_i}(y_i)$. The linearity in the model is imposed by relating the mean to a twice continuously differentiable but otherwise arbitrary function g of a linear combination of unknown parameters $\boldsymbol{\beta}$, namely, $E_{\theta_i}(y_i) = g(\boldsymbol{\beta}'\boldsymbol{x}_i)$; x_i's assuming the role of covariates. In the literature, g is termed as the link function, with special attention being devoted to natural link functions which occur by taking g to be the identity function. Note that in our model, the independent data $\{N_i\}_{i=1,\dots,n}$ conform to the generalized linear model structure with $g(\cdot) = \exp(\cdot)$ appearing as the (nonnatural) link function.

Asymptotic properties of the MLEs of the parameters in generalized linear models have been well studied in the last decade. The most comprehensive treatment appears in the work of Fahrmeir and Kaufmann (1985), who provide a unified approach for handling both natural and nonnatural link functions. We shall adopt their approach in proving the large sample properties of the MLEs in our model.

Towards that end, we let $\boldsymbol{\beta}_0 = (\beta_{10}, \beta_{20})$ denote the true parameter vector and introduce the matrix

$$\boldsymbol{F}_n(\boldsymbol{\beta}) = E_{\boldsymbol{\beta}}(\boldsymbol{A}_n(\boldsymbol{\beta})) = \sum_{i=1}^{n} v_i(\boldsymbol{\beta})\boldsymbol{x}_i\boldsymbol{x}_i'$$

where $v_i(\boldsymbol{\beta}) = [1 - \exp(-\boldsymbol{\beta}'\boldsymbol{x}_i)]^{-1}$, $i = 1, \dots, n$. For notational simplicity, henceforth we shall drop the $\boldsymbol{\beta}_0$ from $\boldsymbol{F}_n(\boldsymbol{\beta}_0)$, $v_i(\boldsymbol{\beta}_0)$, $\boldsymbol{l}_n(\boldsymbol{\beta}_0)$, and denote them by $\boldsymbol{F}_n$, v_i, and $\boldsymbol{l}_n$, respectively. Also, in the sequel, $\boldsymbol{F}_n^{T/2}(\boldsymbol{F}_n^{1/2})$ will be taken to denote the unique upper (lower) triangular square root of $\boldsymbol{F}_n$, and $\boldsymbol{F}_n^{-T/2}(\boldsymbol{F}_n^{-1/2})$ will specify the respective inverses.

The crucial step in the MLE asymptotics is a Taylor series expansion of $\boldsymbol{l}_n(\hat{\boldsymbol{\beta}}_n) = \boldsymbol{0}$ around $\boldsymbol{\beta}_0$, which yields

$$\boldsymbol{l}_n = \boldsymbol{A}_n(\boldsymbol{\zeta}_n)(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0) \tag{3.4}$$

where $\boldsymbol{\zeta}_n$ is a random point on the line segment joining $\hat{\boldsymbol{\beta}}_n$ and $\boldsymbol{\beta}_0$. Premultiplying both sides of (3.4) by $\boldsymbol{F}_n^{-1/2}$, we have

$$\boldsymbol{F}_n^{-1/2}\boldsymbol{l}_n = \boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n(\boldsymbol{\zeta}_n)\boldsymbol{F}_n^{-T/2}\boldsymbol{F}_n^{T/2}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0). \tag{3.5}$$

The following lemma provides the groundwork for the consistency and asymptotic normality of $\hat{\boldsymbol{\beta}}_n$.

LEMMA 3.1. (a) *$\lambda_{\min}(\boldsymbol{F}_n) \to \infty$ as $n \to \infty$, where $\lambda_{\min}(\boldsymbol{F}_n)$ denotes the minimum eigenvalue of $\boldsymbol{F}_n$.*

(b) $\boldsymbol{F}_n^{-1/2}\boldsymbol{l}_n \xrightarrow{d} N_2(\boldsymbol{0}, \boldsymbol{I})$.

(c) *For all $\delta > 0$,*

$$\| \boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n(\boldsymbol{\beta})\boldsymbol{F}_n^{-T/2} - \boldsymbol{I} \| \xrightarrow{p} 0$$

uniformly in $\boldsymbol{\beta} \in \boldsymbol{M}_n^\delta(\boldsymbol{\beta}_0)$ where $\boldsymbol{M}_n^\delta(\boldsymbol{\beta}_0)$ denotes the sequence of neighborhoods

$$\boldsymbol{M}_n^\delta(\boldsymbol{\beta}_0) = \{\boldsymbol{\beta} : \| \boldsymbol{F}_n^{T/2}(\boldsymbol{\beta} - \boldsymbol{\beta}_0) \| \le \delta\}, \qquad n = 1, 2, \dots.$$

For a smoother reading, the proof of Lemma 3.1 is collected in an appendix at the end of the article. Part (a) of the lemma ensures that the sequence of neighborhoods $\boldsymbol{M}_n^\delta(\boldsymbol{\beta}_0)$ shrink to $\boldsymbol{\beta}_0$ (Fahrmeir and Kaufmann (1985)), a fact which is utilised in proving part (c). Both parts (b) and (c) of the lemma constitute important steps in establishing the asymptotic normality result which we present next.

