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THE CONVERGENCE RATE OF FIXED-WIDTH SEQUENTIAL CONFIDENCE INTERVALS FOR A PARAMETER OF AN EXPONENTIAL DISTRIBUTION

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Abstract. In this paper we consider sequential fixed-width confidence interval estimation for a parameter $\theta = a\mu + b\sigma$ with a and b being given constants when the location parameter μ and the scale parameter σ of the negative exponential distribution are unknown. We investigate the rate of convergence of the coverage probability for fixed-width sequential confidence intervals of θ.

Key words and phrases: Two-parameter exponential, sequential confidence intervals, coverage probability, convergence rate.

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables having the probability density function (p.d.f.)

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right) I_{(x\geq\mu)} \tag{1.1}$$

where the location parameter $\mu \in (-\infty, \infty)$ and the scale parameter $\sigma \in (0, \infty)$ are both unknown and $I_{(\cdot)}$ denotes the indicator function. We want to find a fixed-width confidence interval of

$$\theta = a\mu + b\sigma \qquad \text{for given constants} \quad a \geq 0 \text{ and } b > 0. \tag{1.2}$$

When we take $a = b = 1$, $\theta = \mu + \sigma$ is the population mean of (1.1). For given $p \in (0, 1)$, $\theta = \mu - \sigma \log(1-p)$ is the p-th percentile of (1.1). In the case $a = 0$ and $b = 1$, $\theta = \sigma$ is the standard deviation of (1.1).

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Let $d > 0$ and $\alpha \in (0,1)$ be given constants. We choose the constant $z > 0$ such that $\Phi(z) - \Phi(-z) = 1 - \alpha$ where Φ denotes the standard normal distribution function. For $n \geq 2$, set

$$\hat{T}_n = \min(X_1, \ldots, X_n), \qquad \hat{\sigma}_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{T}_n),$$

$$\theta_n = a\left(\hat{T}_n - \frac{\hat{\sigma}_n}{n}\right) + b\hat{\sigma}_n, \qquad I_n = I_n(d) = [\theta_n - d, \theta_n + d]$$

and

(1.3) $$n_0 = \left(\frac{bz}{d}\right)^2 \sigma^2, \qquad n^* = n_0 + 1.$$

For simplicity, n_0 is assumed to be a positive integer. Then it is known that n^* is the asymptotic optimal sample size for large n. But σ is unknown, so is n^*. Thus, to find a sequential confidence interval with length $2d$ and coverage probability $1 - \alpha$ for the parameter θ of (1.2), Isogai *et al.* (1995) proposed the following stopping rule:

$$M = M(d) = \inf\left\{n \geq m : n \geq \left(\frac{bz_n}{d}\right)^2 \hat{\sigma}_n^2 + 1\right\} \qquad \text{for} \quad d > 0$$

where $m \geq 2$ is the starting sample size, $\eta(x)$ is a positive continuous function on $[0, \infty)$ for which $\eta(x) = 1 + \Delta x^{-1} + o(x^{-1})$ as $x \to \infty$ with a constant Δ, and $z_n = z\eta(n-1)$. Then Isogai *et al.* (1995) showed the asymptotic consistency of the confidence intervals $I_{M(d)}(d) = [\theta_{M(d)} - d, \theta_{M(d)} + d]$ of θ, i.e.

(1.4) $$\lim_{d \to 0} P\{\theta \in I_{M(d)}(d)\} = 1 - \alpha$$

and gave the second order asymptotic efficiency of the procedure. For the scale parameter $\theta = \sigma$, Govindarajulu (1985) considered this problem. However, we are not aware of the convergence rate for the coverage probability of sequential confidence intervals for θ of (1.2).

