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START-UP DEMONSTRATION TESTS UNDER MARKOV DEPENDENCE MODEL WITH CORRECTIVE ACTIONS*

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Abstract. A general probability model for a start-up demonstration test is studied. The joint probability generating function of some random variables appearing in the Markov dependence model of the start-up demonstration test with corrective actions is derived by the method of probability generating function. By using the probability generating function, several characteristics relating to the distribution are obtained.

Key words and phrases: Probability generating function, discrete distributions, run, start-up demonstration test, Markov chain, waiting time.

1. Introduction

In this paper we deal with a general probability model for a start-up demonstration test. A start-up demonstration test is a mechanism by which a vender demonstrates to a customer the reliability of a equipment with regard to its starting. The vender repeats start ups of the equipment until a specified number of consecutive successful start-ups are observed.

By assuming that individual start-ups are independent events with a constant probability p, Hahn and Gage (1983) showed a recurrence formula for the probabilities of the waiting time for the consecutive successes of the specified length. Viveros and Balakrishnan (1993) derived the moments of the distribution and developed statistical inference of the parameter p. A Markov dependence model for the problem is also introduced by Viveros and Balakrishnan (1993). Balakrishnan *et al.* (1995) derived joint probability generating functions for different statistics involved in start-up demonstration testing under the Markov dependence structure. Some corrective action models, such as single, double and sequential action models are also introduced by Balakrishnan *et al.* (1995) in independent trials. In a single corrective action model, one corrective action on the equipment is allowed

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resulting in a change in the probability of success. The equipment intervension takes place at the time of the first failure. In a double corrective action model, two corrective actions are allowed at the times of the first two failures. In a sequential corrective action model, corrective action is allowed at the time of every failure.

Independently of these studies, statistical distribution theory on runs is developed by many authors. The start-up demonstration tests are closely related to the geometric distribution of order k and its modifications (cf. e.g. Feller ((1968), pp. 303–341), Philippou *et al.* (1983), Aki and Hirano (1989, 1993, 1994), Johnson *et al.* ((1992), pp. 426–432), Balasubramanian *et al.* (1993), Mohanty (1994) and Godbole and Papastavridis (1994)).

The purpose of this paper is to combine these various approaches and to extend a corrective action model from independent trials to Markov dependent trials.

2. Probability generating function

First of all we study a single corrective action model under the Markov dependence structure. Denoting a success and a failure by S and F respectively, let $X_1, X_2, \ldots$ be a sequence of $\{S, F\}$-valued random variables with the following probabilities: before the first F,

$$p_{00} = P(S \text{ in the first trial}) = 1 - q_{00},$$
$$p_{01} = P(S \mid \text{previous trial is } S) = 1 - q_{01},$$

and after the first F (i.e. after the corrective action),

$$p_{10} = P(S \text{ in the trial just after the first } F) = 1 - q_{10},$$
$$p_{11} = P(S \mid \text{previous trial is } S) = 1 - q_{11},$$

and

$$p_{12} = P(S \mid \text{previous trial is } F) = 1 - q_{12}.$$

Let c be the required number of consecutive successful start-ups to achieve acceptance. We introduce the following 10 random variables: before the corrective action (i.e. before the first F), let

S_0 = the number of S in the first trial.

F_0 = the number of F in the first trial $= 1 - S_0$.

S_{01} = the number of S's whose previous trial is S.

F_{01} = the number of F's whose previous trial is S.

After the corrective action,

S_{10} = the number of S in the trial just after the corrective action.

F_{10} = the number of F in the trial just after the corrective action $= 1 - S_{10}$.

S_{11} = the number of S's whose previous trial is S.

F_{11} = the number of F's whose previous trial is S.

S_{12} = the number of S's whose previous trial is F.

F_{12} = the number of F's whose previous trial is F.

Let

$$Z \equiv (S_0, F_0, S_{01}, F_{01}, S_{10}, F_{10}, S_{11}, F_{11}, S_{12}, F_{12}),$$

and let $G(t_0, u_0, t_1, u_1, v_0, w_0, v_1, w_1, v_2, w_2)$ be the probability generating function (p.g.f.) of $\boldsymbol{Z}$.

Suppose we have currently S-run of length i ($i = 1, 2, \ldots, c-1$) and the first F has not yet occurred. Let ϕ_i denote the conditional p.g.f. of $\boldsymbol{Z}$ from this time. Suppose the first F has just occurred. Then, let H denote the conditional p.g.f. of $\boldsymbol{Z}$ from this time. Suppose that the first F has already occurred and we have currently S-run of length j ($j = 0, 1, 2, \ldots, c-1$). Then, let ξ_j denote the conditional p.g.f. of $\boldsymbol{Z}$ from this time.