THEOREM 3.1. *For the sequence $\hat{\boldsymbol{\beta}}_n$ as above, the vector $(n^{1/2}(\log n)^{-1} \cdot (\hat{\beta}_{1n} - \beta_{10}), n^{1/2}(\hat{\beta}_{2n} - \beta_{20}))'$ is asymptotically bivariate (singular) normal with mean $\mathbf{0}$ and variance-covariance matrix*

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

PROOF. Note that in view of (3.5) and Lemma 3.1, we have

$$\boldsymbol{F}_n^{T/2}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0) \xrightarrow{d} N_2(\mathbf{0}, \boldsymbol{I}). \tag{3.6}$$

The nonzero elements of $\boldsymbol{F}_n^{T/2} \equiv (f_{ij}(n))_{i,j=1,2}$ are given by

$$f_{11}(n) = \left(\sum_{i=1}^{n} v_i\right)^{1/2},$$
$$f_{12}(n) = \left(\sum_{i=1}^{n} v_i\right)^{-1/2} \sum_{i=1}^{n} v_i \log i,$$
$$f_{22}(n) = \left(\sum_{i=1}^{n} v_i(\log i)^2 - \frac{(\sum_{i=1}^{n} v_i \log i)^2}{\sum_{i=1}^{n} v_i}\right)^{1/2}.$$

To transform (3.6) to our required result, we define the matrix $\boldsymbol{C}_n = \operatorname{diag}(n^{1/2}(\log n)^{-1}, n^{1/2})$ and observe that

$$\begin{aligned}(n^{1/2}(\log n)^{-1}(\hat{\beta}_{1n} - \beta_{10}), n^{1/2}(\hat{\beta}_{2n} - \beta_{20}))' \\ = \boldsymbol{C}_n(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0) = \boldsymbol{C}_n \boldsymbol{F}_n^{-T/2} \boldsymbol{F}_n^{T/2}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0)\end{aligned} \tag{3.7}$$

where $\boldsymbol{F}_n^{-T/2} \equiv (f^{ij}(n))_{i,j=1,2}$ is the inverse of $\boldsymbol{F}_n^{T/2}$ with nonzero entries

$$\begin{aligned} f^{11}(n) &= f_{11}^{-1}(n), \\ f^{12}(n) &= -\frac{f_{12}(n)}{f_{11}(n) f_{22}(n)}, \\ f^{22}(n) &= f_{22}^{-1}(n). \end{aligned} \tag{3.8}$$

Since $v_n \to 1$ as $n \to \infty$ and $\sum_{i=1}^{n} (\log i)^k \sim n(\log n)^k$ for large n, k being any nonnegative integer, it is easy to establish that

$$n^{-1}(\log n)^{-k} \sum_{i=1}^{n} v_i(\log i)^k \to 1 \quad \text{as } n \to \infty; \qquad k = 0, 1, 2. \tag{3.9}$$

Repeated use of (3.9) along with some minor manipulations for the $f_{22}(n)$ term enable us to deduce $f_{11}(n) \sim n^{1/2}$, $f_{12}(n) \sim n^{1/2}(\log n)$, $f_{22}(n) \sim n^{1/2}$ for large n. Thus, we have as $n \to \infty$,

$$C_n F_n^{-T/2} \to \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \equiv D. \tag{3.10}$$

In view of (3.6), (3.7), and (3.10), the required result follows by noting that $DD' = \Sigma_1$. □

Remarks.

(i) An alternative, direct proof of the above theorem is detailed in Sen (1993). It is apparent that due to the singularity of the variance-covariance matrix Σ_1, the classical treatment involving Taylor-expansion of the score function and inversion of the information matrix needs to be modified here. The modification, although simple in principle, faces tedious and complicated manipulations in the present case due to the nonstandard rates of convergence. The major advantage of the approach presented here avoids this complication in an indirect way and arrives at the singularity as an offshoot of a more general result stated in (3.6).

(ii) In terms of the original parameters μ_0 and δ_0, the large sample result translates to

$$\begin{pmatrix} n^{1/2}(\log n)^{-1}(\hat{\mu} - \mu_0) \\ n^{1/2}(\hat{\delta} - \delta_0) \end{pmatrix} \xrightarrow{d} N_2\left(\mathbf{0}, \begin{bmatrix} \mu_0^2 & \mu_0 \\ \mu_0 & 1 \end{bmatrix}\right). \tag{3.11}$$

3.1 *Estimating current system reliability*

In reliability applications, one is often interested in estimating the *current* value of the system reliability. Under the failure-truncated sampling scheme of monitoring the failure process of a single-shot system, the current system reliability at the n-th configuration is given by $R_n = 1 - q_n$. For our NHG model, the current system unreliability q_n can be expressed in terms of the parameters β_1 and β_2 as:

$$-\log q_n = \beta_1 + \beta_2 \log n.$$

An estimate $\hat{q}_n$ of q_n is obtained by replacing the parameters by their MLEs. The statistic $\log \hat{q}_n$, centered at the true parameter value $\boldsymbol{\beta}_0$, yields

$$\log(\hat{q}_n / q_n) = (\beta_{10} - \hat{\beta}_1) + (\beta_{20} - \hat{\beta}_2) \log n. \tag{3.12}$$

An argument similar to that for part (b) of Theorem 3.1, with C_n replaced by the vector $c_n = (n^{1/2}, n^{1/2} \log n)'$, readily establishes the following theorem.

THEOREM 3.2. *The sequence of random variables $n^{1/2} \log(\hat{q}_n / q_n)$ converges in distribution to a normal random variable with mean zero and variance* 2.

