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KERNEL-TYPE DENSITY AND FAILURE RATE ESTIMATION FOR ASSOCIATED SEQUENCES

ISHA BAGAI[1] AND B. L. S. PRAKASA RAO[2]

[1] *Indian Statistical Institute, 7, S. J. S. Sansanwal Marg, New Delhi-110016, India*
[2] *Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India*

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Abstract. Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of associated random variables defined on a probability space $(\Omega, \mathcal{B}, \mathcal{P})$ with probability density function $f(x)$ and failure rate function $r(x)$ for X_1. Let $f_n(x)$ be a kernel-type estimator of $f(x)$ based on $X_1, \dots, X_n$. Properties of $f_n(x)$ are studied. Pointwise strong consistency and strong uniform consistency are established under a certain set of conditions. An estimator $r_n(x)$ of $r(x)$ based on $f_n(x)$ and $\bar{F}_n(x)$, the empirical survival function, is proposed. The estimator $r_n(x)$ is shown to be pointwise strongly consistent as well as uniformly strongly consistent over some sets.

Key words and phrases: Density estimator, failure-rate estimator, kernel estimators, associated sequences.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of associated random variables defined on a probability space $(\Omega, \mathcal{B}, \mathcal{P})$ with density function f, distribution function F, survival function $\bar{F} = 1 - F$ and failure rate function $r_F = f/\bar{F}$, respectively, for X_1. The random variables $X_1, X_2, \dots, X_n$ are said to be associated if for every pair of functions $h(\boldsymbol{x})$ and $g(\boldsymbol{x})$ from R^n to R, which are nondecreasing componentwise,

$$\mathrm{Cov}(h(\boldsymbol{X}), g(\boldsymbol{X})) \geq 0,$$

whenever it is finite, where $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$. An infinite family is said to be associated if every finite subfamily is associated.

Several types of estimators for a density function of i.i.d. observations have been proposed in the literature. However, the most commonly used density estimator is the kernel-type estimator. It has been extensively studied (see, for example, Rosenblatt (1956), Parzen (1962), Prakasa Rao (1983), Silverman (1986)) especially when $X_1, X_2, \dots$ is a sequence of i.i.d. random variables. Roussas (1969) and Prakasa Rao (1978), among others, considered density estimation for stationary Markov processes satisfying Doeblin's condition. However, most often in reliability studies, the random variables, which are generally lifetimes of components, are not independent but are associated. For example, if the failure times of a system follow the multivariate exponential distribution (cf. Marshall and Olkin (1967)), then they are associated. If independent components of a system are subject to the same stress, then their lifetimes are associated. Another example is when the failure of one component increases the chance of failure for its neighbours and the related lifetimes are associated. Thus, there is a need to study the problem of density estimation for a sequence of associated random variables. First we establish the strong law of large numbers for sums of functions of stationary associated random variables which is used to study the properties of an estimator $f_n(x)$ of $f(x)$. Then, the kernel-type estimator $f_n(x)$ of $f(x)$ is proposed and its properties are discussed in the next section. Finally, an estimator $r_n(x)$ for the failure rate function $r(x)$ based on $f_n(x)$ and $\bar{F}_n(x)$, the empirical survival function, is proposed. It is shown to be strongly consistent pointwise as well as uniformly strongly consistent over certain sets.

Roussas (1991) studied strong uniform consistency of kernel estimates of r-th order derivative of f under some regularity conditions on the kernel and bandwidth. He has also obtained rates of convergence. A preliminary version of this paper was prepared independently around the time Roussas (1991) appeared. Techniques of proofs given here are essentially the same as in Roussas (1991) and Bagai and Prakasa Rao (1991).

2. Preliminaries

First we obtain a strong law of large numbers for functions of stationary associated random variables $\{X_n, n \geq 1\}$. For a sequence of stationary associated random variables, a strong law was obtained by Newman (1984) and another one for nonstationary sequence of associated random variables by Birkel (1989).

Let c denote a generic positive constant in the sequel. For simplicity we write $\mathrm{Var}(Z) = \mathrm{Cov}(Z, Z)$.