The aim of this paper is to investigate the rate of convergence of (1.4). Since the term $\hat{\sigma}_n/n$ of the definition of statistic θ_n above does not play any role as far as asymptotic theory goes, in this paper we use

$$\hat{\theta}_n = a\hat{T}_n + b\hat{\sigma}_n$$

as an estimator of θ, and define a confidence interval for θ as

$$I_n = I_n(d) = [\hat{\theta}_n - d, \hat{\theta}_n + d].$$

To find the convergence rate we propose the following stopping rule:

$$N = N(d) = \inf\left\{n \geq m : n \geq \left(\frac{bz}{d}\right)^2 \hat{\sigma}_n^2 + 1\right\} \qquad \text{for} \quad d > 0$$

where

$$m = m(d) = \max\{2, [m_0 d^{-\gamma}]^* + 1\} \tag{1.5}$$

for any fixed $m_0 \in (0, \infty)$ and $\gamma \in (0, 2)$, and $[x]^*$ denotes the largest integer not greater than x. For the location parameter μ, Mukhopadhyay (1974) and Swanepoel and van Wyk (1982) considered this problem. Problems of this type for population means have been generally studied by Chow and Robbins (1965), Csenki (1980), Callaert and Janssen (1981) and Mukhopadhyay and Datta (1996). In Section 2 we shall derive the rate of convergence of (1.4). In Section 3 we shall give the estimates of the terms in the proof of Theorem 2.1 in Section 2.

Remark 1. By Lemma 1 of Chow and Robbins (1965) we have that $P\{N(d) < \infty\} = 1$ for all $d > 0$.

2. Main result

In this section we shall give the main result concerning the rate of convergence of the coverage probability for θ of (1.2) which will be derived by the method similar to that of Callaert and Janssen (1981).

The following lemma is Lemma 9 of Landers and Rogge (1976) which, together with (2.6), (2.7) and (2.8) below, yields the convergence rate.

LEMMA 2.1. *Let $\{X_k, k \geq 1\}$ and $\{Y_k, k \geq 1\}$ be two sequences of random variables. Assume that for a sequence of positive numbers $\{a_k, k \geq 1\}$*

$$\sup_{-\infty < t < \infty} |P\{X_k \leq t\} - \Phi(t)| = O(a_k) \quad \text{and} \quad P\{|Y_k| > a_k\} = O(a_k).$$

Then

$$\sup_{-\infty < t < \infty} |P\{X_k + Y_k \leq t\} - \Phi(t)| = O(a_k).$$

We shall now give the main result.

THEOREM 2.1. *For any fixed $\delta \in (0, \gamma/4)$,*

$$P\{\theta \in I_{N(d)}(d)\} = 1 - \alpha + O(d^{1/2-\delta}) \quad \text{as} \quad d \to 0.$$

Remark 2. Replacing $\hat{\theta}_n$ by θ_n defined in Section 1, this theorem also holds.

PROOF OF THEOREM 2.1. Let any $\delta \in (0, \gamma/4)$ be fixed and set

$$\beta = 1 - 2\delta. \tag{2.1}$$

Since

$$|P\{\theta \in I_{N(d)}(d)\} - (1 - \alpha)| \leq 2 \sup_{-\infty < x < \infty} \left| P\left\{ \hat{\theta}_{N(d)} - \theta \leq x\frac{d}{z} \right\} - \Phi(x) \right|,$$

in order to prove the theorem it is sufficient to show that

$$\sup_{-\infty<x<\infty}\left|P\left\{\hat{\theta}_{N(d_k)}-\theta\le x\frac{d_k}{z}\right\}-\Phi(x)\right|=O(d_k^{\beta/2})\quad\text{as}\quad d_k\to 0 \tag{2.2}$$

for β of (2.1) and any sequence of small positive numbers $\{d_k, k\ge 1\}$ such that $\lim_{k\to\infty} d_k=0$ and $0<d_k<\min\{1/2,(2bz\sigma)^{2/(\beta-2)},(bz\sigma/\sqrt{2})^{2/(2-\beta)}\}$. Let any sequence $\{d_k\}$ above be fixed. For simplicity, let d_k and $N_k=N(d_k)$ be denoted by d and N with dropping the suffix k throughout the remainder of this paper except for lemmas. Let $X_{n(1)}\le X_{n(2)}\le\cdots\le X_{n(n)}$ denote the order statistics for $X_1,\dots,X_n$. Put