In a classical multiparameter setup, it is more appropriate to look at joint confidence regions for the parameters rather than individual confidence intervals for statistical inference purposes, especially in the presence of dependence among the respective estimators. Three traditional approaches for finding joint confidence regions are based on (i) likelihood-ratio statistic, (ii) Wald statistic, and (iii) Rao's score statistic. Evaluated at the true parameter vector, either of the three statistics is asymptotically distributed as a χ^2 with k degrees of freedom, k being the number of parameters. Lacking an invertible asymptotic variance-covariance matrix, none of the standard results will be applicable in the current situation. It is however, interesting to note that a two-dimensional asymptotic confidence region for $\boldsymbol{\beta}$ can be obtained directly from Theorem 3.2. Specifically, in view of (3.12), the statement of Theorem 3.2 readily translates into the $100(1-\alpha)\%$ large-sample confidence region

$$\boldsymbol{Q}_n(\boldsymbol{\beta}) = \{\boldsymbol{\beta} : (\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})' \boldsymbol{H}_n (\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \le \chi^2_{1-\alpha,1}\}$$

where $\chi^2_{1-\alpha,1}$ is the $100(1-\alpha)$-th percentile of a χ^2 random variable with 1 degree of freedom and

$$\boldsymbol{H}_n = (1/2) \begin{bmatrix} n & n(\log n) \\ n(\log n) & n(\log n)^2 \end{bmatrix}.$$

4. Construction of simple estimators—C-F Model Analog

In order to motivate a simple construction of estimators for the parameters (μ, δ) of the NHG model, we first provide a connection between the NHG and the C-F model. Consider initially the C-F model where each configuration consists of a single trial (as in Finkelstein (1983)). The expected number of failures up to the i-th configuration equals $\mu_1 i^{\delta_1}$, where the serial number i can now be interpreted as the cumulative number of trials. This establishes a correspondence between the cumulative number of trials and the cumulative number of failures until the i-th configuration as:

$$\text{cumulative number of trials} = \left[\frac{\text{cumulative number of failures}}{\mu_1}\right]^{1/\delta_1}.$$

Under the NHG model, the left hand side is considered to be the random variable T_i, and the right hand side becomes $(i/\mu_1)^{1/\delta_1}$. Consequently, $N_i = T_i - T_{i-1}$ would correspond to $[(i/\mu_1)^{1/\delta_1} - ((i-1)/\mu_1)^{1/\delta_1}]$. We match this last expression with $E(N_i)$ to arrive at

$$q_i = \mu_1^{1/\delta_1}[i^{1/\delta_1} - (i-1)^{1/\delta_1}]^{-1} \approx \frac{\mu_1^{1/\delta_1}}{\delta_1} i^{1-\delta_1^{-1}}. \tag{4.1}$$

The above approximation is good even for moderately large i. An inspection of (2.1) and (4.1) yields the parameter relations

$$\delta = 1/\delta_1, \qquad \mu = \mu_1^{1/\delta_1}, \tag{4.2}$$

which can now be used as a basis for constructing estimators for the NHG model from those for the C-F model. In particular, we consider the *Continuous Analog Estimators* (CAEs) for the parameters in the C-F model, which were proposed by Finkelstein (1983) from an analogy of the C-F model and its NHPP counterpart. In terms of the present notation, the CAEs (henceforth denoted by (μ_1^*, δ_1^*)) assume the form

$$\delta_1^* = n\left[\sum_{i=1}^{n} \log(T_n/T_i)\right]^{-1}, \quad \mu_1^* = nT_n^{-\delta_1^*}. \tag{4.3}$$

They are of particular interest because of their structural simplicity, and also for the model (1.1), their asymptotic equivalence to the MLEs (Bhattacharyya and Ghosh (1991)). Guided by the correspondence in (4.2), we define the estimators (μ^*, δ^*) for the parameters of the NHG model as

$$\delta^* = 1/\delta_1^* = \frac{1}{n}\sum_{i=1}^{n} \log(T_n/T_i), \qquad \mu^* = (\mu_1^*)^{1/\delta_1^*} = \frac{n^{\delta^*}}{T_n}. \tag{4.4}$$

The pair of estimators (μ^*, δ^*) has an alternative interesting interpretation from the point of view of misspecification. If the C-F model is "wrongly" fitted to a dataset conforming in reality to the NHG model, and the CAEs (μ_1^*, δ_1^*) are used for statistical inference, then (μ^*, δ^*) represent the estimated values of the true parameters (μ, δ).

In what follows, we let (μ_0, δ_0) denote the true parameter values for the NHG model and refer to (μ^*, δ^*) as the C-F model Analog Estimators (CFAEs). Since T_is are random variables with growing expectations in the order of i^{δ_0}, it is easier to work with the scaled random variables $X_i = \mu_0 T_i / i^{\delta_0}$, $i = 1, \ldots, n$. Necessary large-sample results concerning the sequence of X_is are detailed in the following lemma.

LEMMA 4.1. *As* $n \to \infty$,
(a) $n^{1/2}(X_n - 1) \xrightarrow{d} N(0, \delta_0^2/(2\delta_0 - 1))$.
(b) $n^{1/2}(\bar{X}_n - X_n) \xrightarrow{d} N(0, \delta_0^2/(2\delta_0 - 1))$.
(c) $n^{-1/2}\sum_{i=1}^{n} \log(X_n/X_i) = n^{1/2}(X_n - \bar{X}_n) + o_p(1)$.

The asymptotic properties shared by the sequence of X_is are clearly nonstandard and the derivation depends heavily on a refined treatment of the first two moments of certain specific random variables such as $\sum_{i=1}^{n}(X_i - X_n)$, $\sum_{i=1}^{n}(X_i - 1)(X_i^{-1} - 1)$ etc. We refrain from presenting a detailed proof which follows along similar lines of argument to that adopted for a corresponding result laid out for the PEXP model in Sen and Bhattacharyya (1993). It may be worth noting, however, that in the aforementioned article the authors dealt with continuous random variables and there is no direct way of translating the results obtained there to the present case in spite of the obvious analogy between an exponential and a geometric random variable.

