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VON MISES ω^2-STATISTIC AND THE GENERALIZED BAYESIAN BOOTSTRAPS*

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Abstract. The uniform distance between the distribution functions of the von Mises ω^2-statistic for sampling from a continuous distribution and of the "generalized Bayesian ω^2-statistic" for sampling from the uniform distribution on a finite number of points is estimated. Application to the generalized Bayesian bootstraps is discussed.

Key words and phrases: von Mises statistic, Bayesian bootstrap, central limit theorem in Hilbert space, rate of approximation.

1. Introduction and notation

In a recent paper Lo and Sazonov (1995) we estimated the uniform distance between the distribution functions of the von Mises ω^2-statistics ω_n^2, constructed for sampling from a continuous distribution, and $\omega_{n,m}^2$, constructed for sampling from a uniform distribution on a set consisting of m different points. The relation of this problem to the question of closeness of the von Mises ω^2-statistic (for sampling from a continuous distribution) and its bootstrap version was also indicated and an estimate of this closeness was given.

In the present paper we consider a similar problem for the case when $\omega_{n,m}^2$ is replaced by a "generalized Bayesian ω^2-statistic" $(\omega_{u,n}^2)^B$ (this amounts to the replacement of the empirical distribution function in the construction of $\omega_{n,m}^2$ with $m = n$ by the distribution function of a "randomly uniform" distribution on a set of n different points). This problem is connected with the precision of the generalized Bayesian bootstrap and an estimate of this precision is deduced below.

The approach we are using now is similar to that employed in Lo and Sazonov (1995) and is based on Berry-Esseen type estimates for Hilbert space valued random variables. The application of this technique to the present problem, and in

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particular to the generalized Bayesian bootstrap, however, is complicated by the appearance of an additional factor which accounts for a lower order estimate as compared to what was obtained in Lo and Sazonov (1995).

Another way to tackle the problem of asymptotic behavior of bootstrapped statistics is to use the strong approximation technique (cf. Lo (1987, 1991)). In trying this approach however one meets some essential technical difficulties. But, even if one overcomes this difficulties, the general known now results on strong approximation in application to the generalized Bayesian bootstrap will at best lead to a marginal improvement (just by a logarithmic factor) of the estimate obtained in the present paper. It is appropriate to mention, though, that the strong approximation approach may cover the general case of not necessarily continuous F.

Let $X, X_1, X_2, \dots$ be a sequence of independent real random variables defined on a probability space $(S, \mathcal{B}, P)$, which a common distribution function $F(x) = P(X \le x)$. Denote $F_n(x) = n^{-1}\sum_{j=1}^n 1_{\{X_j \le x\}}$ the empirical distribution functions corresponding to the sequence $X_1, X_2, \dots$ (as usual 1_A stands for the indicator function of a set A, i.e. $1_A(s) = 1$ if $s \in A$ and $1_A(s) = 0$ if $s \notin A$). The von Mises ω^2-statistic based on $X_1, \dots, X_n$ is

$$\omega_n^2 = n \int_{-\infty}^{\infty} (F_n(x) - F(x))^2 dF(x).$$

Next let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent identically distributed (i.i.d.) nonnegative random variables satisfying the conditions indicated below. The variables $\xi, \xi_1, \xi_2, \dots$ may be thought of as defined on an another probability space, $(S', \mathcal{B}', P')$ say, since otherwise one is led to consider conditional probabilities which are essentially irrelevant here. Regarding $\xi, \xi_1, \xi_2, \dots$ we will assume that $P'(\xi = 0) = 0$,

$$E'\xi = 1, \quad \text{and} \quad E'e^{t\xi} < \infty \quad \text{for some } t > 0. \tag{1.1}$$

Note that condition (1.1) is equivalent to the condition: there exist $g > 0$, $T > 0$ such that

$$E'e^{t(\xi-1)} \le e^{-gt^2/2} \quad \text{for all } |t| \le T \tag{1.1'}$$

(see Petrov (1975), Chapter 3, Lemma 5). The generalized Bayesian bootstrap version of ω_n^2 is defined as (cf. Lo (1991))

$$(\omega_n^2)^B = n\sigma^{-2} \int_{-\infty}^{\infty} (F_n^B(x) - F_n(x))^2 dF_n(x),$$

where $\sigma^2 = E'(\xi - 1)^2$, $F_n^B(x) = \sum_{j=1}^n \Delta_j 1_{\{X_j \le x\}}$, and

$$\Delta_j = \xi_j \left(\sum_{i=1}^n \xi_i\right)^{-1}, \qquad j = 1, \dots, n.$$

Obviously $(\omega_n^2)^B$ is a measurable function of $X_1, \ldots, X_n, \xi_1, \ldots, \xi_n$.