LEMMA 2.1. (Lemma 3 in Newman (1980)) *Let (X, Y) be associated random variables with finite variance. Then, for any two differentiable functions f and g,*

$$|\,\mathrm{Cov}(f(X), g(Y))| \leq \sup_x |f'(x)| \sup_y |g'(y)| \,\mathrm{Cov}(X, Y) \tag{2.1}$$

where f' and g' denote the derivatives of f and g respectively.

We now prove a strong law of large numbers for sums of functions of associated random variables.

LEMMA 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables. Let $S_{m_n,n} = \sum_{j=1}^{m_n} f_n(X_j)$ where f_n is differentiable with $\sup_n \sup_x |f_n'(x)| \leq c < \infty$. Further suppose that*

$$\sum_{j=1}^{\infty} \mathrm{Cov}(X_1, X_j) \leq c < \infty. \tag{2.2}$$

Then,

$$\mathrm{Var}(S_{m_n,n}) \leq 2cm_n.$$

PROOF. Observe that

$$\begin{aligned}
\mathrm{Var}(S_{m_n,n}) &= \mathrm{Var}\left(\sum_{j=1}^{m_n} f_n(X_j)\right) \\
&= m_n \mathrm{Var}(f_n(X_1)) + 2\sum\sum_{1\leq i<j\leq m_n} \mathrm{Cov}(f_n(X_i), f_n(X_j)) \\
&\quad \text{(by stationarity)} \\
&\leq 2m_n \sum_{j=1}^{m_n} \mathrm{Cov}(f_n(X_1), f_n(X_j)),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{Var}(S_{m_n,n}) &\leq 2cm_n \sum_{j=1}^{\infty} \mathrm{Cov}(X_1, X_j) \qquad \text{(by Lemma 2.1)} \\
&\leq 2cm_n \qquad \text{(by (2.2))}.
\end{aligned}$$

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables. Let $S_{k,n} = \sum_{j=1}^{k} f_n(X_j)$ where f_n is differentiable with $\sup_n \sup_x |f_n'(x)| \leq c$. Suppose $E[f_n(X_1)] = 0$, $\mathrm{Var}[f_n(X_1)] < \infty$ and condition (2.2) holds. Then,*

$$\frac{S_{n,n}}{n} \to 0 \qquad a.s.\ as \quad n \to \infty.$$

PROOF. Using Chebychev's inequality and Lemma 2.2, we note that $\sum_n \Pr[|S_{n^2,n}| > n^2\varepsilon] < \infty$ for all $\varepsilon > 0$ and hence, by Borel-Cantelli lemma, it follows that

$$\frac{S_{n^2,n}}{n^2} \to 0 \qquad \text{a.s. as} \quad n \to \infty. \tag{2.3}$$

Let

$$D_n = \max_{n^2<k\leq(n+1)^2} |S_{k,n} - S_{n^2,n}|.$$

Then, by Chebychev's inequality,

$$\Pr\{D_n \geq n^2\varepsilon\} \leq \frac{1}{n^4\varepsilon^2} E(D_n^2). \tag{2.4}$$

Furthermore

$$
\begin{aligned}
E[D_n^2] &= E\left[\left\{\max_{n^2<k\le(n+1)^2}|S_{k,n}-S_{n^2,n}|\right\}^2\right]\\
&= E\left[\max_{n^2<k\le(n+1)^2}|S_{k,n}-S_{n^2,n}|^2\right]\\
&\le E\left[\sum_{k=n^2+1}^{(n+1)^2}|S_{k,n}-S_{n^2,n}|^2\right]\\
&= E[(f_n(X_{n^2+1}))^2+(f_n(X_{n^2+1})+f_n(X_{n^2+2}))^2+\cdots\\
&\quad +(f_n(X_{n^2+1})+f_n(X_{n^2+2})+\cdots+f_n(X_{(n+1)^2}))^2]\\
&\le 2nE\left[\sum_{k=n^2+1}^{(n+1)^2} f_n^2(X_k)+2\sum\sum_{n^2+1\le i,j,i\ne j\le(n+1)^2} f_n(X_i)f_n(X_j)\right]\\
&= 2n\operatorname{Var}[S_{(n+1)^2,n}-S_{n^2,n}]\\
&\le cn^2 \qquad \text{(by Lemma 2.2)}.
\end{aligned}
$$

