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SUCCESS RUNS OF LENGTH k IN MARKOV DEPENDENT TRIALS

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Abstract. The geometric type and inverse Polýa-Eggenberger type distributions of waiting time for success runs of length k in two-state Markov dependent trials are derived by using the probability generating function method and the combinatorial method. The second is related to the minimal sufficient partition of the sample space. The first two moments of the geometric type distribution are obtained. Generalizations to ballot type probabilities of which negative binomial probabilities are special cases are considered. Since the probabilities do not form a proper distribution, a modification is introduced and new distributions of order k for Markov dependent trials are developed.

Key words and phrases: Ballot problem, success runs, Markov dependent trials, discrete distributions.

1. Introduction

The study of success runs of length k in independent Bernoulli trials (see Feller (1968), p. 322) has led to a series of papers on discrete distributions of order k and their properties, more particularly, the relation with Fibonacci sequences (see Turner (1979), Philippou and Muwafi (1982), Philippou *et al.* (1983), Philippou (1984), Aki *et al.* (1984), Charalambides (1986), Hirano (1986), Philippou and Makri (1986)). Some multiparametric generalizations and extensions to Polýa and inverse Polýa distributions (also called Polýa-Eggenberger distributions, see Johnson and Kotz (1977), pp. 176–194) are dealt with by Philippou (1988) and by Aki and Hirano (1987), Philippou *et al.* (1989) respectively.

A generalization of independent Bernoulli trials has led to the consideration of two-state Markov dependent trials under which various probability distributions have been derived. Our interest in this paper being on binomial and negative binomial type of distributions (which include ballot type or first passage type of distributions (see Mohanty (1979), p. 128)), we only refer to Gabriel (1959), Narayana (1959), Narayana *et al.* (1960), Narayana and Sathe (1961), Seth (1963), Mohanty (1966*a*) (1979, p. 132), Jain (1971, 1973), Klotz (1972, 1973), Nain and Sen (1979), Bhat and Lal (1988) and Lal and Bhat (1989). Some authors (Seth, Jain, Nain and Sen, Bhat and Lal) have formulated Markov dependent trial as correlated random walk problems and some others (Narayana, Narayana *et al.*, Mohanty) as two-coin tossing games. In fact, Blackwell and Girshick ((1954), p. 222) provided the two-coin tossing formulation as an example for minimal sufficient statistics and this way of presentation was later followed by others.

In this paper, we extend the study of success runs of length k from the case of independent trials to the two-state Markov dependent trials. Some distributions arising out of this situation are recently obtained by Balakrishnan *et al.* (1992) and Balasubramanian *et al.* (1994) as generalizations of a problem of practical interest in Hahn and Gage (1983) and by Aki and Hirano (1993) and Hirano and Aki (1993). Also see Rajarshi (1974). In our case, we formulate the Markov dependent trials or correlated random walk as a two-coin tossing game in the line of Narayana. In Section 2 we derive the geometric type distributions and the corresponding means and variances. A natural correspondence is shown to exist between geometric types and inverse Polýa-Eggenberger type of distributions. There is some overlapping between this section and results in the first two papers referred in this paragraph. In particular the geometric type distributions and the moments have been derived in those two papers. In Section 3, we generalize the geometric types to ballot type of probabilities of which negative binomial types are special cases. A binomial type of random variable is also considered. Unfortunately these probabilities do not form a distribution. Therefore, in Section 4, a modification is introduced so as to get a distribution. Keeping earlier nomenclature is mind, we call it a ballot type distribution of order k and from it derive other distributions of order k. In their two papers Aki and Hirano have dealth with the geometric type and three binomial types of proper distributions.

A standard approach to analyze these problems is to treat the underlying sequences as Markov chains and to use the probability generating function (p.g.f.) technique right at the outset without looking into the structure of the chain (see Feller (1968)). This is what may be termed as a "top-to-bottom" analysis in which the equation on the p.g.f. has to be solved and then the solution is expanded in order to derive the desired probabilities. On the other hand, we may use tools from discrete mathematics (in particular, enumerative combinatorics) in which a problem is split into sub-problems that are solved to be recombined for giving a solution to the original problem. This approach in contrast to the earlier one, may be called a "bottom-to-top" analysis. The fact that the latter which is combinatorial in nature is quite effective in many situations (see e.g. Mohanty (1979) in counting lattice paths within general boundaries, Böhm and Mohanty (1994) in queueing problems) is not new but being formally emphasized in this paper. One can trace back to the origin of these two approaches viz., the p.g.f. technique essentially comes from the random walk formulation (see Feller (1968)) whereas the combinatorial technique is a natural one to apply when a minimal sufficient partition is under consideration in a coin tossing formulation (see Blackwell and Girshick (1954)). In the second case, the sample space or the set of sequences of outcomes is divided into subsets on the basis of minimal sufficient statistics and the cardinality of each subset is determined combinatorially—this is what we meant by a "bottom-to-top" analysis. (We remark that Bhat and Lal have adopted a different procedure for computing the probability distribution.) Narayana (1959) recognized the efficacy of the combinatorial methods, the use of which was subsequently exploited in papers by Narayana *et al.* (1960, 1961) and Mohanty (1966*a*). In this paper however, we use both methods of analysis and show the respective advantages. Our present combinatorial argument is in the line of Philippou and Muwafi (1982) and is led through the structure of minimal sufficient partition.

