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CHARACTERIZATIONS BASED ON CONDITIONAL EXPECTATIONS OF THE DOUBLED TRUNCATED DISTRIBUTION

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Abstract. The expression of the continuous distribution function $F(x)$ is obtained whenever $m(x,y) = E(X \mid x \leq X \leq y)$ is known. Moreover, we obtain the necessary and sufficient conditions so that any function $m : \mathbb{R}^2 \to \mathbb{R}$ is the conditional expectation $E(X \mid x \leq X \leq y)$ of a random variable X with continuous distribution function. Furthermore, we relate $m(x,y)$ to order statistics.

Key words and phrases: Characterizations, doubled truncated distribution, conditional expectations, mean residual life, order statistics.

1. Introduction

In recent years considerable attention has been paid to the problem of characterizing the probability distribution function (pdf) of a random variable (r.v.) based on conditional expectations in general, and in particular, on its mean residual life function (m.r.l.).

The characterization and representation theorems based on the m.r.l. $e(x) = E(X - x \mid X \geq x)$ or the function $E(X \mid X \geq x)$ can be seen in Cox (1962), Meilijson (1972), Swartz (1973), Laurent (1974), Vartak (1974), Gupta (1975), Galambos and Kotz (1978), Gupta (1979), Hall and Wellner (1981) and Kupka and Loo (1989).

Hamdan (1972), Talwalker (1977), Shanbhag and Bhaskara Rao (1975), Gupta (1975) and Ouyang (1983, 1987) give a representation for absolutely continuous pdf using $E(h(X) \mid X \geq x)$ under certain conditions for function $h(x)$. Kotz and Shanbhag (1980) study the above problem for r.v. in general. Kotlarski (1972) shows that there exists a one-to-one map between functions $\phi(x) = E(h(X) \mid X \geq x)$ and continuous pdf, while Zoroa *et al.* (1990) characterize the set of functions $\phi(x)$ which are conditional expectations of continuous pdf.

The characterization problems for continuous pdf using the conditional expectations $E(X \mid X \leq x)$ can be read in Zoroa and Ruiz (1981, 1982). The results for

$E(h(X) \mid X \leq x)$ are analogous. Using equality $E(X \mid X \leq x) = -E(X^* \mid X^* \geq x^*)$, where $X^* = -X$ and $x^* = -x$, we can relate the results for left truncated variables with the results for right truncated variables and vice versa.

In the present paper, if $\mathcal{F}$ denotes the set of real continuous pdf, for each $F \in \mathcal{F}$ we consider the doubly truncated mean function $m(x,y)$ defined by

$$m(x,y) = E(X \mid x \leq X \leq y) = \frac{1}{F(y) - F(x)} \int_x^y t dF(t) \tag{1.1}$$

whose domain of definition is obviously the set $D = \{(x,y) \in \mathbb{R}^2$ such that $F(x) < F(y)\}$.

Note that $m(x,y)$ exists for all $F \in \mathcal{F}$ since the Riemann-Stieltjes integral of (1.1) is always convergent, while $e(x)$ or $\phi(x)$ cannot exist.

The expression (1.1) enables us to construct a map "ω" from $\mathcal{F}$ into the family of the real functions from $\mathbb{R}^2$ to $\mathbb{R}$, defined by $\omega(F) = m$. Thus, it is natural to phrase the following questions: (i) Is "ω" a one-to-one map? (ii) Let $m(x,y)$ be a given function in $\operatorname{Im}\omega$, what is the explicit expression of $F = \omega^{-1}(m)$? (iii) Which are the necessary and sufficient conditions in order for any real function "m" to be in $\mathcal{M} = \operatorname{Im}\omega$?

Kotlarski ((1972), Theorem 2) and Shanbhag and Bhaskara Rao ((1975), Corollary 2) get an affirmative answer to the proposed question (i) for $E(h(X) \mid x \leq X \leq y)$ under the conditions that $F(x)$ is strictly increasing and $h(x)$ is almost everywhere differentiable and strictly increasing in support (a,b).

Note that in this case (even $h(x)$ is only continuous), $E(h(X) \mid x \leq X \leq y)$ coincides with $m(x,y)$ considering $X^* = h(X)$.

The aim of the present paper is to give solutions to problems (ii) (see Section 3) and (iii) (see Section 4) for $m(x,y)$ and for continuous distribution functions, which allow us to characterize some usual distributions (see examples in Section 5).