By considering the condition of one-step ahead from every condition, we have the following system of equations:

$$G = p_{00}t_0\phi_1 + q_{00}u_0H \tag{2.1}$$

$$\begin{cases} \phi_1 = p_{01}t_1\phi_2 + q_{01}u_1H \\ \phi_2 = p_{01}t_1\phi_3 + q_{01}u_1H \\ \cdots \\ \phi_{c-1} = p_{01}t_1 \cdot 1 + q_{01}u_1H \end{cases} \tag{2.2}$$

$$H = p_{10}v_0\xi_1 + q_{10}w_0\xi_0 \tag{2.3}$$

$$\begin{cases} \xi_1 = p_{11}v_1\xi_2 + q_{11}w_1\xi_0 \\ \xi_2 = p_{11}v_1\xi_3 + q_{11}w_1\xi_0 \\ \cdots \\ \xi_{c-1} = p_{11}v_1 \cdot 1 + q_{11}w_1\xi_0 \end{cases} \tag{2.4}$$

$$\xi_0 = p_{12}v_2\xi_1 + q_{12}w_2\xi_0. \tag{2.5}$$

From (2.4), we have

$$\xi_1 = q_{11}w_1\frac{1-(p_{11}v_1)^{c-1}}{1-p_{11}v_1} \cdot \xi_0 + (p_{11}v_1)^{c-1}.$$

From (2.5), we see that

$$\xi_0 = \frac{p_{12}v_2}{1-q_{12}w_2}\xi_1.$$

Then, we have

$$\xi_1 = \frac{(p_{11}v_1)^{c-1}(1-p_{11}v_1)(1-q_{12}w_2)}{(1-p_{11}v_1)(1-q_{12}w_2) - q_{11}w_1(1-(p_{11}v_1)^{c-1})p_{12}v_2},$$

and

$$\xi_0 = \frac{p_{12}v_2(p_{11}v_1)^{c-1}(1-p_{11}v_1)}{(1-p_{11}v_1)(1-q_{12}w_2)-q_{11}w_1(1-(p_{11}v_1)^{c-1})p_{12}v_2}.$$

Then, (2.3) implies

$$H = \frac{(p_{11}v_1)^{c-1}(1-p_{11}v_1)\{p_{10}v_0(1-q_{12}w_2)+q_{10}w_0p_{12}v_2\}}{(1-p_{11}v_1)(1-q_{12}w_2)-q_{11}w_1(1-(p_{11}v_1)^{c-1})p_{12}v_2}.$$

By solving (2.2), we have

$$\phi_1 = q_{01}u_1\frac{1-(p_{01}t_1)^{c-1}}{1-p_{01}t_1}\cdot H + (p_{01}t_1)^{c-1}.$$

Then, from (2.1), we obtain

$$\begin{aligned}(2.6)\qquad G &= \left(p_{00}t_0q_{01}u_1\frac{1-(p_{01}t_1)^{c-1}}{1-p_{01}t_1}+q_{00}u_0\right)\cdot H + p_{00}t_0(p_{01}t_1)^{c-1}\\ &= p_{00}t_0(p_{01}t_1)^{c-1}\\ &\quad+\frac{1}{R}\{(p_{11}v_1)^{c-1}(1-p_{11}v_1)\{p_{10}v_0(1-q_{12}w_2)+q_{10}w_0p_{12}v_2\}\\ &\qquad\times\{p_{00}t_0q_{01}u_1(1-(p_{01}t_1)^{c-1})+q_{00}u_0(1-p_{01}t_1)\}\},\end{aligned}$$

where

$$R = (1-p_{01}t_1)\{(1-p_{11}v_1)(1-q_{12}w_2)-q_{11}w_1(1-(p_{11}v_1)^{c-1})p_{12}v_2\}.$$

Thus, we have

THEOREM 2.1. *The probability generating function of* $\boldsymbol{Z}$ *is given as* (2.6).

Remark. By setting $p_{00} = p_{01} = p_0$, $q_{00} = q_{01} = q_0 = 1 - p_0$, $p_{10} = p_{11} = p_{12} = p_1$, $q_{10} = q_{11} = q_{12} = q_1 = 1 - p_1$, $t_0 = t_1 = \tau_0$, $u_0 = u_1 = \eta_0$, $v_0 = v_1 = v_2 = \tau_1$ and $w_0 = w_1 = w_2 = \eta_1$, we obtain from (2.6) the joint p.g.f. of (the number of S's before the corrective action, the number of F before the corrective action, the number of S's after the corrective action, the number of F's after the corrective action) in independent trials,

$$p_0^c\tau_0^c + \frac{1-(p_0\tau_0)^c}{1-p_0\tau_0}\,\frac{q_0\eta_0(p_1\tau_1)^c(1-p_1\tau_1)}{1-p_1\tau_1-q_1\eta_1+q_1\eta_1(p_1\tau_1)^c},$$

which agrees with Result 4 of Balakrishnan *et al.* (1995).

Though we have obtained Theorem 2.1 just by solving the system of equations of conditional p.g.f.'s, we can get it also by considering structures of typical sequences as follows.