The large-sample result for the CFAEs are detailed in the follwoing theorem.

THEOREM 4.1. *For the NHG model* (2.1), *the asymptotic distribution of* $(n^{1/2}(\log n)^{-1}(\mu^* - \mu_0), n^{1/2}(\delta^* - \delta_0))'$ *is bivariate (singular) normal with mean* $\mathbf{0}$ *and variance-covariance matrix*

$$\delta_0^2/(2\delta_0 - 1)\begin{bmatrix} \mu_0^2 & \mu_0 \\ \mu_0 & 1 \end{bmatrix}.$$

PROOF. We first express the CFAEs in terms of X_is as:

$$\begin{aligned} \delta^* &= \log(T_n/n^{\delta_0}) - n^{-1}\sum_{i=1}^n \log(T_i/i^{\delta_0}) + \delta_0 n^{-1}\sum_{i=1}^n \log(n/i) \\ &= n^{-1}\sum_{i=1}^n \log(X_n/X_i) + \delta_0[\log n - (\log n!)/n] \end{aligned} \tag{4.5}$$

$$\begin{aligned} \log\mu^* - \log\mu_0 &= \log(n^{\delta_0}/\mu_0 T_n) + (\log n)(\delta^* - \delta_0) \\ &= (\log n)(\delta^* - \delta_0) - \log X_n. \end{aligned} \tag{4.6}$$

An inspection of (4.5) yields

$$\begin{aligned} n^{1/2}(\delta^* - \delta_0) &= n^{-1/2}\sum_{i=1}^n \log(X_n/X_i) + n^{1/2}\delta_0[\log n - (\log n!)/n - 1] \\ &= n^{1/2}(X_n - \bar{X}_n) + n^{1/2}\delta_0[\log n - (\log n!)/n - 1] + o_p(1) \\ &= n^{1/2}(X_n - \bar{X}_n) + o_p(1), \end{aligned} \tag{4.7}$$

where the second equality follows from Lemma 4.1 (c), and the last equality results from an application of the Stirling's formula yielding

$$\log n - (\log n!)/n - 1 = -(\log n)/2n + O(1/n).$$

Also from (4.6),

$$n^{1/2}(\log n)^{-1}(\log\mu^* - \log\mu_0) = n^{1/2}(\delta^* - \delta_0) - n^{1/2}(\log n)^{-1}\log X_n. \tag{4.8}$$

In view of Lemma 4.1 (a) we have $n^{1/2}(\log n)^{-1}\log X_n$ converging to zero in probability so

$$n^{1/2}(\log n)^{-1}(\mu^* - \mu_0) = \mu_0 n^{1/2}(\delta^* - \delta_0) + o_p(1).$$

The stated theorem then follows from the asymptotic normality of $n^{1/2}(X_n - \bar{X}_n)$. □

The measure of unreliability $q_n = \mu n^{1-\delta}/\delta$ can be estimated using the CFAEs as $q_n^* = \mu^* n^{1-\delta^*}/\delta^*$. As a consequence of the above derivations, we obtain the following

THEOREM 4.2. $n^{1/2}\log(q_n^*/q_n) \xrightarrow{d} N(0, (\delta_0^2 + 2\delta_0 - 1)/(2\delta_0 - 1))$.

PROOF. We observe that

$$
\begin{aligned}
n^{1/2}\log(q_n^*/q_n) &= n^{1/2}(\log\mu^* - \log\mu_0) - n^{1/2}(\log\delta^* - \log\delta_0) \\
&\quad - n^{1/2}\log n(\delta^* - \delta_0) \\
&= -n^{1/2}(\log\delta^* - \log\delta_0) - n^{1/2}\log X_n \\
&= -n^{1/2}(\delta^* - \delta_0)/\delta_0 - n^{1/2}(X_n - 1) + o_p(1) \\
&= n^{1/2}(\bar{X}_n - X_n)/\delta_0 - n^{1/2}(X_n - 1) + o_p(1),
\end{aligned} \tag{4.9}
$$

where the second and the fourth equalities are consequences of (4.6) and (4.7), respectively. The third equality results from a combination of Taylor expansion and the facts that both $n^{1/2}(\delta^* - \delta_0)$ and $n^{1/2}(X_n - 1)$ are $O_p(1)$. Recasting the problem back to the original random variables N_i's, we can write the difference of the first two terms in (4.9) as

$$S_n \equiv \mu_0 \sum_{i=1}^{n} [N_i\{n^{-1/2}d_{in}/\delta_0 - n^{1/2-\delta_0}\} - n^{-1/2}\mu_0^{-1}],$$

where $d_{in} = \sum_{j=i}^{n}(j^{-\delta_0} - n^{1-\delta_0}(n-i+1)^{-1})$. Following some tedious algebra (omitted here) it can be shown that as $n \to \infty$,

(i) $E(S_n) = o(1)$,

(ii) $\mathrm{Var}(S_n) \to (\delta_0^2 + 2\delta_0 - 1)/(2\delta_0 - 1)$,

and

(iii) $\displaystyle\sum_{i=1}^{n} E[N_i\{n^{-1/2}\delta_0^{-1}d_{in} - n^{1/2-\delta_0}\} - n^{-1/2}\mu_0^{-1}]^4 = o(1)$.

In proving all three assertions the key role is played by the identification of d_{in} as a Riemann sum

$$d_{in} = n^{1-\delta_0}\left[n^{-1}\sum_{j=[nu]}^{n}\left\{\frac{1}{(j/n)^{\delta_0}} - \frac{1}{1-u}\right\}\right],$$

by viewing the index $i = [nu]$, $0 < u < 1$. The asymptotic normality of $n^{1/2}\log(q_n^*/q_n)$ now follows as a consequence of Lindeberg-Feller central limit theorem.