2. Basic properties

It is easy to show that $m \in \mathcal{M}$ with domain D fulfills the following properties

Property 2.1. D is an open and not empty set such that if $(x,y) \in D$ then $x \leq y$ and it holds that $(x,t) \in D$ or $(t,y) \in D$ for all $t \in \mathbb{R}$.

Property 2.2. Function $m(x,y)$ is continuous in both variables.

Property 2.3. $x < m(x,y) < y$ for all $(x,y) \in D$.

Property 2.4. $m(x,y)$ is increasing in both variables. Moreover, $m(x,y)$ is constant in $[a,b]$ if, and only if, $(a,b) \notin D$.

Let us give some lemmas which will enable us to give other properties of function $m(x,y)$.

LEMMA 2.1. *Let $F \in \mathcal{F}$, $m = \omega(F)$ and $a, b, x \in \mathbb{R}$, with $a \leq b$ and $(x, a) \in D$. Then, Riemann-Stieltjes integrals dependent on the "x" parameter*

$$I(x) = \int_a^b \frac{d_2 m(x,z)}{z - m(x,z)} \quad \text{and} \quad J(x) = \int_a^b \frac{dF(z)}{F(z) - F(x)}$$

exist, are continuous in "a" and "b" and are equal.

LEMMA 2.2. *Let $F \in \mathcal{F}$, $m = \omega(F)$ and $a, b, y \in \mathbb{R}$, with $a \leq b$ and $(b, y) \in D$. Then, Riemann-Stieltjes integrals dependent on the "y" parameter*

$$I^*(y) = \int_a^b \frac{d_1 m(z,y)}{m(z,y) - z} \quad \text{and} \quad J^*(y) = \int_a^b \frac{dF(z)}{F(y) - F(z)}$$

exist, are continuous in "a" and "b" and are equal.

As a consequence of the previous two lemmas we obtain the following properties

Property 2.5. Let $F \in \mathcal{F}$ and $m = \omega(F)$. Then, Riemann-Stieltjes integrals

$$I_1(x,y) = \int_{-\infty}^{x} \frac{d_1 m(z,y)}{m(z,y) - z} \tag{2.1}$$

$$I_2(x,y) = \int_{y}^{\infty} \frac{d_2 m(x,z)}{z - m(x,z)} \tag{2.2}$$

are convergent for all $(x, y) \in D$.

Property 2.6. Let $F \in \mathcal{F}$, $m = \omega(F)$ and $(x, y) \in D$. Then, the function

$$E_x(y) = \frac{\exp\{I_1(x,y)\}}{\exp\{I_1(x,y)\} + \exp\{I_2(x,y)\} - 1} \tag{2.3}$$

does not depend on "x", and the function

$$E_y^*(x) = \frac{\exp\{I_1(x,y)\} - 1}{\exp\{I_1(x,y)\} + \exp\{I_2(x,y)\} - 1} \tag{2.4}$$

does not depend on "y", where $I_1(x, y)$ and $I_2(x, y)$ are given in (2.1) and (2.2).

Property 2.7. Let $F \in \mathcal{F}$ and $m = \omega(F)$. Then, the Riemann-Stieltjes integrals

$$\int_{\pi_1(y)} \frac{d_1 m(z,y)}{m(z,y) - z} \quad \text{and} \quad \int_{\pi_2(x)} \frac{d_2 m(x,z)}{z - m(x,z)}$$

where $\pi_1(y) = \{z \in \mathbb{R}$ such that $(z, y) \in D\}$ and $\pi_2(x) = \{z \in \mathbb{R}$ such that $(x, z) \in D\}$ are divergent for all $(x, y) \in D$.

3. The inversion formula

In this section we obtain the explicit expression of $F(x)$ starting from function $m(x,y)$.

PROPOSITION 3.1. *Let $F \in \mathcal{F}$, $m = \omega(F)$ and $(x,y) \in D$. Then*

$$F(t) = \begin{cases} E_x(t) & \text{if } (x,t) \in D \\ E_y^*(t) & \text{if } (x,t) \notin D \end{cases} \tag{3.1}$$

where $E_x(t)$ and $E_y^*(t)$ are defined as in (2.3) *and* (2.4).