Thus the paper has several objectives which are listed as follows:

(i) to find the geometric type distribution of success runs of length k under two-stage Markov dependent trials;

(ii) to connect these to Polýa-Eggenberger type distributions;

(iii) to use both p.g.f. and combinatorial techniques and to relate the second one to a minimal sufficient partition;

(iv) to generalize the approach in order to get ballot type distributions;

(v) to bridge the gap of communication among (1) those interested in Fibonacci sequences and distributions of order k, (2) those who work through random walks, (3) those who work through coin tossing games, and (4) those who arrive at these types of distributions through applications (see Philippou (1986), Viveros and Balakrishnan (1993) for further references on applications).

2. Waiting time for a run of length k

Consider Markov dependent trials, each trial being either a success (s) or a failure (f), which is governed by the transition probability matrix

	To	
From	s	f
s	p_1	q_1
f	p_2	q_2

where $q_i = 1 - p_i$, $i = 1, 2$. Here we are interested in success runs of length k. We may reformulate the above as a two-coin tossing game (see Mohanty (1979), p. 132). Let there be two coins, coin 1 and coin 2. Toss coin 1(2) if a head (tail) has appeared in the previous trial. If s and f correspond to a head (H) and a tail (T) respectively, it is clear that $P(H \mid \text{coin}\, i) = p_i$, $i = 1, 2$. Because of this equivalence, henceforth we use the terminology of the coin tossing game (sometimes both without causing any ambiguity).

Let X_i represent the number of trials needed to get k successive H's for the first time given that the game starts with coin i, $i = 1, 2$. We will find the probability generating function (p.g.f.) of X_i for which let us introduce the following notations:

$G_i(z)$: p.g.f. of X_i,

$\Pi_{n,i}$: P (in n trials the number of successive heads at the end is i, $i = 0, \ldots, k-1$ and at no time there are k successive heads),

$\psi_i(z)$: $\sum_n \Pi_{n,i} z^n$.

If $\{\Pi_{n,i}\}$ is a proper distribution, then $\psi_i(z)$ is the p.g.f.

Suppose the game starts with coin 2. Then

$$\Pi_{0,0} = 1, \tag{2.1}$$

$$\Pi_{n,0} = q_2\Pi_{n-1,0} + q_1 \sum_{i=1}^{k-1} \Pi_{n-1,i}, \tag{2.2}$$

$$\Pi_{n,1} = p_2\Pi_{n-1,0}, \tag{2.3}$$

$$\Pi_{n,i} = p_1\Pi_{n-1,i-1}, \qquad 2 \le i \le k-1, \tag{2.4}$$

$$P(X_2 = n) = p_1\Pi_{n-1,k-1}. \tag{2.5}$$

From these relations, we get

$$\psi_0(z) = q_2 z\psi_0(z) + q_1 z \sum_{i=1}^{k-1} \psi_i(z) + \Pi_{00}, \tag{2.6}$$

$$\psi_1(z) = p_2 z\psi_0(z), \tag{2.7}$$

$$\psi_i(z) = p_1 z\psi_{i-1}(z), \qquad 2 \le i \le k-1, \tag{2.8}$$

$$G_2(z) = p_1 z\psi_{k-1}(z). \tag{2.9}$$

Therefore

$$G_2(z) = p_2 p_1^{k-1} z^k \psi_0(z) \tag{2.10}$$

where

$$\left(1 - q_2 z - q_1 p_2 \sum_{j=0}^{k-2} p_1^j z^{j+2}\right) \psi_0(z) = \Pi_{0,0} = 1. \tag{2.11}$$

Hence we obtain

$$G_2(z) = \frac{p_2 p_1^{k-1} z^k}{1 - q_2 z - q_1 p_2 \sum_{j=0}^{k-2} p_1^j z^{j+2}}. \tag{2.12}$$

Notice that we have derived $G_2(z)$ through the probabilities of outcomes at the end of a sequence. For determining $G_1(z)$, such probabilities need further elaboration. On the other hand, if we consider the outcomes at the beginning of a sequence the derivation becomes straight forward for both $G_1(z)$ and $G_2(z)$. Let us call the subsequence $H \cdots HT$ (there are i H's, $i = 0, 1, \ldots, k-1$) the i-th type of subsequence and the subsequence $H \cdots H$ (k in number) the k-th type of subsequence. Divide sequences in $G_1(z)$ into those starting with the i-th type subsequence ($i = 0, 1, \ldots, k-1$) and followed by a sequence in $G_2(z)$, or one sequence of the k-th type. Thus, we have the following lemma:

LEMMA 2.1.

$$G_1(z) = q_1 \left(\sum_{j=0}^{k-1} p_1^j z^{j+1}\right) G_2(z) + p_1^k z^k, \tag{2.13}$$

where $G_2(z)$ *is given by* (2.12).

Note that an argument similar to the one leading to (2.13) gives rise to:

LEMMA 2.2.

$$G_2(z) = \left(q_2 z + q_1 p_2 \sum_{j=1}^{k-1} p_1^{j-1} z^{j+1} \right) G_2(z) + p_2 p_1^{k-1} z^k \tag{2.14}$$

which leads to (2.12).