5. Comparative study of estimator performance

This paper concludes with a report on the relative performance of the maximum likelihood estimators and the simple C-F model analog estimators developed in Sections 3 and 4, respectively. Two perspectives are taken—large sample efficiency expressed in terms of the ARE, and bias and the mean squared error comparisons for small and moderate sample sizes.

From Theorem 3.1 and Theorem 4.1 it is seen that the simple estimators, although consistent under the NHG model, lose asymptotic efficiency compared to the MLEs. Specifically, comparing the asymptotic variances based on the marginal distributions of the estimators, the numerical value of the ARE turns out to be

$$ARE(\mu^* : \hat{\mu}) = ARE(\delta^* : \hat{\delta}) = (2\delta_0 - 1)/\delta_0^2,$$

which is less than 1 and approaches 1(0) as δ_0 approaches $1(\infty)$. The corresponding loss in asymptotic efficiency in measuring the current system unreliability q_n is measured by the ratio of the asymptotic variances stated in Theorem 3.2 and Theorem 4.2, respectively, which equals $2(2\delta_0 - 1)/(\delta_0^2 + 2\delta_0 - 1)$. Figure 1 shows the plot of these AREs as a function of δ, which indicates that the loss in efficiency in estimating the unreliability is substantially smaller compared to the individual estimators for a moderate range of δ.

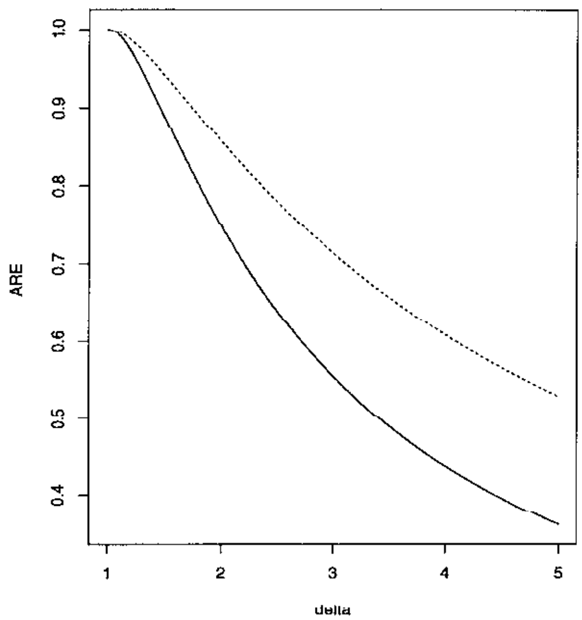

Fig. 1. Relative efficiencies of the CFAEs w.r.t the MLEs when estimating the individual parameters (solid curve) and the current system unreliability (dotted curve).

Monte Carlo simulation techniques were employed to study the performances of the MLE's $(\hat{\mu}, \hat{\delta})$ and the CFAE's estimators (μ_1^*, δ_1^*) in both small and moderate sized samples. Four pairs of (μ_0, δ_0) values, $(0.3, 1.5)$, $(0.3, 2.0)$, $(0.5, 2.5)$ and $(0.8, 2.5)$ were used for the study. For each case one thousand realizations of the two sets of estimators were obtained with the sample sizes $n = 10$, 25, and 50. The MLEs were computed using the two-variable Newton-Raphson procedure with the consistent estimators (μ_1^*, δ_1^*) as the initial value for the iteration.

Table 1. Estimated bias and mean squared error of the MLE's and the CFAE's.

Sample Size (n)	Maximum Likelihood Estimators				Simple Estimators			
	Bias($\hat{\mu}$)	MSE($\hat{\mu}$)	Bias($\hat{\delta}$)	MSE($\hat{\delta}$)	Bias(μ^*)	MSE(μ^*)	Bias(δ^*)	MSE(δ^*)
	$\mu_0 = 0.3$, $\delta_0 = 1.5$							
10	0.147	0.071	0.126	0.064	−0.138	0.024	−0.265	0.097
25	0.066	0.046	0.008	0.036	−0.105	0.022	−0.169	0.055
50	0.058	0.040	0.009	0.021	−0.069	0.019	−0.096	0.027
	$\mu_0 = 0.3$, $\delta_0 = 2.0$							
10	0.059	0.055	−0.032	0.110	−0.203	0.044	−0.524	0.339
25	0.054	0.044	− 0.006	0.042	−0.152	0.031	−0.259	0.104
50	0.051	0.035	0.005	0.021	−0.109	0.021	−0.144	0.041
	$\mu_0 = 0.5$, $\delta_0 = 2.5$							
10	−0.036	0.060	−0.147	0.134	−0.397	0.162	−0.744	0.638
25	− 0.012	0.051	−0.051	0.036	−0.330	0.120	−0.379	0.184
50	−0.0001	0.041	−0.027	0.017	−0.252	0.081	−0.211	0.068
	$\mu_0 = 0.8$, $\delta_0 = 2.5$							
10	−0.259	0.122	−0.268	0.160	−0.670	0.453	−0.840	0.777
25	− 0.205	0.089	−0.136	0.043	−0.578	0.348	−0.443	0.230
50	−0.159	0.067	−0.080	0.018	−0.479	0.252	−0.264	0.090

Geometric random variables $G(q)$ with the p.d.f

$$P(G(q) = x) = (1 - q)^{x-1} q, \qquad x = 1, 2, \ldots$$

were generated by exploiting the following relationship between the exponential and the geometric distribution. Note that if $\mathcal{E}(\lambda)$ is an exponential random variable with mean λ^{-1}, then

$$\begin{aligned} P(r \le \mathcal{E}(\lambda) \le r + 1) &= \exp(-\lambda r)(1 - \exp(-\lambda)) \\ &= P[G(e^{-\lambda}) = r + 1], \qquad r = 0, 1, \ldots. \end{aligned}$$

To generate a value x from $G(q)$ we first generate y from $\mathcal{E}(\lambda)$ with $\lambda = -\log(1-q)$ and then take $x = [y + 1]$, the nearest integer less than or equal to $y + 1$.