PROOF. Using $X_y = (X \mid X \le y)$, $X_x = (X \mid X \ge x)$ and relation

$$m(x,y) = E(X_y \mid X_y \ge x) = E(X_x \mid X_x \le y)$$

the proof results from inversion formula for m.r.l. □

Remark 3.1. If $F(x)$ is derivable (except, at most, in a finite number of points), it is easy to show that $m(x,y)$ is derivable in the same points and I_1 and I_2 are Riemann integrals.

Remark 3.2. Function $m(x,y)$ allow us to generalize the m.r.l. $e(x)$ to the double truncation through functions

$$e_1(x,y) = m(x,y) - x = E(X - x \mid x \le X \le y)$$
$$e_2(x,y) = y - m(x,y) = E(y - X \mid x \le X \le y).$$

The first can be interpreted as the expected additional life for a "unit" which was functioning at an age "x" and ceased to function before an age "y", and the second as the inactivity expected time for this "unit". Using Proposition 3.1, we have that both $e_1(x,y)$ and $e_2(x,y)$ univocally determine $F(x)$. In this context, $m(x,y)$ can be interpreted as total expected mean life for the said unit.

Remark 3.3. Let $X_{(1)} \le \cdots \le X_{(n)}$ be the order statistics from a sample of X r.v., it is easy to confirm that

$$E\left(\frac{1}{s-r-1}\sum_{i=r+1}^{s-1} h(X_{(i)}) \mid X_{(r)} = x, X_{(s)} = y\right) = E(h(X) \mid x \le X \le y)$$

for all $(x,y) \in D$, if $1 \le r < s \le n$, and hence we can characterize $F(x)$ using (3.1), which enables us to extend previous results given by Balasubramanian and Beg (1992) (for particular functions $h(x)$) and Gupta *et al.* (1993) (for $r = 1$ and $s = n$).

4. The main result

In this section, we characterize the set $\mathcal{M}$, formed by all the functions from $\mathbb{R}^2$ to $\mathbb{R}$ which are the conditional expectation $E(X \mid x \le X \le y)$ for some continuous distribution. First we will give a technical lemma.

LEMMA 4.1. *Let $\bar{m} : \bar{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$ any real function verifying Properties 2.1 to 2.6. Then, for $x, y, s, t \in \mathbb{R}$ with $s \le t \le y$ and $(x, s) \in D$, we have*

$$\int_s^t u d\bar{E}_x(u) = \bar{m}(x,t)[\bar{E}_x(t) - \bar{E}_y^*(x)] - \bar{m}(x,s)[\bar{E}_x(s) - \bar{E}_y^*(x)]$$

where $\bar{E}_x(y)$ and $\bar{E}_y^*(x)$ are defined as in (2.3) *and* (2.4).

PROOF. The proof of this lemma can be obtained using a similar way to the proof of Lemma 2.1 (Zoroa *et al.* (1990)) using the r.v. $X_x = (X \mid X \ge x)$. □

The following theorem gives the just conditions in order for any real function $\bar{m}(x, y)$ to be in $\mathcal{M} = \operatorname{Im} \omega$.

THEOREM 4.1. *Let $\bar{m} : \bar{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$ any real function. Then, $\bar{m} \in \mathcal{M}$ if, and only if, $\bar{m}$ verifies Properties* 2.1 *to* 2.7.

PROOF. In Section 2 we have seen that Properties 2.1 to 2.7 are necessary. To show that they are sufficient we define the function

$$\bar{F}(t) = \begin{cases} \bar{E}_x(t) & \text{if } (x,t) \in \bar{D} \\ \bar{E}_y^*(t) & \text{if } (x,t) \notin D \end{cases}$$

where $(x, y) \in \bar{D}$ and $\bar{E}_x(t)$ and $\bar{E}_y^*(t)$ are defined as in (2.3) and (2.4), using now any real function $\bar{m}$.

Let us to see that $\bar{F}(t)$ is a continuous pdf. First, definition of $\bar{F}(t)$ makes sense for all $t \in \mathbb{R}$ from Properties 2.1, 2.3, 2.4 and 2.5. Moreover, from Properties 2.3 and 2.4 we have that $\bar{F}(t) \in [0, 1]$.