Let X be the total number of attempted start-ups until the item is accepted (i.e. the number of trials until the first consecutive c successes). Corresponding to $X = c$, we have the contribution to the p.g.f. to be

$$p_{00}(p_{01})^{c-1}t_0{t_1}^{c-1}. \tag{2.7}$$

For values greater than c, a typical sequence of start-ups leading to the acceptance of the item with the first trial being a S and with k (≥ 1) sequences of failures, is one of the following two forms:

(A) $S\underbrace{S\cdots S}_{a_1}\underbrace{F}_{b_1=1}\underbrace{S\cdots S}_{a_2}\underbrace{F\cdots F}_{b_2}\cdots\underbrace{S\cdots S}_{a_k}\underbrace{F\cdots F}_{b_k}\underbrace{S\cdots S}_{c}$,
$0 \leq a_1 \leq c-2,\ 1 \leq a_i \leq c-1$ for $i = 2, 3, \ldots, k$,
$1 \leq b_i$ for $i = 2, 3, \ldots, k$;

(B) $S\underbrace{S\cdots S}_{a_1}\underbrace{FF\cdots F}_{b_1}\underbrace{S\cdots S}_{a_2}\underbrace{F\cdots F}_{b_2}\cdots\underbrace{S\cdots S}_{a_k}\underbrace{F\cdots F}_{b_k}\underbrace{S\cdots S}_{c}$,
$0 \leq a_1 \leq c-2,\ 1 \leq a_i \leq c-1$ for $i = 2, 3, \ldots, k$,
$2 \leq b_1,\ 1 \leq b_i$ for $i = 2, 3, \ldots, k$;

The contribution of (A) to the p.g.f. for fixed k, is:

$$\begin{aligned}
&p_{00}t_0(q_{01}u_1 + p_{01}t_1q_{01}u_1 + \cdots + (p_{01}t_1)^{c-2}q_{01}u_1)(p_{10}v_0)\\
&\quad\times (q_{11}w_1 + p_{11}v_1q_{11}w_1 + \cdots + (p_{11}v_1)^{c-2}q_{11}w_1)^{k-1}\\
&\quad\times (p_{12}v_2 + q_{12}w_2p_{12}v_2 + (q_{12}w_2)^2p_{12}v_2 + \cdots)^{k-1}{p_{11}}^{c-1}{v_1}^{c-1}\\
&= p_{00}q_{01}p_{10}{p_{11}}^{c-1}t_0u_1v_0{v_1}^{c-1}\frac{1-(p_{01}t_1)^{c-1}}{1-p_{01}t_1}\\
&\quad\times\left\{q_{11}p_{12}w_1v_2\frac{1}{1-q_{12}w_2}\cdot\frac{1-(p_{11}v_1)^{c-1}}{1-p_{11}v_1}\right\}^{k-1}.
\end{aligned}$$

Summing up from $k = 1$ to ∞, we obtain the contribution of (A) to the p.g.f. as

$$\Pi_A = \frac{p_{00}q_{01}p_{10}{p_{11}}^{c-1}t_0u_1v_0{v_1}^{c-1}(1-p_{11}v_1)(1-q_{12}w_2)(1-(p_{01}t_1)^{c-1})}{(1-p_{01}t_1)\{(1-p_{11}v_1)(1-q_{12}w_2) - q_{11}p_{12}w_1v_2 + q_{11}p_{12}{p_{11}}^{c-1}w_1v_2{v_1}^{c-1}\}}. \tag{2.8}$$

Similarly, we obtain the contribution of (B) to the p.g.f. as

$$\Pi_B = \frac{p_{00}q_{01}q_{10}p_{12}{p_{11}}^{c-1}t_0u_1w_0v_2{v_1}^{c-1}(1-p_{11}v_1)(1-(p_{01}t_1)^{c-1})}{(1-p_{01}t_1)\{(1-p_{11}v_1)(1-q_{12}w_2) - q_{11}p_{12}w_1v_2 + q_{11}p_{12}{p_{11}}^{c-1}w_1v_2{v_1}^{c-1}\}}. \tag{2.9}$$

A typical sequence of start-ups leading to the acceptance of the item with the first trial being a F and with k (≥ 1) failures is one of the following two forms:

(C) $F\underbrace{S\cdots S}_{a_1}\underbrace{F\cdots F}_{b_1}\underbrace{S\cdots S}_{a_2}\underbrace{F\cdots F}_{b_2}\cdots\underbrace{S\cdots S}_{a_k}\underbrace{F\cdots F}_{b_k}\underbrace{S\cdots S}_{c}$,
$1 \leq a_i \leq c-1$ for $i = 1, 2, \ldots, k$;
$1 \leq b_i$ for $i = 1, 2, \ldots, k$ and $k \geq 0$;

(D) $F\underbrace{F\cdots F}_{b_1}\underbrace{S\cdots S}_{a_1}\underbrace{F\cdots F}_{b_2}\underbrace{S\cdots S}_{a_2}\cdots\underbrace{S\cdots S}_{a_{k-1}}\underbrace{F\cdots F}_{b_k}\underbrace{S\cdots S}_{c}$,
$1 \le a_i \le c-1$ for $i = 1, 2, \dots, k-1$;
$1 \le b_i$ for $i = 1, 2, \dots, k$ and $k \ge 1$.