Table 1 gives the estimated bias and mean squared errors for both the MLEs and the CFAEs. Overall, the MLEs for both parameters perform better than the CFAEs. As is evidenced from Table 1, μ^* and δ^* show a strong tendency of underestimation over all ranges of (μ_0, δ_0) values considered. The magnitudes of bias are substantially smaller for $\hat{\mu}$ compared to μ^*. For almost all values of the sample sizes, the estimated mean squared errors for the MLEs are either comparable to or smaller than those of the corresponding CFAEs.

In conclusion, it appears that the MLEs outperform the simple estimators even in small samples. The trade-off seems to be the ease of obtaining the estimate versus a better performance in terms of bias and variance. In practical applications, the simple estimators can in the least provide an initial guess about the parameters and can be used to demonstrate reliability growth ($\delta > 1$) or otherwise. Moreover, if the estimation of reliability is the ultimate goal, choosing the simple estimator q_n^* will not be unadvisable in view of its stabler behavior compared to the individual estimators.

Acknowledgements

The authors would like to thank the two anonymous referees for their suggestions which led to a substantially improved version of the article.

Appendix

PROOF OF LEMMA 3.1. (a) Since $\lambda_{\min}(\boldsymbol{F}_n) = (\lambda_{\max}(\boldsymbol{F}_n^{-1}))^{-1}$, we need to establish that $\lambda_{\max}(\boldsymbol{F}_n^{-1}) \to 0$, as $n \to \infty$. Recall that

$$\boldsymbol{F}_n = \begin{bmatrix} \sum_{i=1}^n v_i & \sum_{i=1}^n v_i \log i \\ \sum_{i=1}^n v_i \log i & \sum_{i=1}^n v_i (\log i)^2 \end{bmatrix},$$

where $v_i = [1 - \exp(-\boldsymbol{\beta}_0'\boldsymbol{x}_i)]^{-1}$. Using (3.9), it follows that $\sum_{i=1}^n v_i \sim n$, $\sum_{i=1}^n v_i \log i \sim n \log n$, $\sum_{i=1}^n v_i (\log i)^2 \sim n(\log n)^2$. Further,

$$\begin{aligned} d_n \equiv |\boldsymbol{F}_n| &= \sum_{i=1}^n v_i \sum_{i=1}^n v_i (\log i)^2 - \left(\sum_{i=1}^n v_i \log i\right)^2 \\ &= \sum_{i=1}^n v_i \sum_{i=1}^n v_i [\log(i/n)]^2 - \left[\sum_{i=1}^n v_i \log(i/n)\right]^2 \\ &\sim n^2. \end{aligned}$$

Let $\lambda_n^{(1)}, \lambda_n^{(2)}$ be the eigenvalues of $\boldsymbol{F}_n^{-1}$. As a consequence of the above discussion,

$$\text{(A.1)} \qquad \lambda_n^{(1)} + \lambda_n^{(2)} = \operatorname{tr}(\boldsymbol{F}_n^{-1}) = d_n^{-1}\left(\sum_{i=1}^n v_i + \sum_{i=1}^n v_i (\log i)^2\right) \sim n^{-1}(\log n)^2.$$

Since $\boldsymbol{F}_n^{-1}$ is a positive definite matrix, $\lambda_n^{(1)}, \lambda_n^{(2)}$ are positive for all n, and thus (A.1) entails that both $\lambda_n^{(1)}, \lambda_n^{(2)}$ and consequently $\max(\lambda_n^{(1)}, \lambda_n^{(2)})$ converge to zero as $n \to \infty$.

(b) We define a sequence of random variables e_i, $i = 1, 2, \ldots, n$, as

$$\text{(A.2)} \qquad e_i = (N_i - 1)[\exp(\boldsymbol{\beta}_0'\boldsymbol{x}_i) - 1]^{-1} - 1.$$

Note that e_i's are independent with mean zero and $\operatorname{Var}(e_i) = v_i$. Denote $\boldsymbol{l}_n = (l_{1n}, l_{2n})'$, and consider the random vactor

$$n^{-1/2}\begin{bmatrix} 1 & 0 \\ (1 - \log n) & 1 \end{bmatrix}\begin{pmatrix} l_{1n} \\ l_{2n} \end{pmatrix} \equiv \boldsymbol{D}_n \boldsymbol{l}_n.$$

Since $l_{1n} = \sum_{i=1}^n e_i$ and $l_{2n} = \sum_{i=1}^n e_i \log i$ can both be expressed as a sum of independent mean zero random variables one can apply the Lindeberg-Feller form of central limit theorem to demonstrate that $\boldsymbol{c}'\boldsymbol{D}_n\boldsymbol{l}_n\boldsymbol{c} \to \boldsymbol{c}'\boldsymbol{c}$ as $n \to \infty$ for any $\boldsymbol{c} \neq 0$, which yields the result

$$\text{(A.3)} \qquad \boldsymbol{D}_n \boldsymbol{l}_n \xrightarrow{d} N_2(\boldsymbol{0}, \boldsymbol{I}).$$