$$U_{in}=(n-i+1)(X_{n(i)}-X_{n(i-1)})\quad\text{for}\quad 2\le i\le n.$$

Then it is known that $U_{2n},\dots,U_{nn}$ are i.i.d. random variables with p.d.f. $f_{0,\sigma}$ in (1.1) and that

$$\hat{\sigma}_n=\frac{1}{n-1}\sum_{i=2}^{n}U_{in}. \tag{2.3}$$

We define the following stopping rule:

$$N^*=N^*(d)=\inf\left\{n\ge m-1:\sum_{i=2}^{n+1}U_{in+1}\le\frac{d}{bz}n^{3/2}\right\},$$

where m is given by (1.5). It is clear that

$$N=N^*+1. \tag{2.4}$$

For all $n\ge 2$, set

$$Q_n=\frac{1}{n-1}\sum_{i=2}^{n}(U_{in}-\sigma)\quad\text{and}\quad R_n=a(\hat{T}_n-\mu). \tag{2.5}$$

Then, from (2.3) and (2.5) we have

$$\hat{\theta}_n-\theta=bQ_n+R_n.$$

Let any real number x be fixed. It follows from (2.4) that

$$\left|P\left\{\hat{\theta}_N-\theta\le x\frac{d}{z}\right\}-\Phi(x)\right|=|P\{(bz/d)Q_{N^*+1}+(z/d)R_{N^*+1}\le x\}-\Phi(x)|. \tag{2.6}$$

It will be shown in Section 3 that

$$\sup_{-\infty<x<\infty}|P\{(bz/d)Q_{N^*+1}\le x\}-\Phi(x)|=O(d^{\beta/2}) \tag{2.7}$$

and

$$P\{|(z/d)R_{N^*+1}|>d^{\beta/2}\}=O(d^{\beta/2}). \tag{2.8}$$

Thus, by Lemma 2.1, (2.6), (2.7) and (2.8) we obtain (2.2). Therefore the proof is complete.

3. Proofs of (2.7) and (2.8)

In this section we shall prove (2.7) and (2.8) along lines similar to those of Callaert and Janssen (1981). Theorem 1 of Landers and Rogge (1976) plays a crucial role in showing (2.7). Throughout this section, let $V_1, V_2, \ldots$ be a sequence of i.i.d. random variables with p.d.f. $f_{0,\sigma}$ in (1.1) and let c_1, c_2 and c_3 denote appropriate positive constants.

We shall provide four lemmas concerning two sequences of random variables $\{\sum_{i=2}^{n} U_{in}, n \geq 2\}$ and $\{\sum_{i=1}^{n-1} V_i, n \geq 2\}$. Lombard and Swanepoel (1978) give

LEMMA 3.1. *Two sequences of random variables*

$$\left\{\sum_{i=2}^{n} U_{in}, n \geq 2\right\} \quad \text{and} \quad \left\{\sum_{i=1}^{n-1} V_i, n \geq 2\right\}$$

have the same distribution.

We define the following stopping rule:

$$T = T(d) = \inf\left\{n \geq m-1 : \sum_{i=1}^{n} V_i \leq \frac{d}{bz} n^{3/2}\right\} \quad \text{for} \quad d > 0$$

where m is given by (1.5).

LEMMA 3.2. *We have the following two facts.*

(i) *$N^*(d)$ and $T(d)$ have the same distribution.*

(ii) *For any fixed integer $k(\geq m-1)$ satisfying $P\{N^*(d) = k\} > 0$, we get*

$$P\left\{\sum_{i=2}^{k+1} U_{i\,k+1} \leq x \;\middle|\; N^*(d) = k\right\} = P\left\{\sum_{i=1}^{k} V_i \leq x \;\middle|\; T(d) = k\right\} \quad \text{for all} \quad x > 0,$$

where m is given by (1.5).