Similarly as in (A), we obtain the contributions of (C) and (D) to the p.g.f. respectively as

$$\Pi_{\mathrm{C}} = \frac{q_{00}p_{10}{p_{11}}^{c-1}u_0 v_0 {v_1}^{c-1}(1-p_{11}v_1)(1-q_{12}w_2)}{(1-p_{11}v_1)(1-q_{12}w_2) - q_{11}p_{12}w_1v_2 + q_{11}p_{12}{p_{11}}^{c-1}w_1v_2{v_1}^{c-1}}, \tag{2.10}$$

and

$$\Pi_{\mathrm{D}} = \frac{q_{00}q_{10}p_{12}{p_{11}}^{c-1}u_0 w_0 v_2 {v_1}^{c-1}(1-p_{11}v_1)}{(1-p_{11}v_1)(1-q_{12}w_2) - q_{11}p_{12}w_1v_2 + q_{11}p_{12}{p_{11}}^{c-1}w_1v_2{v_1}^{c-1}}. \tag{2.11}$$

Finally, by making use of the expressions in (2.7), (2.8), (2.9), (2.10) and (2.11), we derive the joint p.g.f. as

$$G = p_{00}{p_{01}}^{c-1}t_0{t_1}^{c-1} + \Pi_{\mathrm{A}} + \Pi_{\mathrm{B}} + \Pi_{\mathrm{C}} + \Pi_{\mathrm{D}}.$$

Of course, this expression agrees with (2.6).

Before closing the section we note that our method can be applied directly to study more complicated models such as double or sequential corrective action models, though the corresponding results may not be so simple.

Here, we consider a double corrective action model under the Markov dependence structure. Let $X_1, X_2, \dots$ be a sequence of $\{S, F\}$-valued random variables with the following probabilities: before the first F,

$$p_{00} = P(S \text{ in the first trial}) = 1 - q_{00},$$
$$p_{01} = P(S \mid \text{previous trial is } S) = 1 - q_{01},$$

between the first and the second F (i.e. between two corrective actions),

$$p_{10} = P(S \text{ in the trial just after the first } F) = 1 - q_{10},$$
$$p_{11} = P(S \mid \text{previous trial is } S) = 1 - q_{11},$$

and after the second F (i.e. after two corrective actions),

$$p_{20} = P(S \text{ in the trial just after the second } F) = 1 - q_{20},$$
$$p_{21} = P(S \mid \text{previous trial is } S) = 1 - q_{21},$$

and

$$p_{22} = P(S \mid \text{previous trial is } F) = 1 - q_{22}.$$

Let c be the required number of consecutive successful start-ups to achieve acceptance. We introduce the following 14 random variables: before the corrective actions (i.e. before the first F), let

S_0 = the number of S in the first trial.
F_0 = the number of F in the first trial $= 1 - S_0$.
S_{01} = the number of S's whose previous trial is S.
F_{01} = the number of F's whose previous trial is S.

Between two corrective actions,

S_{10} = the number of S in the trial just after the first corrective action.
F_{10} = the number of F in the trial just after the first corrective action $= 1 - S_{10}$.
S_{11} = the number of S's whose previous trial is S.
F_{11} = the number of F's whose previous trial is S.

After two corrective actions,

S_{20} = the number of S in the trial just after the second corrective action.
F_{20} = the number of F in the trial just after the second corrective action $= 1 - S_{20}$.
S_{21} = the number of S's whose previous trial is S.
F_{21} = the number of F's whose previous trial is S.
S_{22} = the number of S's whose previous trial is F.
F_{22} = the number of F's whose previous trial is F.

Let

$$\boldsymbol{Z}_2 \equiv (S_0, F_0, S_{01}, F_{01}, S_{10}, F_{10}, S_{11}, F_{11}, S_{20}, F_{20}, S_{21}, F_{21}, S_{22}, F_{22})$$

and let $G_2(v_{00}, w_{00}, v_{01}, w_{01}, v_{10}, w_{10}, v_{11}, w_{11}, v_{20}, w_{20}, v_{21}, w_{21}, v_{22}, w_{22})$ be the probability generating function (p.g.f.) of $\boldsymbol{Z}_2$.

Suppose we have currently S-run of length i ($i = 1, 2, \ldots, c-1$) and the first F has not yet occurred. Let ϕ_{0i} denote the conditional p.g.f. of $\boldsymbol{Z}_2$ from this time. Suppose the first F has just occurred. Then, let H_1 denote the conditional p.g.f. of $\boldsymbol{Z}_2$ from this time. Suppose that the first F has already occurred, the second F has not yet occurred, and we have currently S-run of length j ($j = 1, 2, \ldots, c-1$). Then, let ϕ_{1j} denote the conditional p.g.f. of $\boldsymbol{Z}_2$ from this time. Suppose that the second F has just occurred. Then, let H_2 denote the conditional p.g.f. of $\boldsymbol{Z}_2$ from this time. Suppose that the second F has already occurred, and we have currently S-run of length j ($j = 0, 1, 2, \ldots, c-1$). Then, let ϕ_{2j} denote the conditional p.g.f. of $\boldsymbol{Z}_2$ from this time.