Therefore $\sum_n P\{D_n\ge n^2\varepsilon\}<\infty$ for all $\varepsilon>0$. Again, using Borel-Cantelli lemma, we get that

$$
\frac{D_n}{n^2}\to 0 \quad \text{a.s. as} \quad n\to\infty. \tag{2.5}
$$

Furthermore

$$
\frac{|S_{k,n}|}{k}\le\frac{|S_{n^2,n}|+D_n}{n^2} \quad \text{for} \quad n^2<k\le(n+1)^2.
$$

Hence, from (2.3) and (2.5), it follows that

$$
\frac{S_{n,n}}{n}\to 0 \quad \text{a.s. as} \quad n\to\infty.
$$

Other results which will be used later are stated below for completeness.

THEOREM 2.2. *For every $\alpha\in J$, an index set, let $\{X_j(\alpha), j\ge 1\}$ be an associated sequence. Let f_n, $n\ge 1$ be functions of bounded variation which are differentiable and suppose that $\sup_{n\ge1}\sup_x|f_n'(x)|\le c<\infty$. Let $E(f_n(X_j(\alpha)))=0$ for every $n\ge 1$, $j\ge 1$ and $\alpha\in J$. Suppose there exist $r>2$ and $\delta>0$ (independent of α, j and n) such that*

$$
\sup_{n\ge1}\sup_{\alpha\in J}\sup_{j\ge1}E|f_n(X_j(\alpha))|^{r+\delta}<\infty. \tag{2.6}
$$

Let

$$
u(n,\alpha)=\sup_{k\ge1}\sum_{j:|j-k|\ge n}\operatorname{Cov}(X_j(\alpha),X_k(\alpha)). \tag{2.7}
$$

Suppose that there exists $c > 0$ independent of $\alpha \in J$ such that

$$u(n, \alpha) \leq cn^{-(r-2)(r+\delta)/2\delta}.$$

Then there exists a constant B not depending on n, m and α, such that

$$\sup_{m \geq 1} \sup_{\alpha \in J} \sup_{k \geq 0} E|S_{n+k,m}(\alpha) - S_{k,m}(\alpha)|^r \leq Bn^{r/2} \tag{2.8}$$

where

$$S_{m_n,n}(\alpha) = \sum_{j=1}^{m_n} f_n(X_j(\alpha)).$$

PROOF. Since f_n is a function of bounded variation, we can express f_n as

$$f_n(x) = f_{n,1}(x) - f_{n,2}(x)$$

where $f_{n,1}$ and $f_{n,2}$ are two monotone functions. Observe that monotone functions of associated random variables are associated (Esary *et al.* (1967)). Note that

$$\begin{aligned} &E[|S_{n+k,n}(\alpha) - S_{k,n}(\alpha)|^r] \\ &= E\left[\left|\sum_{j=k+1}^{n+k} f_n(X_j(\alpha))\right|^r\right] \\ &= E\left[\left|\sum_{j=k+1}^{n+k} (f_{n,1}(X_j(\alpha)) - f_{n,2}(X_j(\alpha)))\right|^r\right] \\ &\leq c\left\{E\left[\left|\sum_{j=k+1}^{n+k} f_{n,1}(X_j(\alpha))\right|^r\right] + E\left[\left|\sum_{j=k+1}^{n+k} f_{n,2}(X_j(\alpha))\right|^r\right]\right\} \end{aligned}$$

(by C_r-inequality, Rao (1973)). The result then follows from Lemma 2.1. and a uniform version of Theorem 1 of Birkel (1988*a*).