On taking derivatives of $G_1(z)$ and $G_2(z)$ and putting $z = 1$, we obtain

$$G_2'(1) = \frac{q_1 + p_2 - p_2 p_1^{k-1}}{q_1 p_2 p_1^{k-1}}, \tag{2.15}$$

$$\begin{aligned} G_2''(1) = {} & \frac{2p_2(1 - kp_1^{k-1} + (k-1)p_1^k)}{q_1^2 p_2 p_1^{k-1}} \\ & + 2\left(\frac{q_1 + p_2 - p_2 p_1^{k-1}}{q_1} - kp_2 p_1^{k-1} \right) G_2'(1), \end{aligned} \tag{2.16}$$

$$G_1'(1) = \frac{(1 - p_1^k)(q_1 + p_2)}{q_1 p_2 p_1^{k-1}} \tag{2.17}$$

and

$$\begin{aligned} G_1''(1) = {} & \frac{2p_1(1 - kp_1^{k-1} + (k-1)p_1^k)}{q_1^2} \\ & + \frac{2(1 - (k+1)p_1^k + kp_1^{k-1})}{q_1} G_2'(1) + (1 - p_1^k) G_2''(1). \end{aligned} \tag{2.18}$$

Obviously (2.14) and (2.17) give the means and expressions for the variances can be obtained from (2.16) and (2.18) by using the well known formula

$$\text{Variance} = G_i''(1) + G_i'(1) - [G_i'(1)]^2.$$

So far as the exact distribution of X_i is concerned, an expansion of $G_i(z)$ as a power series in z will give $P(X_i = n)$. It is easier to start with $G_2(z)$ in (2.12) and get

$$\begin{aligned} G_2(z) = {} & \sum_{n_0=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \binom{n_0 + \cdots + n_{k-1}}{n_0, \ldots, n_{k-1}} (q_2 z)^{n_0} \\ & \cdot \left[\prod_{i=1}^{k-1} (q_1 p_2 p_1^{i-1} z^{i+1})^{n_i} \right] p_2 p_1^{k-1} z^k \\ = {} & p_2 p_1^{k-1} z^k \Bigg[\sum_{j=0}^{\infty} \sum_{\{(n_0, \ldots, n_{k-1}) : n_0 + \cdots + n_{k-1} = j\}} \binom{j}{n_0, \ldots, n_{k-1}} \\ & \cdot (q_2 z)^{n_0} \sum_{i=1}^{k-1} (q_1 p_2 p_1^{i-1} z^{i+1})^{n_i} \Bigg] \end{aligned} \tag{2.19}$$

from which we get

PROPOSITION 2.1.

$$P(X_2 = x) = p_2 p_1^{k-1}\left[\sum_{\{(n_0,\ldots,n_{k-1}):n_0+2n_1+\cdots+kn_{k-1}+k=x\}} \binom{n_0+\cdots+n_{k-1}}{n_0,\ldots,n_{k-1}} \cdot q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_{k-1}} p_1^{n_2+2n_3+\cdots+(k-2)n_{k-1}}\right], \quad x = k, k+1, \ldots. \tag{2.20}$$

An examination of (2.19) suggests that n_i represents the number of the i-th type subsequences ($i = 0, 1, \ldots, k-1$) in a sequence of G_2. Let a subsequence HT be called a right turn and its number be denoted by Y. (Since a sequence can be represented by a lattice path when H and T correspond to a horizontal unit and a vertical unit, we have used the word "turn", being appropriate for a path.) Also let W denote the number of T's. If $X_2 = x$, $Y = y$ and $W = w$, then the following relations hold good:

$$\begin{cases} n_0 + n_1 + \cdots + n_{k-1} = w, \\ n_1 + \cdots + n_{k-1} = y, \\ n_0 + 2n_1 + \cdots + kn_{k-1} + k = x. \end{cases} \tag{2.21}$$

Thus we have

PROPOSITION 2.2.

$$P(X_2 = x, Y = y, W = w) = q_1^y p_2^{y+1} q_2^{w-y} p_1^{x-y-w-1} \sum_{A(x,y,w)} \binom{w}{n_0, n_1, \ldots, n_{k-1}} \tag{2.22}$$

where $A(x,y,w) = \{(n_0,\ldots,n_{k-1}) : (2.21)$ *is satisfied*$\}$.

COMBINATORIAL PROOF. Any sequence of outcomes is an arrangement of n_i subsequences of type i, $i = 0, 1, \ldots, k-1$, followed by a k-th type subsequence, such that (2.21) is satisfied. Similar to a right turn let us call a subsequence TH a left turn. Observe that (i) every left turn following the first right turn contributes q_1p_2 to the probability of each sequence (T in a right turn HT contributes q_1 and H in the following left turn TH contributes p_2) and there are y of them; (ii) for each of the remaining $w-y$ T's (which do not appear as a part of a right turn) the probability is q_2; (iii) the first H has probability p_2; (iv) the probability of each of the remaining H's is p_1. Therefore the probability of each sequence is

$$(q_1p_2)^y q_2^{w-y} p_2 p_1^{x-w-y-1}.$$

Expression (2.22) is verified when we realize that the number of such sequences is given by the sum in (2.22). The proof is complete.