PROOF. First, we shall prove (i). By Lemma 3.1 we have

$$\begin{aligned}
&P\{N^* > k\} \\
&\quad = P\left\{\sum_{i=2}^{n+1} U_{in+1} > \frac{d}{bz} n^{3/2} \text{ for all } n = m-1, \ldots, k\right\} \\
&\quad = P\left\{\sum_{i=1}^{n} V_i > \frac{d}{bz} n^{3/2} \text{ for all } n = m-1, \ldots, k\right\} \\
&\quad = P\{T > k\} \quad \text{for all} \quad k \geq m-1
\end{aligned}$$

and $P\{N^* = m-1\} = P\{T = m-1\}$, which conclude (i). Next, we shall show (ii). By (i) above, the definitions of N^* and T and Lemma 3.1 we get

$$\begin{aligned}
&P\left\{\sum_{i=2}^{k+1} U_{ik+1} \leq x \,\middle|\, N^* = k\right\} \\
&= P\left\{\sum_{i=2}^{j+1} U_{ij+1} > \frac{d}{bz} j^{3/2} \text{ for all } j = m-1, \ldots, k-1,\right. \\
&\qquad\qquad \left.\sum_{i=2}^{k+1} U_{ik+1} \leq \frac{d}{bz} k^{3/2} \text{ and } \sum_{i=2}^{k+1} U_{ik+1} \leq x\right\} \Big/ P\{N^* = k\} \\
&= P\left\{\sum_{i=1}^{j} V_i > \frac{d}{bz} j^{3/2} \text{ for all } j = m-1, \ldots, k-1,\right. \\
&\qquad\qquad \left.\sum_{i=1}^{k} V_i \leq \frac{d}{bz} k^{3/2} \text{ and } \sum_{i=1}^{k} V_i \leq x\right\} \Big/ P\{T = k\} \\
&= P\left\{\sum_{i=1}^{k} V_i \leq x \,\middle|\, T = k\right\}
\end{aligned}$$

which gives (ii). This completes the proof.

By a theorem on the moments of U-statistics (see Lemma A of Serfling (1980), p. 185), we have the following lemma.

LEMMA 3.3. *Let*

$$A_n = \frac{1}{n}\sum_{i=1}^{n}(V_i - \sigma) \quad \text{for} \quad n \geq 1. \tag{3.1}$$

Then

$$E|A_n|^r = O(n^{-r/2}) \quad \text{as } n \to \infty \quad \text{for any } r \geq 2.$$

By Lemma 3.3 and the method analogous to Lemma 4 of Callaert and Janssen (1981), we can obtain the following lemma.

LEMMA 3.4. *Let A_n be as in (3.1), ν and c arbitrary positive constants, λ any real constant and $g(d)$ be positive numbers tending to a positive constant as $d \to 0$. Then*

$$P\left\{\sup_{n \geq [g(d)d^{-\nu}]^*} |A_n| \geq cd^{\lambda}\right\} = O(d^{(\nu/2-\lambda)r}) \quad \text{as} \quad d \to 0$$

for any $r \geq 2$.

First of all, we shall show (2.8). Throughout the remainder of this section, β is given by (2.1). Let

$$J_1 = P\{|(z/d)R_{N^*+1}| > d^{\beta/2}\}.$$

Then, for $a > 0$

$$\begin{aligned} J_1 &\le P\{\hat{T}_{N^*+1} - \mu > c_1 d^{\beta/2+1}, |N^* - n_0| \le d^\beta n_0\} \\ &\quad + P\{|N^* - n_0| > d^\beta n_0\} \\ &\le P\{\hat{T}_{N^*+1} - \mu > c_1 d^{\beta/2+1}, m_1 \le N^* \le m_2\} + J_2, \end{aligned} \tag{3.2}$$

where $c_1 = (az)^{-1}$, $J_2 = P\{|N^* - n_0| > d^\beta n_0\}$, n_0 is given in (1.3) and

$$m_1 = [(1 - d^\beta)n_0]^* \quad \text{and} \quad m_2 = [(1 + d^\beta)n_0]^*. \tag{3.3}$$