For $(x, y) \in \bar{D}$, we have

$$\bar{E}_x(y) = \frac{1}{1+\alpha(y)} = \frac{\beta(x) + \exp\{-I_2(x,y)\}}{1+\beta(x)} \tag{4.1}$$

$$\bar{E}_y^*(x) = \frac{1}{1+\beta^{-1}(x)} = \frac{1 - \exp\{-I_1(x,y)\}}{1+\alpha(y)} \tag{4.2}$$

where

$$\alpha(y) = \frac{\exp\{I_2(x,y)\} - 1}{\exp\{I_1(x,y)\}}$$

only depends on "y" from (4.1) and Property 2.6, and where

$$\beta(x) = \frac{\exp\{I_1(x,y)\} - 1}{\exp\{I_2(x,y)\}}$$

only depends on "x" from (4.2) and Property 2.6.

If for $t \in \mathbb{R}$, $(x,t) \in \bar{D}$ and $(t,y) \in \bar{D}$, using (4.1) and (4.2), we have

$$\bar{E}_x(u) - \bar{E}^*_y(t) = \bar{E}_t(u) - \bar{E}^*_u(t) = \frac{\exp\{-I_2(t,u)\}}{1+\beta(t)}$$

for all $u > a_t$, where $\pi_2(t) = (a_t, \infty)$ from Property 2.1. As $(t, a_t) \notin \bar{D}$, $\bar{m}(x,y)$ will be constant in $[t, a_t]$ and, in consequence, $\bar{E}_x(t) = \bar{E}_x(a_t)$. Using Property 2.7, if we make $u \to a_t$, we have

$$\bar{E}_x(t) = \bar{E}^*_y(t) \tag{4.3}$$

and hence, definition of $\bar{F}(t)$ does not depend on the (x,y) point chosen in $\bar{D}$ and, hence, from Properties 2.2 and 2.3 $\bar{F}(t)$ is a continuous function.

On the other hand, from (4.1) and as $\lim_{t\to\infty} I_2(x,t) = 0$, we have $\lim_{t\to\infty} \cdot \bar{F}(t) = 1$. Likewise, we have $\lim_{t\to-\infty} \bar{F}(t) = 0$.

From (4.1), (4.2) and (4.3), we have the monotonicity of $\bar{F}(t)$. Moreover, it verifies

$$\bar{F}(s) < \bar{F}(t) \Leftrightarrow (s,t) \in \bar{D}. \tag{4.4}$$

To summarize, $\bar{F}(t)$ is a continuous pdf, that is to say $F \in \mathcal{F}$. Let us to see that $\omega(\bar{F}) = \bar{m}$. First, from (4.4) the definition domain of $\omega(\bar{F})$ is equal to $\bar{D}$. Let $(s,t) \in \bar{D}$, as $(s, a_s) \notin D$, by (4.4), $\bar{F}(s) = \bar{F}(a_s)$ and therefore

$$\int_s^t u d\bar{F}(u) = \lim_{z\to a_s^+} \int_z^t u d\bar{F}(u)$$

and using Lemma 4.1, we obtain

$$\int_s^t u d\bar{F}(u) = \bar{m}(s,t)[\bar{F}(t) - \bar{F}(s)] - \lim_{z\to a_s^+} \bar{m}(s,z)[\bar{F}(z) - \bar{F}(s)].$$

By Property 2.2 and the fact that both $\bar{F}(t)$ and $\bar{m}(s,t)$ are continuous, we have

$$\int_s^t u d\bar{F}(u) = \bar{m}(s,t)[\bar{F}(t) - \bar{F}(s)]$$

and, consequently $\omega(\bar{F}) = \bar{m}$. □

5. Examples

Remark 5.1. For distributions with mass concentrated in $[a,b]$ we will only give $m(x,y)$ in $[a,b]$, since if $x \le a$ then $m(x,y) = m(a,y)$ and if $y \ge b$ $m(x,y) = m(x,b)$.

Example 5.1. Given $a, b, p \in \mathbb{R}$ with $a < b$ and $0 < p < 1$, we consider the function

$$\bar{m}(x,y) = px + (1-p)y$$

defined for $a \leq x < y \leq b$ (and extended using Remark 5.1). It is easy to check that $\bar{m}$ satisfies Properties 2.1, 2.2, 2.3, 2.4, 2.5 and 2.7, but $\bar{m}(x,y)$ only satisfies the Property 2.6 if $p = 1/2$. In this case using (3.1) we obtain the uniform distribution (see Power distribution with $c = 1$ in Table 1). If $p \neq 1/2$ the functions determined from $\bar{m}(x,y)$ by (3.1), are not in $\mathcal{F}$ or $\omega(\bar{F}) \neq \bar{m}(x,y)$.