THEOREM 2.3. *For any $\alpha \in J$, an index set, let $\{X_j(\alpha), j \geq 1\}$ be an associated sequence. Let f_n, $n \geq 1$ be functions of bounded variation which are differentiable and suppose that $\sup_{n\geq 1} \sup_x |f_n(x)| < \infty$, and $\sup_{n \geq 1} \sup_x |f_n'(x)| \leq c < \infty$. Let $E(f_n(X_j(\alpha)) = 0$, $n \geq 1$, $\alpha \in J$ and $j \geq 1$. Assume that there exists $r > 2$ such that*

$$u(n, \alpha) = O(n^{-(r-2)/2}).$$

Then (2.8) holds.

PROOF. The proof follows from Theorem 2.2 and Theorem 2 in Birkel (1988*a*).

3. Kernel-type density estimator

Here we propose a kernel-type estimator for the unknown density function f of X_1, when $\{X_n, n \geq 1\}$ is a stationary sequence of associated random variables. We assume that the support of f is a closed interval $I = [a, b]$ in the real line. Let us consider

$$f_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right), \qquad x \in I \tag{3.1}$$

as an estimator for $f(x)$, where $K(\cdot)$ is a suitable kernel and h_n is a bandwidth sequence.

The asymptotic behaviour of $f_n(x)$ is discussed later under the assumptions (A) listed below.

(A1) $K(\cdot)$ is a bounded density function and of bounded variation on R satisfying (i) $\lim_{|u|\to\infty} |u|K(u) = 0$, (ii) $\int_{-\infty}^{\infty} u^2 K(u)du < \infty$.

(A2) $K(x)$ is differentiable and $\sup_x |K'(x)| \leq c < \infty$.

Further it is assumed that the covariance structure of $\{X_n\}$ satisfies the following condition.

(B) For all ℓ and $r \geq 0$, $\sum_{j:|\ell-j|\geq r} \operatorname{Cov}(X_j, X_\ell) \leq u(r)$, where $u(r) = e^{-\alpha r}$ for some $\alpha > 0$.

Remark 3.1. Cox and Grimmet (1984) and Birkel (1988*a*, 1988*b*), among others, observed that, in any asymptotic study of a sequence of associated random variables, the covariance structure plays an important role. Cox and Grimmet (1984), while considering the asymptotic normality of a triangular array of associated random variables, assume that there exists a function,

$$u(r) \to 0 \quad \text{as} \quad r \to \infty$$

such that

$$\sum_{j:|\ell-j|\geq r} \operatorname{Cov}(X_{nj}, X_{n\ell}) \leq u(r) \quad \text{for all} \quad \ell, n, r \geq 0. \tag{3.2}$$

The condition (B) imposes restrictions on the covariance structure of $\{X_n\}$ analogous to (3.2). It can be easily seen that (B) implies (2.2). Roussas (1991) assumes a weaker condition similar to that in Bagai and Prakasa Rao (1991).

Assume that f is thrice differentiable and the third derivative is bounded. Let $h_n \to 0$ and $nh_n^4 \to \infty$ as $n \to \infty$. Then, under (A) and (B), following Parzen (1962), it can be checked that $E[f_n(x)]$ and $B_n(x)$, the bias of $f_n(x)$, are given by

$$E[f_n(x)] = f(x) - h_n f'(x)\gamma_1 + \frac{h_n^2}{2} f''(x)\gamma_2 + O(h_n^3) \tag{3.3}$$

where

$$\gamma_j = \int_{-\infty}^{\infty} x^j K(x)dx, \qquad j = 1, 2; \tag{3.4}$$

and

$$\begin{aligned} B_n(x) &= E[f_n(x)] - f(x) \\ &= -h_n f'(x)\gamma_1 + \frac{h_n^2}{2} f''(x)\gamma_2 + O(h_n^3). \end{aligned} \tag{3.5}$$

Furthermore

$$\begin{aligned} \operatorname{Var} f_n(x) &= \frac{1}{nh_n^2} \operatorname{Var}\left(K\left(\frac{x-X_1}{h_n}\right)\right) \\ &\quad + \frac{1}{n^2h_n^2} \sum\sum_{1\le i\ne j\le n} \operatorname{Cov}\left(K\left(\frac{x-X_i}{h_n}\right), K\left(\frac{x-X_j}{h_n}\right)\right). \end{aligned} \tag{3.6}$$