Let us estimate J_2. Clearly

$$J_2 < P\{N^* > m_2\} + P\{N^* < m_1\} \equiv J_{21} + J_{22}, \quad \text{say}. \tag{3.4}$$

First, we shall estimate J_{21}. By the definition of N^*, Lemma 3.1 and (3.1) we get

$$J_{21} \le P\left\{A_{m_2-1} > \frac{d}{bz}(m_2 - 1)^{1/2} - \sigma\right\}. \tag{3.5}$$

From (1.3) and the fact that $1 > 2c_3 d^{2-\beta} = 2(d^\beta n_0)^{-1} \to 0$ as $d \to 0$ we have

$$\frac{d}{bz}(m_2 - 1)^{1/2} - \sigma > \sigma d^\beta \frac{1 - 2(d^\beta n_0)^{-1}}{(1 + d^\beta - 2n_0^{-1})^{1/2} + 1} > \frac{\sigma}{4} d^\beta. \tag{3.6}$$

Set $g(d) = (1 + d^\beta)(bz\sigma)^2 - d^2$. Then $m_2 - 1 = [g(d)d^{-2}]^*$, which, together with (3.5) and (3.6), implies

$$J_{21} \le P\left\{\sup_{j \ge [g(d)d^{-2}]^*} |A_j| \ge \frac{\sigma}{4} d^\beta\right\}.$$

Hence it follows from Lemma 3.4 that

$$J_{21} = O(d^{r(1-\beta)}) \quad \text{for any} \quad r \ge 2. \tag{3.7}$$

We shall now estimate J_{22}.

$$\begin{aligned} J_{22} &= P\{m - 1 \le N^* \le m_3\} + P\{m_3 < N^* \le m_1\} \\ &\equiv J_{221} + J_{222}, \quad \text{say}, \end{aligned}$$

where $m_3 = [(1 - d^{\beta_0})n_0]^*$, $\beta_0 = \gamma/2 + \beta - 1$ and γ is given in (1.5). We shall estimate J_{221}. Since

$$\frac{d}{bz} m_3^{1/2} - \sigma < -\frac{\sigma}{4} d^{\beta_0},$$

it follows from Lemma 3.2 and (1.5) that

$$J_{221} \le P\left\{A_j \le \frac{d}{bz} m_3^{1/2} - \sigma \text{ for some } m - 1 \le j \le m_3\right\}$$
$$\le P\left\{\sup_{j \ge [m_0 d^{-\gamma}]^*} |A_j| \ge \frac{\sigma}{4} d^{\beta_0}\right\}.$$

Then by Lemma 3.4 we get

$$J_{221} = O(d^{r(1-\beta)}) \quad \text{for any} \quad r \ge 2. \tag{3.8}$$

By the same argument as (3.8) we have

$$J_{222} = O(d^{r(1-\beta)}) \quad \text{for any} \quad r \ge 2. \tag{3.9}$$

Hence it follows from (3.8) and (3.9) that

$$J_{22} = O(d^{r(1-\beta)}) \quad \text{for any} \quad r \ge 2. \tag{3.10}$$

Thus, in view of (3.4), (3.7) and (3.10) we have that $J_2 = O(d^{r(1-\beta)})$ for any $r \ge 2$. By choosing a constant r satisfying $r > \max\{2, \beta/2(1-\beta)\}$ we obtain