Example 5.2. Analogously, it is easy to confirm that the function

$$\bar{m}(x,y) = x^p y^{1-p}$$

defined for $0 \leq a \leq x < y \leq b$, where $p \in (0,1)$ and "b" can be infinite, verifies Properties 2.1, 2.2, 2.3, 2.4, 2.5 and 2.7, but only verifies Property 2.6 if $p = 1/2$. In this case, using (3.1) we obtain the following continuous distributions

$$F_1(t) = \frac{\sqrt{t} - \sqrt{a}}{\sqrt{b} - \sqrt{a}} \frac{\sqrt{b}}{\sqrt{t}} \quad \text{if} \quad t \in [a,b] \text{ and } b \in \mathbb{R}$$

$$F_2(t) = 1 - \frac{\sqrt{a}}{\sqrt{t}} \quad \text{if} \quad t \geq a \text{ and } b = \infty \text{ (Pareto distribution).}$$

For special forms of $m(x,y)$, the following results can be obtained using Theorem 4.1.

Remark 5.2. If $h(x)$ is a continuous and strictly monotonic function in (a,b), where "a" and "b" can be infinite, then using inversion formula (3.1) and the equivalence of $m(x,y)$ and $E(h(X) \mid x \leq X \leq y)$, it is easy to check that

$$F(x) = \left(\frac{h(x) - h(a)}{h(b) - h(a)} \right)^c \quad \text{for} \quad x \in (a,b)$$

if, and only if

$$E(h(X) \mid x \leq X \leq y) = h(a) + \frac{c}{c+1} \frac{(h(y) - h(a))^{c+1} - (h(x) - h(a))^{c+1}}{(h(y) - h(a))^c - (h(x) - h(a))^c} \quad \text{for} \quad a \leq x < y \leq b.$$

In particular, for $c = 1$ and from Remark 3.3 we obtain the results given by Balasubramanian and Beg (1992) and Gupta *et al.* (1993).

In Tables 1, 2 and 3 some examples of usual distributions can be seen. It is easy to show that the next functions $m(x,y)$ verify the properties of Theorem 4.1, and hence, they are in $\mathcal{M}$ and determine continuous distribution functions by (3.1). Note that for Cauchy and Pareto ($c < 1$) distributions the m.r.l. $e(x)$ does not exist.

Table 1. Examples for $m(x,y) = E(X \mid x \le X \le y)$.

Name	$f(x)$	Support	$m(x,y)$
Exponential	$\frac{1}{\sigma}\exp\left(-\frac{x-a}{\sigma}\right)$	$[a,\infty)$	$\dfrac{(x+\sigma)\exp\left(-\frac{x-a}{\sigma}\right)-(y+\sigma)\exp\left(-\frac{y-a}{\sigma}\right)}{\exp\left(-\frac{x-a}{\sigma}\right)-\exp\left(-\frac{y-a}{\sigma}\right)}$
Power	$\frac{c}{b-a}\left(\frac{x-a}{b-a}\right)^{c-1}$	$[a,b]$	$\frac{1}{c+1}\,\frac{(cy+a)(y-a)^c-(cx+a)(x-a)^c}{(y-a)^c-(x-a)^c}$
Laplace	$\frac{1}{2}\exp\{-\lvert x\rvert\}$	$\mathbb{R}$	$\begin{cases} \frac{y\cdot e^y-x\cdot e^x}{e^y-e^x}-1 & \text{if } x<y\le 0 \\ \frac{(y+1)e^{-y}+(x-1)e^x}{e^{-y}+e^x-2} & \text{if } x<0<y \\ \frac{y\cdot e^{-y}-x\cdot e^{-x}}{e^{-y}-e^{-x}}+1 & \text{if } 0\le x<y \end{cases}$
Cauchy	$\frac{\sqrt{k}}{\pi(k+(x-a)^2)}$	$\mathbb{R}$	$a+\frac{\sqrt{k}}{2}\,\frac{\log(k+(y-a)^2)-\log(k+(x-a)^2)}{\tan^{-1}\left(\frac{y-a}{\sqrt{k}}\right)-\tan^{-1}\left(\frac{x-a}{\sqrt{k}}\right)}$
Pareto	$c\cdot a^c x^{-c-1}$	$[a,\infty)$	$\begin{cases} \frac{c}{1-c}\,\frac{x^c y-y^c x}{y^c-x^c} & \text{if } c\ne 1 \\ \frac{x\cdot y}{y-x}\log\left(\frac{y}{x}\right) & \text{if } c=1 \end{cases}$

Table 2. Examples using Remark 5.2 with $c = 1$.