For obtaining $P(X_1 = x)$, we expand (2.13) and use (2.19) to get

$$(2.23)\quad G_1(z) = p_1^k z^k + p_2 p_1^{k-1} q_1 z^{k+1} \sum_{j=0}^{\infty} \left[\sum_{\{(n_0,\ldots,n_{k-1}): \sum_0^{k-1} n_i = j\}} \binom{j}{n_0,\ldots,n_{k-1}} \cdot q_2^{n_0} (q_1 p_2)^{\sum_1^{k-1} n_i} p_1^{n_2+2n_3+\cdots+(k-2)n_{k-1}} z^{n_0+2n_1+\cdots+kn_{k-1}} \right]$$
$$+ \sum_{i=1}^{k-1} p_2 p_1^{k+i-1} q_1 z^{k+i+1} \cdot \sum_{j=1}^{\infty} \left[\sum_{\{(n_1,\ldots,n_{k-1}): \sum_0^{k-1} n_i = j\}} \binom{j}{n_0,\ldots,n_{k-1}} \cdot q_2^{n_0} (q_1 p_2)^{\sum_1^{k-1} n_i} p_1^{n_2+2n_3+\cdots+(k-2)n_{k-1}} z^{n_0+2n_1+\cdots+kn_{k-1}} \right].$$

PROPOSITION 2.3.

$$(2.24\text{a})\quad P(X_1 = k) = p_1^k,$$

$$(2.24\text{b})\quad P(X_1 = x, \textit{starts with a } T) = p_2 p_1^{k-1} q_1 \cdot \left[\sum_{\{(n_0,\ldots,n_{k-1}): \sum_0^{k-1} (j+1)n_j + k + 1 = x\}} \binom{n_0 + \cdots + n_{k-1}}{n_0,\ldots,n_{k-1}} \cdot q_2^{n_0} (q_1 p_2)^{\sum_1^{k-1} n_j} p_1^{\sum_2^{k-1} (j-1)n_j} \right],$$
$$x = k+1, k+2, \ldots,$$

$$(2.24\text{c})\quad P(X_1 = x, \textit{starts with an } H) = \sum_{i=1}^{k-1} p_2 p_1^{k+i-1} q_1 \cdot \left[\sum_{\{(n_0,\ldots,n_{k-1}): \sum_0^{k-1} (j+1)n_j + k + 1 = x - i\}} \binom{n_0 + \cdots + n_{k-1}}{n_0,\ldots,n_{k-1}} \cdot q_2^{n_0} (q_1 p_2)^{\sum_1^{k-1} n_j} p_1^{\sum_2^{k-1} (j-1)n_j} \right],$$

$$x = k+2, k+3, \ldots.$$

Let us determine the contribution of the second part toward $P(X_1 = x, Y = y, W = w)$. For fixed x, y, w, let us consider $(n_0, \ldots, n_{k-1})$ such that

$$\begin{cases} n_0 + 2n_1 + \cdots + kn_{k-1} + k + 1 = x, \\ n_0 + n_1 + \cdots + n_{k-1} + 1 = w, \\ n_1 + \cdots + n_{k-1} = y. \end{cases}$$

In any general term of the second part, clearly the left sides of the first two equations respectively represent the number of trials and the number of T's in a sequence respectively. Also note that $n_1 + \cdots + n_{k-1} + 1$ being the exponent of q_1p_2 is the number of left turns which is one more than the number of right turns. This justifies the third equality. The implication is that the number of sequences, each of which has probability $(q_1p_2)^{y+1}q_2^{w-y-1}p_1^{x-y-w-1}$, is $\sum_{A(x-1,y,w-1)} \binom{w-1}{n_0,\ldots,n_{k-1}}$. We can have a similar implication of the last part. All these lead to the following:

PROPOSITION 2.4. *For $x = k$, $P(X_1 = x) = p_1^k$; and for $x > k$,*

$$\begin{aligned} (2.25)\quad & P(X_1 = x, Y = y, W = w, \textit{starts with a } T) \\ &= (q_1p_2)^{y+1}q_2^{w-y-1}p_1^{x-y-w-1} \sum_{A(x-1,y,w-1)} \binom{w-1}{n_0, \ldots, n_{k-1}}, \\ (2.26)\quad & P(X_1 = x, Y = y, W = w, \textit{starts with an } H) \\ &= (q_1p_2)^{y}q_2^{w-y}p_1^{x-y-w} \sum_{i=1}^{k-1} \sum_{A(x-i-1,y-1,w-1)} \binom{w-1}{n_0, \ldots, n_{k-1}} \end{aligned}$$

where A is defined in Proposition 2.2.

COMBINATORIAL PROOF. It is natural to separate out two possibilities, viz., those sequences starting with a T and those with an H. For sequences starting with a T, the probability of each sequence is obtained as

$$(q_1p_2)^{y+1}q_2^{w-y-1}p_1^{x-w-y-1}, \qquad x > k,$$

when we realize that each of $y+1$ left turns contributes q_1p_2 and there cannot be any more q_1 and p_2. Let n_i's be the same as before. If we fix the first outcome to be a T, then the number of sequences turns out to be the summation in (2.25) and this checks the expression in (2.25).