$$J_2 = o(d^{\beta/2}). \tag{3.11}$$

Since $\hat{T}_n \ge \hat{T}_{n+1}$ a.s. for $n \ge 2$, $m_1 \sim (bz\sigma)^2 d^{-2}$ and $P\{\hat{T}_n - \mu > y\} = \exp(-\frac{n}{\sigma} y)$ for $y > 0$, we get that for $a > 0$

$$\begin{aligned} &P\{\hat{T}_{N^*+1} - \mu > c_1 d^{\beta/2+1}, m_1 \le N^* \le m_2\} \\ &\quad \le P\{\hat{T}_{m_1+1} - \mu > c_1 d^{\beta/2+1}\} \\ &\quad = \exp\{-(c_1/\sigma)(m_1+1) d^{\beta/2+1}\} \\ &\quad \le \exp(-c_2 d^{\beta/2-1}) = O(d^{l(1-\beta/2)}) \quad \text{for any integer} \quad l \ge 1, \end{aligned}$$

which, together with (3.2) and (3.11), implies that $J_1 = O(d^{l(1-\beta/2)}) + o(d^{\beta/2})$. For $a = 0$ this equality holds. Hence, with $l = [\beta/(2-\beta)]^* + 1$ we get $J_1 = O(d^{\beta/2})$, which concludes (2.8).

By Lemma 3.2 and the estimate of J_2 in (3.11), we can get the following lemma which will be used when the assumption of Lemma 3.6 is verified.

LEMMA 3.5. *For any constant $c > 0$,*

$$P\left\{\left|\frac{T(d)}{(bz\sigma)^2 d^{-2}} - 1\right| \ge c d^{\beta}\right\} = O(d^{\beta/2}) \quad \text{as} \quad d \to 0.$$

Now, we shall prove (2.7). The following lemma is given by Landers and Rogge (1976).

LEMMA 3.6. *Let $\{X_n, n \geq 1\}$ be independent and identically distributed random variables with variance $\sigma^2 > 0$ and $E|X_n|^3 < \infty$. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables and assume that*

$$P\left\{\left|\frac{N_n}{\tau n} - 1\right| > \varepsilon_n\right\} = O(\sqrt{\varepsilon_n}) \tag{3.12}$$

for some constant $\tau > 0$ and a sequence $\{\varepsilon_n\}$ with $n^{-1} \leq \varepsilon_n \to 0$ as $n \to \infty$. Then

$$\sup_{-\infty<t<\infty}\left|P\left\{\sum_{i=1}^{N_n}(X_i - E(X_i)) \leq t\sigma\sqrt{N_n}\right\} - \Phi(t)\right| = O(\sqrt{\varepsilon_n}).$$

Set

$$J_3(x) = |P\{(bz/d)Q_{N^*+1} \leq x\} - \Phi(x)|.$$

Then, by (2.5) and Lemma 3.2

$$\begin{aligned} J_3(x) &= \left|\sum_{n=m-1}^{\infty} P\left\{(bz/d)n^{-1}\sum_{i=2}^{n+1}(U_{in+1} - \sigma) \leq x \,\middle|\, N^* = n\right\} \right. \\ &\qquad\qquad \left. \cdot P(N^* = n) - \Phi(x)\right| \\ &= \left|\sum_{n=m-1}^{\infty} P\left\{(bz/d)n^{-1}\sum_{i=1}^{n}(V_i - \sigma) \leq x \,\middle|\, T = n\right\} \right. \\ &\qquad\qquad \left. \cdot P(T = n) - \Phi(x)\right| \\ &= \left|P\left\{\sum_{i=1}^{T}(V_i - \sigma)/(\sigma\sqrt{T}) \leq xd\sqrt{T}/(bz\sigma)\right\} - \Phi(x)\right|. \end{aligned} \tag{3.13}$$