By considering the condition of one-step ahead from every condition, we have the following system of equations:

$$G_2 = p_{00}v_{00}\phi_{01} + q_{00}w_{00}H_1 \tag{2.12}$$

$$\begin{cases} \phi_{01} = p_{01}v_{01}\phi_{02} + q_{01}w_{01}H_1 \\ \phi_{02} = p_{01}v_{01}\phi_{03} + q_{01}w_{01}H_1 \\ \cdots \\ \phi_{0,c-1} = p_{01}v_{01} \cdot 1 + q_{01}w_{01}H_1 \end{cases} \tag{2.13}$$

$$H_1 = p_{10}v_{10}\phi_{11} + q_{10}w_{10}H_2 \tag{2.14}$$

$$\begin{cases} \phi_{11} = p_{11}v_{11}\phi_{12} + q_{11}w_{11}H_2 \\ \phi_{12} = p_{11}v_{11}\phi_{13} + q_{11}w_{11}H_2 \\ \cdots \\ \phi_{1,c-1} = p_{11}v_{11} \cdot 1 + q_{11}w_{11}H_2 \end{cases} \tag{2.15}$$

$$H_2 = p_{20}v_{20}\phi_{21} + q_{20}w_{20}\phi_{20} \tag{2.16}$$

$$\begin{cases} \phi_{21} = p_{21}v_{21}\phi_{22} + q_{21}w_{21}\phi_{20} \\ \phi_{22} = p_{21}v_{21}\phi_{23} + q_{21}w_{21}\phi_{20} \\ \cdots \\ \phi_{2,c-1} = p_{21}v_{21} \cdot 1 + q_{21}w_{21}\phi_{20} \end{cases} \tag{2.17}$$

$$\phi_{20} = p_{22}v_{22}\phi_{21} + q_{22}w_{22}\phi_{20}. \tag{2.18}$$

We set

$$U_i \equiv p_{i0}v_{i0}q_{i1}w_{i1}\frac{1-(p_{i1}v_{i1})^{c-1}}{1-p_{i1}v_{i1}} + q_{i0}w_{i0}$$

and

$$V_i = p_{i0}v_{i0}(p_{i1}v_{i1})^{c-1}$$

for every nonnegative integer i. Then, from (2.12) and (2.13) we have $G_2 = U_0H_1 + V_0$. Similarly, from (2.14) and (2.15) we obtain $H_1 = U_1H_2 + V_1$. By solving (2.16), (2.17) and (2.18) we have $H_2 = U_2\phi_{20} + V_2$ and

$$\phi_{20} = \frac{p_{22}v_{22}(p_{21}v_{21})^{c-1}(1-p_{21}v_{21})}{(1-p_{21}v_{21})(1-q_{22}w_{22}) - p_{22}v_{22}q_{21}w_{21}(1-(p_{21}v_{21})^{c-1})}. \tag{2.19}$$

Consequently, the p.g.f. of $\mathbf{Z}_2$ in a double corrective action model is given as

$$G_2 = V_0 + U_0V_1 + U_0U_1V_2 + U_0U_1U_2\phi_{20},$$

where ϕ_{20} is given in (2.19).

Last in the section we consider a sequential corrective action model under the Markov dependence structure. Let $X_1, X_2, \ldots$ be a sequence of $\{S, F\}$-valued random variables with the following probabilities: before the first F,

$$p_{00} = P(S \text{ in the first trial}) = 1 - q_{00},$$
$$p_{01} = P(S \mid \text{previous trial is } S) = 1 - q_{01},$$

for $i = 1, 2, \ldots$, between the i-th and the $(i+1)$ th F (i.e. between the i-th and $(i+1)$-th corrective actions),

$$p_{i0} = P(S \text{ in the trial just after the } i\text{-th } F) = 1 - q_{i0},$$
$$p_{i1} = P(S \mid \text{previous trial is } S) = 1 - q_{i1}.$$

Let c be the required number of consecutive successful start-ups to achieve acceptance. We introduce the following sequence of random variables: before the corrective actions (i.e. before the first F), let

S_0 = the number of S in the first trial.
F_0 = the number of F in the first trial $= 1 - S_0$.
S_{01} = the number of S's whose previous trial is S.
F_{01} = the number of F's whose previous trial is S.

For $i = 1, 2, \ldots$, between the i-th and $(i+1)$-th corrective actions,

S_{i0} = the number of S in the trial just after the i-th corrective action.
F_{i0} = the number of F in the trial just after the i-th corrective action $= 1 - S_{i0}$.
S_{i1} = the number of S's whose previous trial is S.
F_{i1} = the number of F's whose previous trial is S.

Let
$$\boldsymbol{Z_S} \equiv (S_0, F_0, S_{01}, F_{01}, S_{10}, F_{10}, S_{11}, F_{11}, S_{20}, F_{20}, S_{21}, F_{21}, \ldots)$$
and let $G_S(v_{00}, w_{00}, v_{01}, w_{01}, v_{10}, w_{10}, v_{11}, w_{11}, v_{20}, w_{20}, v_{21}, w_{21}, \ldots)$ be the p.g.f. of $\boldsymbol{Z_S}$.