In the case of a sequence starting with a H, it can be verified that the probability for each sequence is

$$(q_1p_2)^{y}q_2^{w-y}p_1^{x-y-w}, \qquad x > k.$$

However, for finding the number of sequences consider arrangements of n_i subsequences of type i, $i = 1, \ldots, k-1$, such that $n_1 + \cdots + n_{k-1} = y$. The number of arrangements is

$$\binom{y}{n_1, \ldots, n_{k-1}}.$$

Given any such arrangement, we place $w - y$ remaining T's in between these subsequences and after the last subsequence, the number of placement being

$$\binom{w-1}{w-y}.$$

Therefore the total number of sequences which start with an H is

$$\sum_{A(x,y,w)} \frac{y}{w}\binom{w}{n_0, n_1, \ldots, n_{k-1}}. \tag{2.27}$$

What remains to prove for checking (2.26) is the following identity:

$$\sum_{i=1}^{k-1} \sum_{A(x-i-1,y-1,w-1)} \binom{w-1}{n_0, n_1, \ldots, n_{k-1}} = \sum_{A(x,y,w)} \frac{y}{w}\binom{w}{n_0, n_1, \ldots, n_{k-1}}. \tag{2.28}$$

If we put $m_j = n_j$ for $j \neq i$ and $m_i = n_i + 1$ of the inner summand of the left side, then it becomes

$$\sum_{A(x,y,w)} \binom{w-1}{m_0, m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_{k-1}}.$$

Therefore, the left side can be seen to be

$$\sum_{A(x,y,w)} \left[\binom{w}{w-y, n_1, \ldots, n_{k-1}} - \binom{w-1}{w-y-1, n_1, \ldots, n_{k-1}}\right]$$

which simplifies to the right side of (2.28) and this completes the proof.

A direct combinatorial proof of (2.26) can be given by considering sequences beginning with a subsequence of type i, $i = 1, \ldots, k-1$. However, our approach helps to simplify (2.26).

A generalization of the inverse Polýa-Eggenberger urn problem along the line of dependent trials or two-coin tossing game can be formulated as follows:

Consider two urns, urn I and urn II. Urn I contains a_1 white balls and b_1 black balls whereas urn II contains a_2 white balls and b_2 black balls. Balls are drawn at random sequentially. If at any stage a white (black) ball is picked up, then the next draw is from urn I(II). When a ball is picked up from urn I(II), it is returned to the same urn with s_1 (s_2) balls of the same colour added.

We may easily identify urn I(II) as coin 1(2) and white (black) ball as $H(T)$. If we retain notations X_i, Y, W for the new formulation without any confusion, then any sequence in $\{X_2 = x, Y = y, W = w\}$ will have the probability

$$\frac{\beta_1^{[y]}\alpha_1^{[x-y-w-1]}\alpha_2^{[y+1]}\beta_2^{[w-y]}}{(\alpha_1+\beta_1)^{[x-w-1]}(\alpha_2+\beta_2)^{[w+1]}} \tag{2.29}$$

where $\alpha_i = \frac{a_i}{s_i}$, $\beta_i = \frac{b_i}{s_i}$, $i = 1, 2$, and $m^{[u]} = m(m+1)\cdots(m+u-1)$. However, the expression for the number of sequences will not change. Thus $P(X_2 = x, Y = y, W = w)$ will be (2.22) where $q_1^y p_2^{y+1} q_2^{w-y} p_1^{x-y-w-1}$ is replaced by (2.29). We can state the generalization as

PROPOSITION 2.5. *The joint probabilities $P(X_i = x, Y = y, W = w)$ $i = 1, 2$ in the inverse Polýa-Eggenberger urn model are obtained by replacing*

$$(2.30) \qquad p_j^r q_j^s \quad by \quad \frac{\alpha_j^{[r]} \beta_j^{[s]}}{(\alpha_j + \beta_j)^{[r+s]}} \qquad j = 1, 2$$

in the corresponding expressions in Propositions 2.2 *and* 2.3.

At the end we offer the following remarks:

(i) We assert that X_i is a proper random variable in the sense that $\sum_x P(X_i = x) = 1$. This will be proved in Section 4.

(ii) If we think of the problem as of two-state Markov dependent trials with initial probabilities $P(s) = P_0 = 1 - P(f)$ then the distribution of X_i provides the conditional distribution of the number of trials. To make it unconditional, it is only an elementary step. Thus we have lost no information by considering a coin tossing game as an alternative.

(iii) Propositions 2.2 and 2.4 are indicative of the fact (X_2, Y, W) and (X_1, Y, W, U) where $U = 1$ or 0 if the first trial is a T or H, form minimal sufficient partitions in the respective games (Blackwell and Girshick (1954)). By combinatorial methods what is being done is to count sequences for which the value of minimal sufficient statistics is fixed.

(iv) From the relations between Proposition 2.1 and Proposition 2.2 and between Proposition 2.3 and Proposition 2.4 it is not difficult to obtain the distribution of X_2 from Proposition 2.2 or X_1 from Proposition 2.4.

(v) The replacement given in (2.30) can be utilized in an obvious manner to obtain further generalization. This is precisely what is suggested at the end of Section 3. Proposition 2.5 has the striking similarity with the relation between the binomial and the hypergeometric distributions.

3. Generalizations

An immediate generalization is to consider (negative binomial type) the probability of the number of trials needed in order to get the r-th k successive H's for the first time (i.e. at no stage there are more than k successive heads). We may go one step further to have ballot type of constraints on the outcomes. Let us introduce the following notations:

n_{ij}: number of subsequences of type i in the first j outcomes. (When $i = k$, the subsequence is $H \cdots HT$ (there are k H's) except the last subsequence which consists of k H's without any T; this modification is necessary because every run of H's of length k except the last one is always followed by a T),

$X_i(r)$: $\min\{j : n_{kj} = \sum_{l=0}^{k-1} \mu_l n_{lj} + r$, game starts with coin $i\}$, where μ_l's are non-negative integers and r a positive integer,

$G_i(r,z)$: $\sum_n P(X_i(r)=n)z^n$.