Using the notation in (3.8) for $\boldsymbol{F}_n^{-T/2}$ we have

$$\boldsymbol{F}_n^{-1/2}\boldsymbol{D}_n^{-1} = n^{1/2}\begin{bmatrix} f^{11}(n) & 0 \\ (\log n - 1)f^{22}(n) - f^{12}(n) & f^{22}(n) \end{bmatrix}.$$

Since $f^{11}(n) \sim n^{-1/2}$, $f^{22}(n) \sim n^{-1/2}$, it is immediate that both the diagonal terms in the above matrix converge to 1. As for the positive off-diagonal term, observe that

$$n^{1/2}[(\log n - 1)f^{22}(n) - f^{12}(n)] = n^{1/2}f^{22}(n)\left[-1 - \frac{\sum_{i=1}^n v_i \log(i/n)}{\sum_{i=1}^n v_i}\right] = o(1)$$

by virtue of the facts that $n^{-1}\sum_{i=1}^{n} v_i \to 1$, $n^{-1}\sum_{i=1}^{n} v_i \log(i/n) \to -1$, and $n^{1/2}f^{22}(n) \to 1$ as $n \to \infty$, and so

$$\boldsymbol{F}_n^{-1/2}\boldsymbol{D}_n^{-1} \overset{n\to\infty}{\to} \boldsymbol{I}. \tag{A.4}$$

Since $\boldsymbol{F}_n^{-1/2}\boldsymbol{l}_n = \boldsymbol{F}_n^{-1/2}\boldsymbol{D}_n^{-1}\boldsymbol{D}_n\boldsymbol{l}_n$, we have our required result in view of (A.3) and (A.4).

(c) Denoting $\boldsymbol{A}_n \equiv \boldsymbol{A}_n(\boldsymbol{\beta}_0) = (a_{ij})_{i,j=1,2}$, and writing

$$\boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n(\boldsymbol{\zeta}_n)\boldsymbol{F}_n^{-T/2} = \boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n\boldsymbol{F}_n^{-T/2} + \boldsymbol{F}_n^{-1/2}[\boldsymbol{A}_n(\boldsymbol{\zeta}_n) - \boldsymbol{A}_n]\boldsymbol{F}_n^{-T/2} \tag{A.5}$$

we observe that it suffices to establish (i) the probability convergence of the first term on the right of (A.5) to the identity matrix and (ii) uniform negligibility of the second term in the prescribed neighborhood.

In order to show that $\boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n\boldsymbol{F}_n^{-T/2} \xrightarrow{p} \boldsymbol{I}$, we simply need to demonstrate that expectation of each term in $\boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n\boldsymbol{F}_n^{-T/2}$ converges to the corresponding entry in $\boldsymbol{I}$, while the respective variances are $o(1)$ as $n \to \infty$. We shall only establish this for the off-diagonal term and indicate that the treatment is similar for the other two. Observe that the off-diagonal element of $\boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n\boldsymbol{F}_n^{-T/2}$ can be written as

$$\begin{aligned} &f_{11}^{-1}f_{22}^{-1}(a_{12} - f_{11}^{-1}f_{12}a_{11}) \\ &\quad = \left(\sum_{i=1}^{n} v_i\right)^{-1} d_n^{-1/2}\left(a_{12}\sum_{i=1}^{n} v_i - a_{11}\sum_{i=1}^{n} v_i \log i\right) \\ &\quad \sim n^{-2}\left(a_{12}\sum_{i=1}^{n} v_i - a_{11}\sum_{i=1}^{n} v_i \log i\right) \end{aligned} \tag{A.6}$$

using arguments as in part(a) of the lemma, where d_n as before represents the determinant of $\boldsymbol{F}_n$. Exploiting (3.2), (3.3), (A.2), and the expressions for v_i, we can express a_{11} and a_{12} as

$$\begin{aligned} a_{11} &= n + l_{1n} + \sum_{i=1}^{n}(e_i + 1)(v_i - 1), \\ a_{12} &= \sum_{i=1}^{n}(\log i) + l_{2n} + \sum_{i=1}^{n}(e_i + 1)(v_i - 1)\log i, \end{aligned}$$

and thus we can present (A.6) (after some simplifications) as

$$\begin{aligned} &n^{-2}\left(a_{12}\sum_{i=1}^{n} v_i - a_{11}\sum_{i=1}^{n} v_i \log i\right) \\ &\quad - \left[n^{-2}\sum_{i=1}^{n} v_i \sum_{i=1}^{n} v_i \log(i/n) - n^{-1}\sum_{i=1}^{n} v_i \log(i/n)\right] \end{aligned}$$

$$+\left[\left(n^{-3/2}\sum_{i=1}^{n}v_i\right)n^{-1/2}(l_{2n}-l_{1n}\log n)\right]$$
$$+\left[n^{-2}\sum_{i=1}^{n}(e_i+1)(v_i-1)\left\{\log(i/n)\sum_{j=1}^{n}v_j-\sum_{j=1}^{n}v_j\log(j/n)\right\}\right]$$
$$\equiv K_1+K_2+K_3.$$

Using arguments as before, it is easy to establish that $K_1 = o(1)$. The result stated in (A.3) entails that $n^{-1/2}(l_{2n} - l_{1n}\log n) \xrightarrow{d} N(0,2)$. Since $\sum_{i=1}^{n} v_i \sim n$, we conclude $K_2 = o_p(1)$. As for K_3, once again, we use the limiting properties of the $\{v_n\}$ to conclude that both $E(K_3)$ and $\mathrm{Var}(K_3)$ converge to zero as $n \to \infty$.