Suppose we have currently S-run of length j $(j = 1, 2, \ldots, c-1)$ and the first F has not yet occurred. Let ϕ_{0j} denote the conditional p.g.f. of $\boldsymbol{Z_S}$ from this time. For every nonnegative integer i, suppose the i-th F has just occurred. Then, let H_i denote the conditional p.g.f. of $\boldsymbol{Z_S}$ from this time. Suppose that the i-th F has already occurred, the $(i+1)$-th F has not yet occurred, and we have currently S-run of length j $(j = 1, 2, \ldots, c-1)$. Then, let ϕ_{ij} denote the conditional p.g.f. of $\boldsymbol{Z_S}$ from this time.

By considering the condition of one-step ahead from every condition, we have the following system of equations:

$$G_S = p_{00} v_{00} \phi_{01} + q_{00} w_{00} H_1 \tag{2.20}$$

$$\begin{cases} \phi_{01} = p_{01} v_{01} \phi_{02} + q_{01} w_{01} H_1 \\ \phi_{02} = p_{01} v_{01} \phi_{03} + q_{01} w_{01} H_1 \\ \cdots \\ \phi_{0,c-1} = p_{01} v_{01} \cdot 1 + q_{01} w_{01} H_1 \end{cases} \tag{2.21}$$

for every positive integer i,

$$H_i = p_{i0}v_{i0}\phi_{i1} + q_{i0}w_{i0}H_{i+1} \tag{2.22}$$

$$\begin{cases} \phi_{i1} = p_{i1}v_{i1}\phi_{i2} + q_{i1}w_{i1}H_{i+1} \\ \phi_{i2} = p_{i1}v_{i1}\phi_{i3} + q_{i1}w_{i1}H_{i+1} \\ \cdots \\ \phi_{i,c-1} = p_{i1}v_{i1}\cdot 1 + q_{i1}w_{i1}H_{i+1} \end{cases} \tag{2.23}$$

Similarly as in the case of the double corrective action model, we have that the p.g.f. of $\boldsymbol{Z_S}$ is given by

$$G_S = V_0 + \sum_{n=1}^{\infty}\left(\prod_{i=0}^{n-1} U_i\right) V_n.$$

3. Some characteristics

We can derive some properties from the joint p.g.f. G obtained in the previous section.

3.1 *Markov dependence model with no corrective action*

We set in (2.6),

$$\begin{array}{llll} p_{00} = p_0, & q_{00} = q_0, & p_{01} = p_1, & q_{01} = q_1, \\ p_{10} = p_2, & q_{10} = q_2, & p_{11} = p_1, & q_{11} = q_1, \\ p_{12} = p_2, & q_{12} = q_2, & v_0 = t_2, & w_0 = u_2, \\ v_1 = t_1, & w_1 = u_1, & v_2 = t_2, & w_2 = u_2. \end{array}$$

Then, we deduce the joint p.g.f. as

$$\Pi = \frac{(p_1t_1)^{c-1}p_2t_2\{q_0u_0(1-p_1t_1) + p_0q_1t_0u_1(1-(p_1t_1)^{c-1})\}}{(1-p_1t_1)(1-q_2u_2) - q_1p_2u_1t_2 + q_1p_2(p_1t_1)^{c-1}u_1t_2} + p_0t_0(p_1t_1)^{c-1}.$$

3.2 *Number of successes observed until acceptance*

We denote by $\boldsymbol{s}$ the number of successes observed until acceptance. By setting

$$t_0 = t_1 = v_0 = v_1 = v_2 = t$$

and

$$u_0 = u_1 = w_0 = w_1 = w_2 = 1,$$

we obtain from (2.6) the p.g.f. of the distribution of the number of successes observed until acceptance to be

$$\text{(3.1)} \quad \Pi_s(t) = p_{00}p_{01}{}^{c-1}t^c + \frac{p_{11}{}^{c-1}t^c(1-p_{11}t)\{q_{00}-(p_{01}-p_{00})t - p_{00}p_{01}{}^{c-1}q_{01}t^c\}}{(1-p_{01}t)\{1-t+p_{11}{}^{c-1}q_{11}t^c\}}.$$

From (3.1), we obtain, for example, the expected number of successes until acceptance to be

$$\text{(3.2)} \qquad E(\boldsymbol{s}) = \Pi_s'(1) = \frac{p_{00}}{q_{01}}(1-p_{01}^c) + \frac{1}{p_{11}^{c-1}q_{11}}(1-p_{00}p_{01}^{c-1})(1-p_{11}^c).$$

3.3 *Number of failures observed until acceptance*

We denote by $\boldsymbol{f}$ the number of failures observed until acceptance. By setting

$$t_0 = t_1 = v_0 = v_1 = v_2 = 1$$

and

$$u_0 = u_1 = w_0 = w_1 = w_2 = u,$$

we obtain from (2.6) the p.g.f. of the distribution of the number of failures observed until acceptance to be

$$\text{(3.3)} \qquad \Pi_f(u) = p_{00}p_{01}^{c-1} + p_{11}^{c-1}(1-p_{00}p_{01}^{c-1}) \cdot \frac{u\{p_{10}+(p_{12}-p_{10})u\}}{1-(1-p_{12}p_{11}^{c-1})u}.$$