As was pointed out in Lo and Sazonov (1995) when the distribution function F is continuous there is no need to bootstrap ω_n^2. For the same reasons the (generalized) Bayesian bootstrap of ω_n^2 for continuous F has no applied value. Nevertheless it is an interesting theoretical problem to estimate the closeness of the distributions of ω_n^2 and $(\omega_n^2)^B$ even when F is continuous since it gives an insight into the precision of the bootstrap technique. Such an estimate is obtained below as a corollary of a slightly more general result.

Let U be the distribution function of the uniform distribution on a set of n different points $x_1, \ldots, x_n$. Denote

$$U^B(x) = \sum_{j=1}^{n} \Delta_j 1_{\{x_j \le x\}}$$

and

$$(\omega_{u,n}^2)^B = n\sigma^{-2} \int_{-\infty}^{\infty} (U^B(x) - U(x))^2 dU(x).$$

In what follows $c, c(\ldots)$, with or without indices, denote absolute constants and constants depending only on the parameters indicated in the parenthesis respectively. The same symbol may stand for different constants. We use the symbol $|\cdot|$ both for the norm in a Hilbert space and for the absolute value of a real or complex number. The meaning of the symbol is always clear from the context.

2. Theorems and lemmas

THEOREM 2.1. *If F is continuous then for all $n_1, n = 1, 2, \ldots$*

$$\delta = \sup_{y \in R} |P(\omega_{n_1}^2 \le y) - P((\omega_{u,n}^2)^B \le y)| \le cn_1^{-1} + c(P_\xi)(\log n)^{3/2} n^{-1/2}, \tag{2.1}$$

where P_ξ is the distribution of ξ (regarding $c(P_\xi)$ see also Remark 1 below).

To prove the theorem we will use a Berry-Esseen type estimate for sums of independent (but possibly differently distributed) random variables with values in a separable Hilbert space H. Such an estimate was obtained in Ulyanov (1987), and we will state it here as a lemma. If V is a nonnegative symmetric operator with finite trace in H having eigenvalues $v_1^2, v_2^2, \ldots : 0 < v_1^2 \ge v_2^2 \ge \cdots$ and if $\beta > 0$, $r = 1, 2, \ldots$, then the inclusion $V \in G(\beta, r)$ will be understood as $v_r^2 \ge \beta v^2$, where $v^2 = \sum_{s=1}^{\infty} v_s^2$ is the trace of V.

LEMMA 2.1. *Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables with values in H, $S_n = \sum_{j=1}^{n} X_j$, V_n be the covariance operator of S_n, N_n be an H-valued $(0, V_n)$ Gaussian random variable. Then for any $\beta, \epsilon : \beta > 0$, $0 < \epsilon \le 1$, there exist constants $q = q(\epsilon)$, $c(\beta, \epsilon)$ such that if $V \in G(\beta, q)$ then*

$$\sup_{y \in R} |P(|S_n| \le y) - P(|N_n| \le y)| \le c(\beta, \epsilon)(\Lambda_2 + L_3^{2-\epsilon} + L_4),$$

where

$$\Lambda_2 = v_n^{-2} \sum_{l=1}^{n} E|X_l 1_{\{|X_l| > v_n\}}|^2, \qquad L_k = v_n^{-k} \sum_{l=1}^{n} E|X_l 1_{\{|X_l| \le v_n\}}|^k,$$
$$k = 3, 4, \quad v_n = \operatorname{tr} V_n.$$

In the proof of the theorem we will also need

LEMMA 2.2. *Let X, Y, Z be real random variables and $0 \le \epsilon \le 1$. Then*

$$\begin{aligned}\sup_{y \ge 0} |P(X \le yY^2) - P(X \le y)| \le\ & 2 \sup_{y \ge 0} |P(X \le y) - P(Z \le y)| \\ & + \sup_{y \ge 0} P(y(1-\epsilon)^2 \le Z \le y(1+\epsilon)^2) \\ & + P(|Y-1| > \epsilon).\end{aligned}$$