Observe that,

$$\begin{aligned} \frac{1}{nh_n^2} \operatorname{Var} K\left(\frac{x-X_1}{h_n}\right) &= \frac{1}{nh_n^2}\left\{E\left[K^2\left(\frac{x-X_1}{h_n}\right)\right]\right. \\ &\qquad \left. -E^2\left[K\left(\frac{x-X_1}{h_n}\right)\right]\right\} \\ &= \frac{1}{nh_n}[f(x)\beta_0 - f'(x)h_n\beta_1 + f''(x)h_n^2\beta_2] \\ &\quad - \frac{1}{n}[B_n(x)+f(x)]^2 + O\left(\frac{h_n^2}{n}\right) \end{aligned} \tag{3.7}$$

where

$$\beta_j = \int_{-\infty}^{\infty} x^j K^2(x)dx, \qquad j = 0,1,2 \tag{3.8}$$

and

$$\begin{aligned} &\frac{2}{n^2} \sum\sum_{1\le i<j\le n} \left|\operatorname{Cov}\left\{\frac{1}{h_n}K\left(\frac{x-X_i}{h_n}\right), \frac{1}{h_n}K\left(\frac{x-X_j}{h_n}\right)\right\}\right| \\ &\quad \le \frac{2}{n^2} \sum\sum_{1\le i<j\le n} \sup_y \left\{\frac{\partial}{\partial y}\psi_n(x,y)\right\}^2 \operatorname{Cov}(X_i,X_j) \qquad \text{(by Lemma 2.1)} \end{aligned}$$

where

$$\begin{aligned} \psi_n(x,y) &= \frac{1}{h_n}K\left(\frac{x-y}{h_n}\right) \\ &\le \frac{c}{n^2h_n^4} \sum\sum_{1\le i<j\le n} \operatorname{Cov}(X_i,X_j) \qquad \text{(by (A2))} \\ &\le \frac{c}{nh_n^4}u(0) \qquad \text{(by (B) and stationarity)} \\ &\le \frac{c}{nh_n^4}. \end{aligned} \tag{3.9}$$

Substituting (3.7) and (3.9) in (3.6), we have

$$\operatorname{Var} f_n(x) = \frac{1}{nh_n}[f(x)\beta_0 + O(h_n)] + O\left(\frac{1}{nh_n^4}\right). \tag{3.10}$$

From (3.3) and (3.10), it follows that $f_n(x)$ is asymptotically unbiased and weakly consistent for $f(x)$. And, following Prakasa Rao ((1983), pp. 35), it can be easily verified that the optimal choice of h_n, which minimizes the mean square error, is $O(n^{-1/5})$, same as the one in the i.i.d. case.

3.1 *Pointwise strong consistency of $f_n(x)$*

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables. Suppose that* (A) *and* (B) *hold. Then, for $x \in I$,*

$$f_n(x) - Ef_n(x) \to 0 \quad a.s. \text{ as } \quad n \to \infty.$$

PROOF. Set

$$\begin{aligned} X_{ni} &= \frac{1}{h_n}\left\{K\left(\frac{x - X_i}{h_n}\right) - EK\left(\frac{x - X_i}{h_n}\right)\right\} \\ &= \psi_n(x, X_i) - E[\psi_n(x, X_i)], \qquad 1 \leq i \leq n. \end{aligned}$$

Then $E(X_{ni}) = 0$, $\operatorname{Var}(X_{ni}) < \infty$, and

$$\begin{aligned} &\sum_{j:|\ell-j|\geq r} |\operatorname{Cov}(X_{nj}, X_{n\ell})| \\ &\quad = \sum_{j:|\ell-j|\geq r} |\operatorname{Cov}(\psi_n(x, X_j), \psi_n(x, X_l))| \\ &\quad \leq \sup_y \left\{\frac{\partial}{\partial y}\psi_n(x, y)\right\}^2 \sum_{j:|\ell-j|\geq r} \operatorname{Cov}(X_j, X_\ell) \qquad \text{(by Lemma 2.1)}. \end{aligned}$$

The conditions of Theorem 2.1 hold because of (A2) and (B). Hence the result follows.