If $X_i(r)=x$, then it is easy to check that for every $j<x$, $n_{kj}<\sum_{i=0}^{k-1}\mu_i n_{ij}+r$. These are ballot-type restrictions as enunciated through lattice paths in Mohanty (1979) (see exercise 10, p. 25) and $X_i(r)$ represents the number of trials needed to have $n_{kj}=\sum_{i=0}^{k-1}\mu_i n_{ij}+r$ satisfied for the first time. Thus, when μ_i's are zeros we have the situation of the r-th k successive H's occurring for the first time.

For reasons which will be clearer soon, let us first proceed with a combinatorial argument. Assume that there are n_i subsequences of type i, $i=0,1,\ldots,k$ when the game stops. Consider starting with coin 2. For $X_2=x$, $Y=y$ $(y\geq k-1)$, $W=w$ $(w\geq r-1,\ w\geq t)$, the probability of each sequence as in Section 2 is

$$q_1^y q_2^{w-y} p_2^{y+1} p_1^{x-y-w-1}$$

and therefore

$$\left\{\begin{array}{l} n_0+\cdots+n_k-1=w, \\ n_1+\cdots+n_k-1=y, \\ n_0+2n_1+\cdots+(k+1)n_k-1=x, \\ n_k=\displaystyle\sum_{i=0}^{k-1}\mu_i n_i+r. \end{array}\right. \tag{3.1}$$

(Note $n_{ix}=n_i$ for $i=0,1,\ldots,k$.)

The number of sequences which satisfy $n_{kj}<\sum_{i=0}^{k-1}\mu_i n_{ij}+r$, $j<x$ and $n_k=\sum_{i=0}^{k-1}\mu_i n_i+r$ is

$$\frac{r}{n_0+\cdots+n_k}\binom{n_0+\cdots+n_k}{n_0,n_1,\ldots,n_k} \tag{3.2}$$

(see Mohanty (1979), Exercise 10, p. 25).

Thus, we have a generalization of Proposition 2.2.

THEOREM 3.1.

$$\begin{aligned} &P(X_2(r)=x,Y=y,W=w) \\ &\quad= q_1^y q_2^{w-y} p_2^{y+1} p_1^{x-y-w-1}\sum_{B(x,y,w)}\frac{r}{w+1}\binom{w+1}{n_0,\ldots,n_k} \end{aligned} \tag{3.3}$$

where $B(x,y,w)=\{(n_0,\ldots,n_k)$: (3.1) is satisfied$\}$.

When $\mu_i=0$ for all i, then we have

$$\begin{aligned} &P(X_2(r)=x) \\ &=\Bigg[\sum_{\{(n_0,\ldots,n_{k-1}):\sum_{j=0}^{k-1}(j+1)n_j=x-(k+1)r+1\}}\binom{n_0+\cdots+n_{k-1}+r-1}{n_0,\ldots,n_{k-1},r-1} \\ &\qquad\cdot q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_{k+1}+r-1}p_2p_1^{x-n_0-2(n_1+\cdots+n_{k-1}+r-1)-1}\Bigg]. \end{aligned} \tag{3.4}$$

In case of $X_1(r)$ consider those sequences starting with a T. An argument similar to the one in the previous section applies. The only change in (2.25) and (2.26) will be the number of sequences which by the application of (3.2) turns out to be

$$\sum_{B(x-1,y,w-1)} \frac{r+\mu_0}{w}\binom{w}{n_0,\dots,n_k}.$$

Therefore, Proposition 2.4 is generalized as follows:

THEOREM 3.2. *For $x > k$,*

(3.5) $P(X_1(r)=x, Y=y, W=w, \textit{starts with a } T)$

$$= (q_1p_2)^{y+1}q_2^{w-y-1}p_1^{x-y-w-1}\sum_{B(x-1,y,w-1)} \frac{r+\mu_0}{w}\binom{w}{n_0,\dots,n_k},$$

and for $x > k$

(3.6) $P(X_1(r)=x, Y=y, W=w, \textit{starts with an } H)$

$$= (q_1p_2)^{y}q_2^{w-y}p_1^{x-y-w}\sum_{i=1}^{k-1}\sum_{B(x-1,y-1,w-1)} \frac{r+\mu_i}{w}\binom{w}{n_0,\dots,n_k}.$$

If $x=k$, then $r=1$ and $n_i=0$, $i=0,\dots,k-1$. From (3.6) we have $P(X_1(r)=k)=p_1^k$ which is the same as (2.24a). When $\mu_i=0$ for all i, (3.6) has an alternative form as given by

(3.7) $P(X_1(r)=x, Y=y, W=w, \text{starts with an } H)$

$$= (q_1p_2)^{y}q_2^{w-y}p_1^{x-y-w}\sum_{B(x,y,w)} \frac{y}{w}\binom{w}{n_0,\dots,n_{k-1},r-1},$$

which is obtained by following the argument leading to (2.27). An expression for $P(X_1(r)=x)$ similar to (3.4) can be obtained but is omitted.