PROOF. Take an $y \ge 0$ and denote

$$\delta_y = P(X \le yY^2) - P(X \le y).$$

Since $|Y-1| \le \epsilon$ implies $Y^2 \le (1+\epsilon)^2$ and $Y^2 \ge (1-\epsilon)^2$, we have

$$\delta_y \le P(X \le y(1+\epsilon)^2) - P(X \le y) + P(|Y-1| > \epsilon)$$

and

$$\delta_y \ge P(X \le y(1-\epsilon)^2) - P(X \le y).$$

Thus if $\delta_y > 0$ then

$$\begin{aligned}\delta_y \le\ & |P(X \le y(1+\epsilon)^2) - P(Z \le y(1+\epsilon)^2)| \\ & + |P(X \le y) - P(Z \le y)| \\ & + P(y < Z \le y(1+\epsilon)^2) \\ & + P(|Y-1| > \epsilon),\end{aligned} \tag{2.2}$$

and if $\delta_y \le 0$, then

$$\begin{aligned}|\delta_y| \le\ & |P(X \le y(1-\epsilon)^2) - P(Z \le y(1-\epsilon)^2)| \\ & + |P(X \le y) - P(Z \le y)| \\ & + P(y(1-\epsilon)^2 < Z \le y).\end{aligned} \tag{2.3}$$

The lemma follows from (2.2) and (2.3).

3. Proof of the theorem

In the proof of the theorem all the probabilities and expectations will be denoted by P and E respectively (regardless of the probability space considered).

Let $G(y)$, $G_n(y)$ be the distribution functions defined below in (3.2) and (3.10) respectively. Obviously we can write (as a matter of fact, we can write it for any $G(y)$, $G_n(y)$)

$$\delta \leq \delta_{n_1} + \delta_{n1} + \delta_{n2}, \tag{3.1}$$

where

$$\delta_{n_1} = \sup_{y \in R} |P(\omega_{n_1}^2 \leq y) - G(y)|, \qquad \delta_{n1} = \sup_{y \in R} |P((\omega_{u,n}^2)^B \leq y) - G_n(y)|$$
$$\delta_{n2} = \sup_{y \in R} |G(y) - G_n(y)|.$$

The rate of decay of δ_{n_1} as $n_1 \to \infty$ is well known (see e.g. Götze (1979), Korolyuk and Borovskikh (1984)). Namely let $G(y)$ be the distribution function corresponding to the characteristic function

$$g(t) = \prod_{n=1}^{\infty} (1 - 2it(\pi j)^{-2})^{-1/2},$$

or, what is the same,

$$G(y) = P\left(\sum_{n-1}^{\infty} \zeta_j^2 \leq y\right), \tag{3.2}$$

where $\zeta_1, \zeta_2, \ldots$ are independent $(0, (\pi j)^{-2})$ normal real random variables. Then there exists c such that

$$\delta_{n_1} \leq c_1^{-1}, \qquad n_1 = 1, 2, \ldots. \tag{3.3}$$

To estimate δ_{n1} we observe that denoting $\eta_k = \sum_{l=1}^{k} \xi_l$ we have

$$\begin{aligned} (\omega_{u,n}^2)^B &= \sigma^{-2} \sum_{k=1}^{n} (U^B(x_k) - U(x_k))^2 \\ &= \sigma^{-2} \sum_{k=1}^{n} (\eta_k/\eta_n - k/n)^2. \end{aligned}$$

Hence

$$P((\omega_{u,n}^2)^B \leq y) = P(\Sigma_n \leq y\eta_n^2), \tag{3.4}$$

where

$$\Sigma_n = \sigma^{-2}\sum_{k=1}^{n}(((m-k)/n)\eta_k - (k/n)(\eta_n - \eta_k))^2.$$

Define now $L_2[0,1]$-valued random variables Y_{nl}, $l = 1,\dots,n$ as follows

$$Y_{nl}(t) = \begin{cases} -(k/n)\sigma^{-1}(\xi_l - 1), & (k-1)/n < t \le k/n, \quad 1 \le k \le l-1, \\ ((n-k)/n)\sigma^{-1}(\xi_l - 1), & (k-1)/n < t \le k/n, \quad l \le k \le n. \end{cases}$$