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, *$f_n(x) \to f(x)$ a.s. at continuity points x of $f(\cdot)$ as $n \to \infty$.*

PROOF. The result follows from the conclusion of the above theorem and the fact that $E[f_n(x)] \to f(x)$ at continuity points x of $f(\cdot)$.

3.2 *Uniform strong consistency of* $f_n(x)$

THEOREM 3.2. *Let* $\{X_n, n \geq 1\}$ *be a stationary sequence of associated random variables. Suppose that* (A) *and* (B) *hold and there exists* $\gamma > 0$ *such that*

$$h_n^{-4} = O(n^{\gamma}). \tag{3.11}$$

Then, for all $\varepsilon > 0$, $r > 1$,

$$\sup_x \Pr[|f_n(x) - Ef_n(x)| > \varepsilon] \leq c\varepsilon^{-2r} n^{-r}.$$

PROOF. Observe that

$$\begin{aligned}
u(n,x) &\equiv \sup_{\ell \geq 1} \sum_{j:|j-\ell| \geq n} \mathrm{Cov}(\psi_n(x, X_j), \psi_n(x, X_\ell)) \\
&\leq \sup_{\ell \geq 1} \sum_{j:|j-\ell| \geq n} \sup_y \left\{ \frac{\partial}{\partial y} \psi_n(x,y) \right\}^2 \mathrm{Cov}(X_j, X_\ell) \qquad \text{(by Lemma 2.1)} \\
&\leq \sup_x \sup_y \left\{ \frac{\partial}{\partial y} \psi_n(x,y) \right\}^2 \sup_{\ell \geq 1} \sum_{j:|j-\ell| \geq n} \mathrm{Cov}(X_j, X_\ell) \\
&= O(n^{-\beta}), \qquad \text{for any} \quad \beta > 0 \qquad \text{(by (3.11) and (3.12))}.
\end{aligned}$$

Then, using Chebychev's inequality and Theorem 2.3 with

$$\beta = (r-1), \qquad r > 1,$$

we have

$$\begin{aligned}
&\sup_x \Pr[|f_n(x) - Ef_n(x)| > \varepsilon] \\
&\quad = \sup_x \Pr\left[\left| \sum_{j=1}^{n} (\psi_n(x, X_j) - E\psi_n(x, X_j)) \right|^{2r} > (n\varepsilon)^{2r} \right] \\
&\quad \leq \sup_x (n\varepsilon)^{-2r} E\left[\left| \sum_{j=1}^{n} (\psi_n(x, X_j) - E\psi_n(x, X_j)) \right|^{2r} \right] \\
&\quad \leq c(n\varepsilon)^{-2r} n^r \\
&\quad = c\varepsilon^{-2r} n^{-r}.
\end{aligned}$$

THEOREM 3.3. *Let* $\{X_n, n \geq 1\}$ *be a stationary sequence of associated random variables satisfying the conditions* (A), (B) *and* (3.11). *Further, suppose that the following condition holds*:

(C) $|f(x_1) - f(x_2)| \leq c|x_1 - x_2|$, $x_1, x_2 \in I$.

Then

$$\sup[|f_n(x) - f(x)|; x\varepsilon I] \to 0 \qquad a.s.$$

The proof of the theorem is based on the following lemmas which are easy to prove and it is a slight variation of a similar proof given in Roussas (1988).

LEMMA 3.1. *If the condition* (C) *holds, then*

$$\sup\{|f(x) - Ef_n(x)|, x \in I\} \leq ch_n.$$

LEMMA 3.2. *If* (A) *holds then*

$$|f_n(x_1) - f_n(x_2)| \leq ch_n^{-2}|x_1 - x_2|, \qquad x_1, x_2 \in I.$$

PROOF OF THEOREM 3.3. Let $\delta_n = n^{-2\theta}$ where θ is chosen so that $0 < \theta < \frac{r-1}{2}$. Such a choice is possible since $r > 1$. Divide the interval $I = [a, b]$ into b_n subintervals $I_{n\ell} = (x_{n\ell}, x_{n,\ell+1}]$, $\ell = 1, \ldots, b_n = N$ of length δ_n. Notice that