Name	$F(x)$	a	b	$h(x)$
Power	$b^{-p}x^p$	0	b	$b^{-p}x^p$
Pareto	$1 - a^p x^{-p}$	a	∞	$1 - x^{-p}$
Beta of first kind	$1 - (1-x)^p$	0	1	$(1-x)^p$
Weibull	$1 - \exp\{-\alpha x^p\}$	0	∞	$\exp\{-\alpha x^p\}$
Inverse Weibull	$\exp\{-\alpha x^{-p}\}$	0	∞	$\exp\{-\alpha x^{-p}\}$
Burr type II	$(1+\exp(-x))^{-k}$	$-\infty$	∞	$(1+\exp(-x))^{-k}$
Burr type III	$(1+x^{-k})^{-m}$	0	∞	$(1+x^{-k})^{-m}$
Burr type IV	$\left(1+\left(\frac{b-x}{x}\right)^{1/b}\right)^{-k}$	0	b	$\left(1+\left(\frac{b-x}{x}\right)^{1/b}\right)^{-k}$
Burr type V	$(1+m\cdot e^{-\tan(x)})^{-k}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	$(1+m\cdot e^{-\tan(x)})^{-k}$
Burr type VI	$(1+m\cdot e^{-\mathrm{Sh}(x)})^{-k}$	$-\infty$	∞	$(1+m\cdot e^{-\mathrm{Sh}(x)})^{-k}$
Burr type VII	$\left(\frac{1+\mathrm{th}(x)}{2}\right)^k$	$-\infty$	∞	$\left(\frac{1+\mathrm{th}(x)}{2}\right)^k$
Burr type VIII	$\left(\frac{2\tan^{-1}(e^x)}{\pi}\right)^k$	$-\infty$	∞	$\left(\frac{2\tan^{-1}(e^x)}{\pi}\right)^k$
Burr type IX	$1 - \frac{2}{m[(1+e^x)^k - 1] + 2}$	$-\infty$	∞	$\frac{2}{m[(1+e^x)^k - 1] + 2}$
Burr type X	$(1-\exp\{-x^2\})^k$	0	∞	$(1-\exp\{-x^2\})^k$
Burr type XI	$\left(x - \frac{\sin(2\pi x)}{2\pi}\right)^k$	0	1	$\left(x - \frac{\sin(2\pi x)}{2\pi}\right)^k$
Burr type XII	$1-(1+\alpha x^p)^{-m}$	0	∞	$1-(1+\alpha x^p)^{-m}$
Cauchy	$\frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}$	$-\infty$	∞	$\tan^{-1}(x)$

Table 3. Examples using Remark 5.2 with $c \neq 1$.

Name	a	b	c	$h(x)$
Power	0	b	p/q	x^q
Inverse Weibull	0	∞	α	$\exp(-x^{-p})$
Burr type II	$-\infty$	∞	k	$(1+\exp(-x))^{-1}$
Burr type III	0	∞	m	$(1+x^{-k})^{-1}$
Burr type IV	0	b	k	$\left(1+\left(\frac{b-x}{x}\right)^{1/b}\right)^{-1}$
Burr type V	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	k	$(1+m\cdot\exp\{-\tan(x)\})^{-1}$
Burr type VI	$-\infty$	∞	k	$(1+m\cdot\exp\{-\mathrm{Sh}(x)\})^{-1}$
Burr type VII	$-\infty$	∞	k	$\frac{1+\mathrm{th}(x)}{2}$
Burr type X	0	∞	k	$1-\exp\{-x^2\}$
Burr type XI	0	1	k	$x - \sin(2\pi x)/2\pi$

Note: $\mathrm{Sh}(x) = \frac{e^x - e^{-x}}{2}$ and $\mathrm{Th}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

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