Obviously Y_{nl}, $l = 1,\dots,n$, are independent,

$$|Y_{nl}| \le \sigma^{-1}|\xi_l - 1|, \tag{3.5}$$

and if $f \in L_2[0,1]$, then

$$E(f, Y_{nl}) = c(f,n)E(\xi_l - 1) = 0. \tag{3.6}$$

Thus (see (1.1)) $EY_{nl} = 0$ and Y_{nl} has moments of all orders. Moreover, if we define $Y_n = n^{-1/2}\sum_{l=1}^{n} Y_{nl}$, then

$$\begin{aligned}\Sigma_n &= \sigma^{-2}\sum_{k=1}^{n}\left(((n-k)/n)\sum_{j=1}^{k}(\xi_j - 1) - (k/n)\sum_{j=k+1}^{n}(\xi_j - 1)\right)^2 \\ &= n\sum_{k=1}^{n}\left(\sum_{l=1}^{n} Y_{nl}(k/n)\right)^2 /n \\ &= n\int_0^1\left(\sum_{l=1}^{n} Y_{nl}(t)\right)^2 dt \\ &= n^2|Y_n|^2.\end{aligned} \tag{3.7}$$

Our next step is to estimate

$$\Delta_{nl} = \sup_{y \in R}|P(|Y_n|^2 \le y) - G_n(y)|.$$

From the definition of Y_{nl} it follows that Y_n belongs to the subspace of $L_2[0,1]$ generated by its orthonormal elements

$$b_{nk}(t) = \begin{cases} n^{1/2}, & \text{if} \quad (k-1)/n < t \le k/n, \\ 0, & \text{otherwise}, \end{cases}$$

$k = 1,\dots,n-1$. For any $r, s : 1 \le r \le s \le n-1$ we have $E(Y_n, b_{nr}) = 0$ (see (3.6)) and

$$E'(Y_n, b_{nr})(Y_n, b_{ns}) = n^{-2}E\left(\sum_{l=1}^{n} Y_{nl}(r/n)\right)\left(\sum_{l=1}^{n} Y_{nl}(s/n)\right)$$

$$
\begin{aligned}
&= \left(\sum_{l=1}^{r} \frac{n-r}{n\sigma}(\xi_l - 1) - \sum_{l=r+1}^{n} \frac{r}{n\sigma}(\xi_l - 1)\right) \\
&\quad \cdot \left(\sum_{l=1}^{s} \frac{n-s}{n\sigma}(\xi_l - 1) - \sum_{l=s+1}^{n} \frac{s}{n\sigma}(\xi_l - 1)\right) \\
&= n^{-2}\left(r\frac{n-r}{n}\frac{n-s}{n} + (n-s)\frac{r}{n}\frac{s}{n} - (s-r)\frac{r}{n}\frac{n-s}{n}\right) \\
&\quad - n^{-2} r(1 - s/n) \\
&= v_{rs}, \qquad \text{say.}
\end{aligned}
$$

The $(n-1)\times(n-1)$ symmetric matrix W with elements v_{rs} has the trace

$$
w_n^2 = n^{-2}\sum_{r=1}^{n-1} r(1 - r/n) = (n^2-1)/6n^2. \tag{3.8}
$$

The eigenvalues of W are

$$
w_{nr} = (4n^2 \sin^2(\pi r/2n))^{-1}, \qquad r = 1, \dots, n-1 \tag{3.9}
$$

(see Lo and Sazonov (1995)). It follows that the covariance operator V_n of Y_n has the trace w_n^2 and the eigenvalues $w_{n1}, \dots, w_{n(n-1)}, 0, 0, \dots$ with w_{nr} defined by (3.9). Denote now N_n a $L_2[0,1]$-valued $(0, V_n)$ Gaussian random variable and let

$$
G_n(y) = P(|N_n|^2 \le y). \tag{3.10}
$$

Applying Lemma 2.1 with $Z_l = n^{-1/2} Y_{nl}$, any $\epsilon : 0 < \epsilon \le 1$, $n \ge q + 1$ (here $q = q(\epsilon)$) and $\beta = \beta(\epsilon) = 6/(\pi q)^2$, we obtain, since for any $r : 1 \le r \le q$

$$
w_{nr} = \frac{1}{4n^2 \sin^2(\pi r/2n)} \ge \frac{1}{(\pi q)^2} = \frac{\beta}{6} \ge \beta\frac{n^2-1}{6n^2} = \beta w_n^2, \tag{3.11}
$$