For the sequence $\{d_k\}$ given in the proof of Theorem 2.1 let $n(k) = [d_k^{-2}]^*$,

$$N_n = \begin{cases} T(d_k) & \text{if } n = n(k) \quad \text{for some } k \geq 1 \\ [(bz\sigma)^2 n]^* & \text{otherwise} \end{cases} \tag{3.14}$$

and

$$\varepsilon_n = \begin{cases} c_3 d_k^{\beta} & \text{if } n = n(k) \quad \text{for some } k \geq 1 \\ c_3 n^{-\beta} & \text{otherwise} \end{cases} \tag{3.15}$$

where $c_3 = (bz\sigma)^{-2}$. From $\beta \in (0, 1)$, the fact that $d_k < \min\{1/2, (c_3/4)^{1/(2-\beta)}\}$ and the definition of ε_n we get

$$n^{-1} \leq \varepsilon_n \quad \text{for all} \quad n \geq c_3^{1/(\beta-1)}.$$

Let

$$J_4 = P\left\{\left|\frac{N_n}{\tau n} - 1\right| > \varepsilon_n\right\} \qquad \text{with} \quad \tau = c_3^{-1}.$$

Then, for $n = n(k)$

$$\begin{aligned}
(3.16)\quad J_4 &= P\left\{\left|\frac{T(d_k)}{(bz\sigma)^2 n(k)} - 1\right| > c_3 d_k^\beta\right\}\\
&\le P\left\{\left|\frac{T(d_k)}{(bz\sigma)^2 d_k^{-2}} - 1\right| > \frac{c_3}{2} d_k^\beta\right\} + P\left\{\frac{T(d_k)}{(bz\sigma)^2}\left|\frac{1}{n(k)} - \frac{1}{d_k^{-2}}\right| > \frac{c_3}{2} d_k^\beta\right\}\\
&\le P\left\{\left|\frac{T(d_k)}{(bz\sigma)^2 d_k^{-2}} - 1\right| > \frac{c_3}{2} d_k^\beta\right\} + P\left\{\frac{T(d_k)}{(bz\sigma)^2 d_k^{-2}} > \frac{3 d_k^{-2}}{8} c_3 d_k^\beta\right\}\\
&\le 2P\left\{\left|\frac{T(d_k)}{(bz\sigma)^2 d_k^{-2}} - 1\right| \ge \frac{c_3}{2} d_k^\beta\right\},
\end{aligned}$$

because

$$\left|\frac{1}{[d_k^{-2}]^*} - \frac{1}{d_k^{-2}}\right| < \frac{4}{3} d_k^4 < \frac{c_3 d_k^{\beta+2}}{2 + c_3 d_k^\beta} \qquad \text{whenever} \quad d_k < \min\{1/2, (c_3/4)^{1/(2-\beta)}\}.$$

Since

$$\left|\frac{[(bz\sigma)^2 n]^*}{(bz\sigma)^2 n} - 1\right| \le c_3 n^{-\beta} \qquad \text{for} \quad n \ne n(k),$$

we have

$$(3.17)\qquad J_4 = 0 \qquad \text{for} \quad n \ne n(k).$$

Hence, in view of (3.16), (3.17) and Lemma 3.5 we get that $J_4 = O(\sqrt{\varepsilon_n})$, from which (3.12) of Lemma 3.6 is satisfied. Thus, by Lemma 3.6 we have

$$\sup_{-\infty<x<\infty}\left|P\left\{\sum_{i=1}^{N_n}(V_i - \sigma)/(\sigma\sqrt{N_n}) \le x\right\} - \Phi(x)\right| = O(\sqrt{\varepsilon_n}),$$

which, with setting $n = n(k)$ in (3.14) and (3.15), yields that

$$(3.18)\qquad \sup_{-\infty<x<\infty}\left|P\left\{\sum_{i=1}^{T}(V_i - \sigma)/(\sigma\sqrt{T}) \le x\right\} - \Phi(x)\right| = O(d^{\beta/2}),$$

where $T = T(d_k)$ and $d = d_k$. Since $|\sqrt{x} - \sqrt{y}| > b$ for nonnegative constants x, y and b implies $|x - y| > b^2$, by Lemma 3.5 we have

$$(3.19)\qquad P\left\{\left|\frac{d\sqrt{T}}{bz\sigma} - 1\right| > d^{\beta/2}\right\} < P\left\{\left|\frac{T}{(bz\sigma)^2 d^{-2}} - 1\right| > d^\beta\right\} = O(d^{\beta/2}).$$

To show (2.7) we need the following lemma which is Lemma 1 of Michel and Pfanzagl (1971).