From (3.3), we obtain, for example, the expected number of failures until acceptance to be

$$\text{(3.4)} \qquad E(\boldsymbol{f}) = \Pi_f'(1) = (1-p_{00}p_{01}^{c-1})\left\{1+\frac{1-p_{10}p_{11}^{c-1}}{p_{12}p_{11}^{c-1}}\right\}.$$

From (3.2) and (3.4), we simply get

$$\text{(3.5)} \quad E(\boldsymbol{s}+\boldsymbol{f}) = 1 + \frac{p_{00}(1-p_{01}^{c-1})}{q_{01}} + \frac{(1-p_{00}p_{01}^{c-1})}{p_{11}^{c-1}}\left\{\frac{1-p_{10}p_{11}^{c-1}}{p_{12}} + \frac{1-p_{11}^c}{q_{11}}\right\}.$$

3.4 *Number of trials required to terminate the experiment*

Let X be the number of trials required to terminate the experiment. By setting in (2.6)

$$t_0 = t_1 = v_0 = v_1 = v_2 = u_0 = u_1 = w_0 = w_1 = w_2 = t,$$

we obtain the p.g.f. of X to be

$$\text{(3.6)} \qquad \Pi_X(t) = p_{00}p_{01}^{c-1}t^c + \frac{A}{B},$$

where

$$A = p_{11}^{c-1} t^{c+1} (1 - p_{11} t)\{p_{10} + (p_{12} - p_{10})t\}\{q_{00} + (p_{00} - p_{01})t - p_{00} p_{01}^{c-1} q_{01} t^c\},$$

and

$$R = (1 - p_{01} t)\{1 - (p_{11} + q_{12})t + (p_{11} - p_{12})t^2 + q_{11} p_{12} p_{11}^{c-1} t^{c+1}\}.$$

From (3.6), we derive the expected value of X to be

$$E(X) = \Pi_X'(1) = 1 + \frac{p_{00}(1 - p_{01}^{c-1})}{q_{01}} + \frac{(1 - p_{00} p_{01}^{c-1})}{p_{11}^{c-1}} \left\{ \frac{1 - p_{10} p_{11}^{c-1}}{p_{12}} + \frac{1 - p_{11}^c}{q_{11}} \right\}.$$

This agrees with the expression derived in equation (3.5).

Let us write the denominater of the second term in (3.6) as

$$R = 1 - A_1 t + A_2 t^2 - A_3 t^3 + A_4 t^{c+1} - A_5 t^{c+2},$$

where

$$\begin{aligned} A_1 &= p_{11} + q_{12} + p_{01}, \\ A_2 &= p_{11} - p_{12} + p_{01}(p_{11} + q_{12}) = p_{01} + (1 + p_{01})(p_{11} - p_{12}), \\ A_3 &= p_{01}(p_{11} - p_{12}), \\ A_4 &= q_{11} p_{12} p_{11}^{c-1} \end{aligned}$$

and

$$A_5 = p_{01} q_{11} p_{12} p_{11}^{c-1}.$$

Similarly, let us write the numerater of the second term in (3.6) as

$$A = B_1 t^{c+1} + B_2 t^{c+2} + B_3 t^{c+3} - B_4 t^{c+4} - B_5 t^{2c+1} - B_6 t^{2c+2} + B_7 t^{2c+3},$$

where

$$\begin{aligned} B_1 &= p_{11}^{c-1} p_{10} q_{00}, \\ B_2 &= p_{11}^{c-1}\{p_{10}(p_{00} - p_{01}) + q_{00}(p_{12} - p_{10} - p_{10} p_{11})\}, \\ B_3 &= p_{11}^{c-1}\{(p_{00} - p_{01})(p_{12} - p_{10} - p_{10} p_{11}) - q_{00} p_{11}(p_{12} - p_{10})\}, \\ B_4 &= p_{11}^{c}(p_{00} - p_{01})(p_{12} - p_{10}), \\ B_5 &= p_{11}^{c-1} p_{00} p_{01}^{c-1} q_{01} p_{10}, \\ B_6 &= p_{11}^{c-1} p_{00} p_{01}^{c-1} q_{01}(p_{12} - p_{10} - p_{10} p_{11}), \end{aligned}$$

and

$$B_7 = p_{11}^{c} p_{00} p_{01}^{c-1} q_{01}(p_{12} - p_{10}).$$

Let

$$\Pi_X^*(t) = \Pi_X(t) - p_{00} p_{01}^{c-1} t^c.$$

Then, equation (3.6) gives

$$\begin{aligned}\Pi_X^*(t)\{1 - A_1 t + A_2 t^2 - A_3 t^3 + A_4 t^{c+1} - A_5 t^{c+2}\} \\ = B_1 t^{c+1} + B_2 t^{c+2} + B_3 t^{c+3} - B_4 t^{c+4} \\ - B_5 t^{2c+1} - B_6 t^{2c+2} + B_7 t^{2c+3}.\end{aligned} \tag{3.7}$$