that

$$
\begin{aligned}
\Delta_{n1} &= \sup_{y \in R} |P(|Y_n|^2 \le y) - G_n(y)| \\
&\le c_1(\epsilon)(\Lambda_2 + L_3^{2-\epsilon} + L_4) \\
&\le c_1(\epsilon)(((w_n\sigma)^{-3} E|\xi - 1|^3)^{2-\epsilon} + (w_n\sigma)^{-4} E|\xi - 1|^4) n^{-1+\epsilon/2},
\end{aligned} \tag{3.12}
$$

where $c_1(\epsilon) = c(\beta(\epsilon), \epsilon)$; indeed by (3.5)

$$
\begin{aligned}
\Lambda_2 &= w_n^{-2} \sum_{l=1}^{n} E|n^{-1/2} Y_{nl} 1_{\{|n^{-1/2} Y_{nl}| > w_n\}}|^2 \\
&\le w_n^{-4} n^{-2} \sum_{l=1}^{n} E|Y_{nl}|^4 \\
&\le (w_n\sigma)^{-4} E|\xi - 1|^4 n^{-1}, \\
L_3 &= w_n^{-3} \sum_{l=1}^{n} E|n^{-1/2} Y_{nl} 1_{\{|n^{-1/2} Y_{nl}| \le w_n\}}|^3 \\
&\le (w_n\sigma)^{-3} E|\xi - 1|^3 n^{-1/2},
\end{aligned}
$$

and similarly

$$L_4 \le (w_n\sigma)^{-4}E|\xi - 1|^4 n^{-1}.$$

If $n \le q(\epsilon)$ then

$$\Delta_{nl} \le 1 \le q(\epsilon)n^{-1},$$

so that (3.12) is true for all $n = 1, 2, \ldots$. In particular when $\epsilon = 1$ we have

$$\Delta_{nl} \le c_1\sigma^{-4}E|\xi - 1|^4 n^{-1/2}, \qquad n = 1, 2, \ldots, \tag{3.12$'$}$$

since $w_n^2 = (n^2-1)/6n^2 > c$ if $n > 2$, and since by the moments inequalities

$$1 = (\sigma^{-2}E(\xi-1)^2)^{3/2} \le \sigma^{-3}E|\xi-1|^3 \le (\sigma^{-4}E(\xi-1)^4)^{3/4} \le \sigma^{-4}E(\xi-1)^4.$$

Now let us estimate

$$\begin{aligned}\Delta_{n2} &= \sup_{y\in R}|P((\omega_{u,n}^2)^B \le y) - P(|Y_n|^2 \le y)| \\ &= \sup_{y\in R}|P(|Y_n|^2 \le y\eta_n^2/n^2) - P(|Y_n|^2 \le y)|\end{aligned}$$

(see (3.4), (3.7)). By (3.10) and Lemma 2.2 with $X = |Y_n|^2$, $Y = \eta_n/n$, $Z = |N_n|^2$, $\epsilon = \epsilon_n$ (to be specified below), we have for all $n = 1, 2, \ldots$

$$\Delta_{n2} \le I_{n1} + I_{n2} + I_{n3}, \tag{3.13}$$

where

$$\begin{aligned}I_{n1} &= 2\Delta_{n1}, \\ I_{2n} &= \sup_{y\ge 0} P(y(1-\epsilon_n)^2 \le |N_n|^2 \le y(1+\epsilon_n)^2), \\ I_{3n} &= P(|\eta_n/n - 1| > \epsilon_n) \\ &= P\left(\left|\sum_{j=1}^n(\xi_j - 1)\right| > n\epsilon_n\right).\end{aligned}$$

Let g, T be the same as in (1.1$'$). We can assume that

$$0 \le (\log n)/n \le c(g,T) = \min((2-3^{1/2})/3g, gT^2/2), \tag{3.14}$$

since otherwise the theorem is obviously true (with $c(P_\xi) + (c(g,T))^{-1}$). Take now

$$\epsilon_n = ((2g\log n)/n)^{1/2}.$$

The density function of $|N_n|^2$ is not greater then $c(w_{n1}w_{n2})^{-1/2}$ (see e.g. Paulauskas and Rackauskas (1989), p. 65), which in virtue of (3.11) is bounded by an absolute constant. Hence if $y \le \log n$ then

$$\begin{aligned}P(y(1-\epsilon_n)^2 \le |N_n|^2 \le y(1+\epsilon_n)^2) &\le cy\epsilon_n \\ &\le cg^{1/2}(\log n)^{3/2}n^{-1/2}.\end{aligned} \tag{3.15}$$