$$b_n \leq c\delta_n^{-1}, \tag{3.12}$$

and

$$\begin{aligned}\sup_{x\in I}|f_n(x) - f(x)| \leq \max_{\ell}\sup_{x\in I_{n\ell}} &|[f_n(x) - f_n(x^*_{n\ell})] + [f_n(x^*_{n\ell}) - Ef_n(x^*_{n\ell})] \\ &+ [Ef_n(x^*_{n\ell}) - f(x^*_{n\ell})] - [f(x) - f(x^*_{n\ell})]|\end{aligned}$$

where $x^*_{n\ell}$ is an arbitrary point in $I_{n\ell}$. Hence,

$$\begin{aligned}\sup_{x\in I}|f_n(x) - f(x)| \leq &\max_{\ell}\sup_{x\in I_{n\ell}}|f_n(x) - f_n(x^*_{n\ell})| \\ &+ \max_{\ell}|f_n(x^*_{n\ell}) - Ef_n(x^*_{n\ell})| \\ &+ \max_{\ell}|Ef_n(x^*_{n\ell}) - f(x^*_{n\ell})| \\ &+ \max_{\ell}\sup_{x\in I_{n\ell}}|f(x) - f(x^*_{n\ell})|.\end{aligned} \tag{3.13}$$

Note that

$$|x - x^*_{n\ell}| \leq c\delta_n, \qquad \text{which implies that } |f(x) - f(x^*_{n\ell})| \leq c\delta_n. \tag{3.14}$$

Then, by Lemma 3.1, it follows that

$$|f(x^*_{n\ell}) - Ef_n(x^*_{n\ell})| \leq ch_n \tag{3.15}$$

and, by Lemma 3.2, we have

$$|f_n(x) - f_n(x^*_{n\ell})| \leq ch_n^{-2}\delta_n. \tag{3.16}$$

Substituting (3.14) to (3.16) in (3.13), we have

$$\begin{aligned} \sup_{x \in I}\{|f_n(x) - f(x)|\} \\ \leq ch_n^{-2}\delta_n + ch_n + c\delta_n + \max_{\ell}|f_n(x^*_{n\ell}) - Ef_n(x^*_{n\ell})|. \end{aligned} \tag{3.17}$$

Let $\varepsilon > 0$. Choose $h_n = n^{-\theta}$, where $\theta > 0$ and

$$\delta_n = \frac{\varepsilon h_n^2}{4c}. \tag{3.18}$$

Then, for large n,

$$\begin{aligned} ch_n^{-2}\delta_n &\leq \varepsilon/4, \\ c\delta_n &= \frac{c}{4}\varepsilon h_n^2 \leq \varepsilon/4, \end{aligned}$$

and

$$ch_n = cn^{-\theta} < \varepsilon/4.$$

Then, for large n, (3.17) reduces to

$$\sup_{x \in I}\{|f_n(x) - f(x)|\} \leq \frac{3\varepsilon}{4} + \max_{\ell}|f_n(x^*_{n\ell}) - Ef_n(x^*_{n\ell})|.$$

Hence,

$$\begin{aligned} &\Pr\left[\sup_{x \in I}\{|f_n(x) - f(x)|\} > \varepsilon\right] \\ &\quad\leq \Pr\left[\max_{\ell}|f_n(x^*_{n\ell}) - Ef_n(x^*_{n\ell})| > \varepsilon/4\right] \\ &\quad\leq \sum_{\ell=1}^{N}\Pr[|f_n(x^*_{n\ell}) - Ef_n(x^*_{n\ell})| > \varepsilon/4] \\ &\quad\leq cb_n\varepsilon^{-2r}n^{-r} \qquad \text{(by Theorem 3.2)} \\ &\quad\leq c\delta_n^{-1}n^{-r}. \end{aligned}$$

Then the result follows using Borel-Cantelli lemma in view of the fact $\sum \delta_n^{-1}n^{-r} < \infty$.