LEMMA 3.7. *Let $\{X_k, k \geq 1\}$ and $\{Y_k, k \geq 1\}$ be two sequences of random variables. Assume that for a sequence of positive numbers $\{a_k, k \geq 1\}$*

$$\sup_{-\infty<t<\infty} |P\{X_k \leq t\} - \Phi(t)| = O(a_k) \quad \textit{and} \quad P\{|Y_k - 1| > a_k\} = O(a_k).$$

Then

$$\sup_{-\infty<t<\infty} |P\{X_k \leq tY_k\} - \Phi(t)| = O(a_k).$$

We are now in the position to prove (2.7). By (3.13), (3.18), (3.19) and Lemma 3.7 we have

$$\begin{aligned} J_3(x) &\leq \sup_{-\infty<x<\infty} \left| P\left\{ \sum_{i=1}^{T} (V_i - \sigma)/(\sigma\sqrt{T}) \leq xd\sqrt{T}/(bz\sigma) \right\} - \Phi(x) \right| \\ &= O(d^{\beta/2}) \quad \text{for any} \quad x. \end{aligned}$$

Thus

$$\sup_{-\infty<x<\infty} |P\{(bz/d)Q_{N^*+1} \leq x\} - \Phi(x)| = \sup_{-\infty<x<\infty} J_3(x) = O(d^{\beta/2}),$$

which concludes (2.7). Therefore (2.7) and (2.8) are proved.

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REFERENCES

Callaert, H. and Janssen, P. (1981). The convergence rate of fixed-width sequential confidence intervals for the mean, *Sankhyā Ser. A*, **43**, 211–219.

Chow, Y. S. and Robbins, H. (1965). On the asymptotic theory of fixed-width sequential confidence intervals for the mean, *Ann. Math. Statist.*, **36**, 457–462.

Csenki, A. (1980). On the convergence rate of fixed-width sequential confidence intervals, *Scand. Actuar. J.*, 107–111.

Govindarajulu, Z. (1985). Fixed-width confidence interval estimation of scale parameter of exponential distribution with unknown location, *J. Orissa Math. Soc.*, **4**, 77–83.

Isogai, E., Hirose, K. and Uno, C. (1995). Fixed-width confidence interval estimation of a parameter of an exponential distribution (submitted for publication).

Landers, D. and Rogge, L. (1976). The exact approximation order in the central-limit theorem for random summation, *Zeit. Wahrscheinlichkeitsth.*, **36**, 269–283.

Lombard, F. and Swanepoel, J. W. H. (1978). On finite and infinite confidence sequences, *South African Statist. J.*, **12**, 1–24.

Michel, R. and Pfanzagl, J. (1971). The accuracy of the normal approximation for minimum contrast estimates, *Zeit. Wahrscheinlichkeitsth.*, **18**, 73–84.

Mukhopadhyay, N. (1974). Sequential estimation of location parameter in exponential distributions, *Calcutta Statist. Assoc. Bull.*, **23**, 85–95.

Mukhopadhyay, N. and Datta, S. (1996). On sequential fixed-width confidence intervals for the mean and second-order expansions of the associated coverage probabilities, *Ann. Inst. Statist. Math.*, **48**, 497–507.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, Wiley, New York.

Swanepoel, J. W. H. and van Wyk, J. W. J. (1982). Fixed width confidence intervals for the location parameter of an exponential distribution, *Comm. Statist. Theory Methods*, **11**, 1279–1289.