If $y > \log n$, then since for any $z \geq 0$ and any Hilbert space valued Gaussian random variable N

$$P(|N| \geq z) \leq 2\exp\{-z^2/2E|N|^2\},$$

(see e.g. Sazonov *et al.* (1988)), since by (3.8)

$$E|N_n|^2 = \operatorname{tr} V_n = w_n^2 = (n^2-1)/6n^2 < 1/6,$$

and since by (3.14) and the definition of ϵ_n $3y(1-\epsilon_n)^2 \geq \log n$, we have for any $y \geq 0$

$$\begin{aligned}(3.16)\quad P(y(1-\epsilon_n)^2 \leq |N_n|^2 \leq y(1-\epsilon_n)^2) &\leq P(|N_n| \geq y^{1/2}(1-\epsilon_n))\\ &\leq 2\exp\{-y(1-\epsilon_n)^2/2E|N_n|^2\}\\ &\leq 2\exp\{-3y(1-\epsilon_n)^2\}\\ &\leq 2n^{-1}.\end{aligned}$$

From (3.15) and (3.16) it follows that

$$I_{2n} \leq cg^{1/2}(\log n)^{3/2}n^{-1/2}. \tag{3.17}$$

Next observe that our choice of ϵ_n together with (3.14) imply $\epsilon_n \leq gT$, and hence, by (1.1′) and the exponential inequality (see, e.g. Petrov (1975), Chapter 3, Theorem 16)

$$\begin{aligned}I_{3n} &\leq 2e^{-n\epsilon_n^2/2g}\\ &= 2n^{-1}.\end{aligned} \tag{3.18}$$

Combining now (3.13), (3.12), (3.17), (3.18) we obtain

$$\Delta_{n2} \leq c(E|\xi-1|^4 n^{-1/2} + g^{1/2}(\log n)^{3/2}n^{-1/2}), \tag{3.19}$$

and from (3.12′), (3.19) we deduce

$$\begin{aligned}\delta_{n1} &\leq \Delta_{n1} + \Delta_{n2}\\ &\leq c(E|\xi-1|^4 n^{-1/2} + g^{1/2}(\log n)^{3/2}n^{-1/2}).\end{aligned} \tag{3.20}$$

Finally it was shown in Lo and Sazonov (1995) that

$$\delta_{n2} \leq cn^{-1}, \qquad n = 1, 2, \ldots, \tag{3.21}$$

and the theorem follows now from (3.1), (3.3), (3.20), (3.21).

COROLLARY 3.1. *If F is continuous then with probability* 1

$$\delta_n(s) = \sup_{y\in R}|P(\omega_n^2 \leq y) - P'((\omega_n^2)^B \leq y)| \leq c(P_\xi)(\log n)^{3/2}n^{-1/2}, \qquad n = 1, 2, \ldots.$$

To deduce this corollary it is enough to observe that when F is continuous all $X_1(s), X_2(x), \ldots$ are different with probability 1 and apply the theorem with $n_1 = n$.

Remark 1. It follows from the proof of the theorem that when (3.14) is satisfied one may take $c(P_\xi) = c(\sigma^{-4}E|\xi - 1|^4 + g^{1/2})$, and if it is violated we have

$$\delta \leq (c(g,T))^{-1}(\log n)n^{-1}.$$

Note also that g may be taken to be any number greater then σ^2 but the value of T for which (1.1′) is true depends on the choice of g (see the proof of Lemma 5, Chapter 3 in Petrov (1975)).

Remark 2. When the distribution function F is not continuous it may happen that $\delta_n(s) \geq cn^{-1/2}$ for all large n with probability 1, where c is an absolute constant. Indeed in Lo and Sazonov (1995), Remark, it was shown that if

$$F(x) = \begin{cases} 0, & x < 0, \\ 1/2, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases}$$

then the distribution function of ω_n^2 has jumps $\geq (1/2)(\pi n)^{-1/2}$ for all large n and as $n \to \infty$ converges to a continuous distribution function. At the same time if we take ξ_j with the standard exponential distribution, then, as is easy to check, with P probability $1 - 2^{-n+1}$ $(\omega_n^2)^B$ has a continuous distribution function and it is degenerate at 0 with probability 2^{-n+1}. This implies that for this choice of F and ξ_j $\delta_n(s) \geq (1/4)(\pi n)^{-1/2}$ for all large n with P probability 1.

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