Comparing the coefficients of t^x on both sides of (3.7), we get the following relations for the probabilities:

$$\begin{aligned}
&P(c) = p_{00} p_{01}^{c-1}, \\
&P(c+1) = B_1, \\
&P(c+2) - A_1 P(c+1) = B_2, \\
&P(c+3) - A_1 P(c+2) + A_2 P(c+1) = B_3, \\
&P(c+4) - A_1 P(c+3) + A_2 P(c+2) - A_3 P(c+1) = -B_4, \\
&P(x) - A_1 P(x-1) + A_2 P(x-2) - A_3 P(x-3) = 0 \\
&\qquad\qquad \text{for } x = c+5, c+6, \ldots, 2c, \\
&P(2c+1) - A_1 P(2c) + A_2 P(2c-1) - A_3 P(2c-2) = -B_5, \\
&P(2c+2) - A_1 P(2c+1) + A_2 P(2c) \\
&\qquad - A_3 P(2c-1) + A_4 P(c+1) = -B_6, \\
&P(2c+3) - A_1 P(2c+2) + A_2 P(2c+1) - A_3 P(2c) \\
&\qquad + A_4 P(c+2) - A_5 P(c+1) = B_7,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&P(x) - A_1 P(x-1) + A_2 P(x-2) - A_3 P(x-3) \\ &\qquad + A_4 P(x-c-1) - A_5 P(x-c-2) = 0 \\ &\qquad\qquad \text{for } x \geq 2c+4.\end{aligned}$$

Relations in (3.8) can be used to compute all necessary probabilities for X in a simple recursive manner.

Let us rewrite equation (3.7) as

$$\begin{aligned}\Pi_X(t)\{1 - A_1 t + A_2 t^2 - A_3 t^3 + A_4 t^{c+1} - A_5 t^{c+2}\} \\ = p_{00} p_{01}^{c-1} t^c \{1 - A_1 t + A_2 t^2 - A_3 t^3 + A_4 t^{c+1} - A_5 t^{c+2}\} \\ + B_1 t^{c+1} + B_2 t^{c+2} + B_3 t^{c+3} - B_4 t^{c+4} \\ - B_5 t^{2c+1} - B_6 t^{2c+2} + B_7 t^{2c+3} \\ = D_0 t^c + D_1 t^{c+1} + D_2 t^{c+2} + D_3 t^{c+3} + D_4 t^{c+4} \\ + D_5 t^{2c+1} + D_6 t^{2c+2} + D_7 t^{2c+3},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}&D_0 = p_{00} p_{01}^{c-1}, \\ &D_1 = B_1 - p_{00} p_{01}^{c-1} A_1 = B_1 - D_0 A_1,\end{aligned}$$

$$\begin{aligned}
D_2 &= B_2 + p_{00}p_{01}^{c-1}A_2 = B_2 + D_0A_2,\\
D_3 &= B_3 - p_{00}p_{01}^{c-1}A_3 = B_3 - D_0A_3,\\
D_4 &= -B_4,\\
D_5 &= -B_5 + p_{00}p_{01}^{c-1}A_4 = -B_5 + D_0A_4,\\
D_6 &= -B_6 - p_{00}p_{01}^{c-1}A_5 = -B_6 - D_0A_5
\end{aligned}$$

and

$$D_7 = B_7.$$

From (3.9), we derive a recurrence relation for the raw moments of X as

$$\begin{aligned}
E(X^k) &- A_1E(X+1)^k + A_2E(X+2)^k - A_3E(X+3)^k\\
&+ A_4E(X+c+1)^k - A_5E(X+c+2)^k\\
&= D_0c^k + D_1(c+1)^k + D_2(c+2)^k + D_3(c+3)^k + D_4(c+4)^k\\
&\quad + D_5(2c+1)^k + D_6(2c+2)^k + D_7(2c+3)^k.
\end{aligned}$$

Noting that

$$1 - A_1 + A_2 - A_3 + A_4 - A_5 = q_{01}q_{11}p_{12}p_{11}^{c-1},$$

we can write the recurrence relation for the raw moments of X as

$$\begin{aligned}
E(X^k) = {} & \frac{1}{q_{01}q_{11}p_{12}p_{11}^{c-1}}\\
& \cdot \left\{\sum_{i=0}^{k-1} E(X^i)\binom{k}{i}\right.\\
& \quad \cdot \{A_1 - A_2 2^{k-i} + A_3 3^{k-i} - A_4(c+1)^{k-i} + A_5(c+2)^{k-i}\}\\
& \quad + D_0c^k + D_1(c+1)^k + D_2(c+2)^k + D_3(c+3)^k\\
& \quad \left. + D_4(c+4)^k + D_5(2c+1)^k + D_6(2c+2)^k + D_7(2c+3)^k\right\},\\
& \qquad k = 1, 2, \dots.
\end{aligned}$$

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