Turning to the second term on the right side of (A.5), we first reexpress the entries of the matrix $\boldsymbol{F}_n^{-1/2}\boldsymbol{A}_n\boldsymbol{F}_n^{-T/2} \equiv (w_{ij})_{i,j=1,2}$ as

$$w_{11} = [nf_{11}^{-2}]\frac{a_{11}}{n},$$
$$w_{21} = w_{12} = (\log n)\left\{[nf_{11}^{-1}f_{22}^{-1}]\left(\frac{a_{12}}{n\log n}\right) - [n(\log n)^{-1}f_{11}^{-2}f_{22}^{-1}f_{12}]\frac{a_{11}}{n}\right\}, \tag{A.7}$$
$$w_{22} = (\log n)^2\left\{[nf_{22}^{-1}]\frac{a_{22}}{n(\log n)^2} - [n(\log n)^{-1}f_{11}^{-1}f_{22}^{-2}f_{12}]\frac{a_{12}}{n(\log n)} + [n(\log n)^{-2}f_{11}^{-2}f_{22}^{-2}f_{12}^{2}]\frac{a_{11}}{n}\right\}.$$

The expressions are arranged in a way such that the terms within the square brackets converge to unity. In view of (3.3) and (A.7) thus, we need to establish that

$$T_{n,k} = (\log n)^2 n^{-1}(\log n)^{-k}\sum_{i=1}^{n}(\log i)^k(N_i-1)[u_i(\boldsymbol{\beta})-u_i(\boldsymbol{\beta}_0)]$$

converges in probability to 0 uniformly in $\boldsymbol{\beta} \in \boldsymbol{M}_n^{\delta}(\boldsymbol{\beta}_0)$ for $k = 0, 1, 2$, where

$$u_i(\boldsymbol{\beta}) = \exp(\boldsymbol{\beta}'\boldsymbol{x}_i)[\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1]^{-2}.$$

In view of part (a) of the lemma, it is clear that the sequence $\boldsymbol{M}_n^{\delta}(\boldsymbol{\beta}_0)$ shrinks to $\boldsymbol{\beta}_0$ as n becomes large for each fixed δ. Since it suffices to establish the uniform convergence of $T_{n,k}$ in any sequence of neighborhoods shrinking to $\boldsymbol{\beta}_0$, we shall work with a neighborhood with a simpler structure. Specifically, we consider the rectangular neighborhood of the form

$$\boldsymbol{R}_n(\boldsymbol{\beta}_0) = \{(\beta_1,\beta_2) : |\beta_1-\beta_{10}| \le (\log n)n^{-1/2}, |\beta_2-\beta_{20}| \le n^{-1/2}\}.$$

Writing $u_i(\boldsymbol{\beta}) = (\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1)^{-1} + (\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1)^{-2}$, we can decompose $T_{n,k}$ into two parts as

$$T_{n,k} = n^{-1}(\log n)^{2-k}\sum_{i=1}^{n}(\log i)^k(N_i-1)$$

$$\cdot\left[(\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1)^{-1}-(\exp(\boldsymbol{\beta}_0'\boldsymbol{x}_i)-1)^{-1}\right]$$
$$+\,n^{-1}(\log n)^{2-k}\sum_{i=1}^{n}(\log i)^k(N_i-1)$$
$$\cdot\left[(\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1)^{-2}-(\exp(\boldsymbol{\beta}_0'\boldsymbol{x}_i)-1)^{-2}\right]$$
$$\equiv B_1+B_2.$$

Using the triangle inequality on B_1, we have

$$\text{(A.8)}\qquad E_{\boldsymbol{\beta}_0}(|B_1|)\le n^{-1}(\log n)^{2-k}\sum_{i=1}^{n}(\log i)^k$$
$$\cdot\left|[\exp(\boldsymbol{\beta}_0'\boldsymbol{x}_i)-1][\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1]^{-1}-1\right|$$
$$=n^{-1}(\log n)^{2-k}\sum_{i=1}^{n}(\log i)^k\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)[\exp(\boldsymbol{\beta}'\boldsymbol{x}_i)-1]^{-1}$$
$$\cdot\left|1-\exp\{-(\boldsymbol{\beta}-\boldsymbol{\beta}_0)'\boldsymbol{x}_i\}\right|.$$

By exploiting certain bounds for exponential functions (see Sen (1993) for details), it can be established that

$$n^{-1}(\log n)^{2-k}\sum_{i=1}^{n}(\log i)^k|1-\exp\{-(\boldsymbol{\beta}-\boldsymbol{\beta}_0)'\boldsymbol{x}_i\}|\to 0$$

uniformly in $\boldsymbol{\beta}\in\boldsymbol{R}_n(\boldsymbol{\beta}_0)$ as $n\to\infty$. To establish that right side of (A.8) converges uniformly to zero, it now suffices to show the uniform convergence of the sequence $b_n=\exp(\boldsymbol{\beta}'\boldsymbol{x}_n)[\exp(\boldsymbol{\beta}'\boldsymbol{x}_n)-1]^{-1}$ to a finite quantity. For a fixed n, $\boldsymbol{\beta}'\boldsymbol{x}_n$ (and hence b_n viewed as a function of $\boldsymbol{\beta}$) attains its extremums (maximum and minimum) at one of the four corner points of the rectangle $\boldsymbol{R}_n(\boldsymbol{\beta}_0)$. Since b_n converges to 1 at all the four corners, we have $E_{\boldsymbol{\beta}_0}(|B_1|)\to 0$ uniformly for $\boldsymbol{\beta}\in\boldsymbol{R}_n(\boldsymbol{\beta}_0)$. The Markov inequality then yields the uniform probability convergence of B_1 to zero. The proof that $B_2=o_p(1)$ follows along similar lines.

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