Let us return to $G_i(r,z)$. It is well known that the negative binomial distribution is obtained as the r-th convolution of the geometric distribution with itself, i.e., the p.g.f. of the negative binomial distribution is obtained as the r-th power of the p.g.f. of the geometric distribution. However, $G_i(r,z)$ does not seem to have this regenerative property. On the other hand, if the k-th type subsequence which appears at the end becomes the usual k-th type subsequence (i.e., it consists of k H's followed by a T) then it looks reasonable to expect the regenerative convolution property. With this change in mind, let the notations be $\tilde{X}_i(r)$ and $\tilde{G}_i(r,z)$. Then $\tilde{G}_2(r,z)$ is given by (see (2.19))

(3.8) $\tilde{G}_2(r,z) = \sum_x P(\tilde{X}_2(r)=x)z^x$

$$= \sum_x \Bigg[\sum_{\{(n_0,\dots,n_k):\sum_{j=0}^{k-1}(j+1)n_j=x\}} \frac{r}{n_0+\cdots+n_k}\binom{n_0+\cdots+n_k}{n_0,\dots,n_k}$$

$$\left. \cdot q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_k} p_1^{n_2+2n_3+\cdots+(k-1)n_k} z^{n_0+2n_1+\cdots+(k+1)n_k} \right]$$

$$\begin{aligned}
&= (q_1p_2p_1^{k-1}z^{k+1})^r \\
&\quad \cdot \left[\sum_{n_0=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{r}{r+\sum_0^{k-1}(\mu_i+1)n_i} \right. \\
&\quad \cdot \binom{r+\sum_0^{k-1}(\mu_i+1)n_i}{n_0,\ldots,n_k} \\
&\quad \cdot (q_2 z (q_1p_2p_1^{k-1}z^{k+1})^{\mu_0})^{n_0} \\
&\quad \left. \cdot \prod_{j=0}^{k-1} (q_1p_2p_1^j z^{j+2}(q_1p_2p_1^{k-1}z^{k+1})^{\mu_{j+1}})^{n_{j+1}} \right].
\end{aligned}$$

By using (3.1) and the following implicit relation in Mohanty (1966*b*), we have

$$\tilde{G}_2(r,z) = (q_1p_2p_1^{k-1}z^{k+1})^r v^r \tag{3.9}$$

where v satisfies the implicit relation

$$\begin{aligned}
v\Bigg(1 &- q_2 z (q_1p_2p_1^{k-1}z^{k+1})^{\mu_0} v^{\mu_0} \\
&- \sum_{j=0}^{k-2} q_1p_2p_1^j z^{j+2} (q_1p_2p_1^{k-1}z^{k+1})^{\mu_{j+1}} v^{\mu_{j+1}} \Bigg) = 1.
\end{aligned} \tag{3.10}$$

With a renewal argument one can show that $\tilde{X}_2(r)$ is the sum of r $\tilde{X}_2(1)$'s and therefore

$$\tilde{G}_2(r,z) = [\tilde{G}_2(1,z)]^r \tag{3.11}$$

(see Mohanty (1979), p. 133).

From (3.9) and (3.10) it follows that $\tilde{G}_2(1,z)$ satisfies the implicit relation

$$\begin{aligned}
\tilde{G}_2(1,z) &= q_1p_2p_1^{k-1}z^{k+1} + q_2 z (\tilde{G}_2(1,z))^{\mu_0+1} \\
&\quad + \sum_{j=1}^{k-1} q_1p_2p_1^{j-1} z^{j+1} (\tilde{G}_2(1,z))^{\mu_j+1}
\end{aligned} \tag{3.12}$$

which is like (2.14) and can be alternatively established in a similar way by having the relation

$$\tilde{G}_2(1,z) = q_1p_2p_1^{k-1}z^{k+1} + q_2 z \tilde{G}_2(\mu_0+1,z) + \sum_{j=1}^{k-1} q_1p_2p_1^{j-1}z^{j+1}\tilde{G}_2(\mu_j+1,z)$$

and using (3.11).

In case of $\tilde{G}_1(r,z)$, we have

$$\tilde{G}_1(r,z) = \tilde{G}_1(1,z)[\tilde{G}_2(1,z)]^{r-1}, \tag{3.13}$$

with the implicit relation stated in the following lemma.

LEMMA 3.1.

$$\tilde{G}_1(1,z) = q_1 p_1^k z^{k+1} + \sum_{j=0}^{k-1} q_1 p_1^j z^{j+1} (\tilde{G}_2(1,z))^{\mu_j+1}, \tag{3.14}$$

where $\tilde{G}_2(1,z)$ *is given by* (3.12). $\tilde{G}_2(r,z)$ *and* $\tilde{G}_1(r,z)$ *are expressed by* (3.11) *and* (3.13).

Note that

$$\begin{aligned} \tilde{G}_i(r,z) &= q_1 z G_i(r,z), \\ P(\tilde{X}_i(r) = x+1) &= q_1 P(X_i(r) = x). \end{aligned} \tag{3.15}$$

Now we turn to the analogue of the binomial distribution. In a natural binomial type generalization we may require the probability of having x runs of H's of length k in n trials. Denote by T_i the corresponding random variable when the game starts with coin i. Assume that the length of every run of H's is not greater than k. As earlier, let n_i be the number of type i subsequences $i = 0, 1, \ldots, k$ (modified type k subsequences are considered) in a sequence. Then we have

THEOREM 3.3.