4. Kernel-type failure rate estimator

The failure rate $r(x)$ is defined as

$$r(x) = f(x)/\bar{F}(x), \qquad \bar{F} > 0.$$

The distribution function $F(x)$ is uniquely determined by $r(x)$ by the relationship

$$\bar{F}(x) = \exp\left\{-\int_{-\infty}^{x} r(t)dt\right\}.$$

The problem of estimating $r(x)$ on the basis of i.i.d. observations $X_1, X_2, \ldots, X_n$ from F has been discussed by Watson and Leadbetter (1964*a*, 1964*b*), Rice and Rosenblatt (1976) and Prakasa Rao and Van Ryzin (1985), among others. An obvious estimate of $r(x)$ is $r_n(x)$ given by

$$r_n(x) = f_n(x)/\bar{F}_n(x), \tag{4.1}$$

where $f_n(x)$ is the kernel-type estimator of $f(x)$ discussed above and $\bar{F}_n(x)$ is the proportion of X_i, $1 \le i \le n$ that exceed x. The properties of $\bar{F}_n(x)$ based on stationary associated sequence $\{X_n\}$ have been discussed by Bagai and Prakasa Rao (1991).

Roussas (1989) discussed the need for estimating $r(x)$ when the lifetimes $X_1, X_2, \ldots, X_n$ are identically distributed but not independent. He discussed the consistency properties of $r_n(x)$ for a stationary sequence of random variables satisfying any one of the four standard modes of mixing random variables. In what follows, we prove analogous result for a stationary sequence of associated random variables.

It is easy to see that

$$r_n(x) - r(x) = \frac{\bar{F}(x)[f_n(x) - f(x)] - f(x)[\bar{F}_n(x) - \bar{F}(x)]}{D_n(x)} \tag{4.2}$$

where

$$D_n(x) = \bar{F}(x)\bar{F}_n(x),$$
$$\bar{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} Y_i(x),$$

and

$$Y_i(x) = \begin{cases} 1 & \text{if } X_i > x, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $D_n(x) > 0$ almost surely for all x in $S = \{x \in R; \bar{F}(x) > 0\}$.

LEMMA 4.1. (Bagai and Prakasa Rao (1991)) *Let $\{X_n, n \ge 1\}$ be a stationary sequence of associated random variables with bounded density for X_1. Assume that, for some $r > 1$,*

$$\sum_{j=n+1}^{\infty} \{\mathrm{Cov}(X_1, X_j)\}^{1/3} = O(n^{-(r-1)}). \tag{4.3}$$

Then, for every x,

$$\bar{F}_n(x) \to \bar{F}(x) \quad a.s. \text{ as } \quad n \to \infty.$$

THEOREM 4.1. *Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables satisfying the conditions* (A), (B) *and* (4.3). *Then, for all $x \in S$ which are continuity points of f,*

$$r_n(x) \to r(x) \quad a.s. \text{ as } \quad n \to \infty.$$

PROOF. For every $x \in S$ and for all sufficiently large n, $D_n(x) > 0$ a.s. Using Corollary 3.1, $f_n(x) \to f(x)$ a.s., when x is a continuity point of f. Then, the result follows from Lemma 4.1 and Corollary 3.1.

LEMMA 4.2. (Bagai and Prakasa Rao (1991)) *Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables satisfying the conditions of Lemma 4.1. Then*

$$\sup[|\bar{F}_n(x) - \bar{F}(x)|, x \in J] \to 0 \quad a.s.,$$

where J is any compact subset of S.

THEOREM 4.2. *Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables satisfying the conditions* (A), (B), (C), (3.11) *and* (4.3). *Then,*

$$\sup\{|r_n(x) - r(x)| : x \in J\} \to 0 \quad a.s.$$

where J is any compact subset of S.

PROOF. The proof follows from the following facts by means of the relation (4.2):

$$\sup[|f_n(x) - f(x)| : x \in J] \to 0 \quad \text{a.s.},$$
$$\sup[|\bar{F}_n(x) - \bar{F}(x)| : x \in J] \to 0 \quad \text{a.s.},$$
$$\sup[f(x) : x \in J] < \infty,$$

and

$$\inf[\bar{F}(x) : x \in J] > 0.$$

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