$$\begin{aligned} P(T_2 = x) = \sum_{j=1}^{k-1} &\Bigg[\sum_{\{(n_0,\ldots,n_k): \sum_{i=0}^k (i+1)n_i = n-j, n_k = x\}} \binom{n_0 + \cdots + n_k}{n_0, \ldots, n_k} \\ &\qquad \cdot q_2^{n_0} (q_1 p_2)^{n_1 + \cdots + n_k} p_2 p_1^{n - n_0 - 2(n_1 + \cdots + n_k) - 1} \Bigg] \\ &+ \Bigg[\sum_{\{(n_0,\ldots,n_k): \sum_{i=0}^k (i+1)n_i = n, n_k = x\}} \binom{n_0 + \cdots + n_k}{n_0, \ldots, n_k} \\ &\qquad \cdot q_2^{n_0} (q_1 p_2)^{n_1 + \cdots + n_k} p_1^{n - n_0 - 2(n_1 + \cdots + n_k)} \Bigg] \\ &+ \Bigg[\sum_{\{(n_0,\ldots,n_k): \sum_{i=0}^k (i+1)n_i = n-k, n_k = x-1\}} \binom{n_0 + \cdots + n_k}{n_0, \ldots, n_k} \\ &\qquad \cdot q_2^{n_0} (q_1 p_2)^{n_1 + \cdots + n_k} p_2 p_1^{n - n_0 - 2(n_1 + \cdots + n_k) - 1} \Bigg]. \end{aligned} \tag{3.16}$$

In (3.16) j ($j = 0, 1, \ldots, k$) *represents the number of H's at the end of the sequence.*

It is remarked that the derivation of $P(T_2 = x)$ is not difficult and the same is true for $P(T_1 = x)$. Because of its length, the expression for $P(T_1 = x)$ is not given.

Next we present another binomial type generalization. For this purpose, denote by $L_i(n)$ the length of the largest run of H's in n trials, given that the game starts with coin i. Let n_i's and j have the same meaning as in the previous case. For the event $\{L_2(n) \le k\}$, the probability of each sequence is

$$q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_k}p_2p_1^{n_2+2n_3+\cdots+(k-1)n_k+j-1}, \qquad \text{if} \quad j = 1, \ldots, k$$

and is

$$q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_k}p_1^{n_2+2n_3+\cdots+(k-1)n_k}, \qquad \text{if} \quad j = 0.$$

A similar expression for $L_1(n)$ is obtained.

Therefore, we have

THEOREM 3.4.

$$(3.17) \quad P(L_2(n) \le k) = \left[\sum_{\{(n_0,\ldots,n_k):\sum_{i=0}^{k}(i+1)n_i=n\}} \binom{n_0+\cdots+n_k}{n_0,\ldots,n_k} \cdot q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_k}p_1^{n-n_0-2(n_1+\cdots+n_k)}\right]$$
$$+\sum_{j=1}^{k}\left[\sum_{\{(n_0,\ldots,n_k):\sum_{i=0}^{k}(i+1)n_i=n-j\}} \binom{n_0+\cdots+n_k}{n_0,\ldots,n_k} \cdot q_2^{n_0}(q_1p_2)^{n_1+\cdots+n_k}p_2p_1^{n_2+2n_3+\cdots+(k-1)n_k+j-1}\right]$$

and

$$(3.18) \quad P(L_1(n) \le k) = \sum_{\{(n_0,\ldots,n_k):\sum_{i=0}^{k}(i+1)n_i=n\}} \left[\binom{n_0+\cdots+n_k-1}{n_0-1,n_1,\ldots,n_k} \cdot (q_1p_2)^{n_1+\cdots+n_k}q_1q_2^{n_0-1}p_1^{n-n_0-2(n_1+\cdots+n_k)}\right.$$
$$\left. +\sum_{u=1}^{k}\left\{\binom{n_0+\cdots+n_k-1}{n_0,\ldots,n_{u-1},n_u-1,n_{u-1},\ldots,n_k} \cdot (q_1p_2)^{n_1+\cdots+n_k-1}q_1q_2^{n_0}p_1^{n-n_0-1-2(n_1+\cdots+n_k-1)}\right\}\right]$$
$$+\sum_{j=1}^{k}\sum_{\{(n_0,\ldots,n_k):\sum_{i=0}^{k}(i+1)n_i=n-j\}}$$

$$\left[\binom{n_0+\cdots+n_k-1}{n_0-1,n_1,\ldots,n_k}\right.$$
$$\cdot (q_1p_2)^{n_1+\cdots+n_k+1}q_2^{n_0-1}p_1^{n-n_0+1-2(n_1+\cdots+n_k+1)}$$
$$+\sum_{u=1}^{k}\left\{\binom{n_0+\cdots+n_k-1}{n_0,\ldots,n_{u-1},n_u-1,n_{u+1},\ldots,n_k}\right.$$
$$\left.\left.\cdot (q_1p_2)^{n_1+\cdots+n_k}q_0^{n_0}p_1^{n-n_0-2(n_0+\cdots+n_k)}\right\}\right].$$

Here and elsewhere some of the combinatorial methods are adaptations of those in Philippou and Muwafi (1982) and Philippou and Makri (1985). Following remark (v) at the end of Section 2, we can obtain extensions of the inverse Polýa-Eggenberger type by the use of (2.30).

4. Distribution of order k

So far, we have been deriving probabilities of events resembling geometric, negative binomial and binomial distributions without checking whether or not such probabilities form a distribution. For instance let us examine (2.20). It is easily seen from (2.19) that

$$(4.1)\quad G_2(1)=\left[\sum_{n_0=0}^{\infty}\cdots\sum_{n_{k-1}=0}^{\infty}\frac{1}{n_0+\cdots+n_{k-1}+1}\binom{n_0+\cdots+n_{k-1}+1}{n_0,\ldots,n_{k-1},1}\right.$$
$$\left.\cdot q_2^{n_0}\left\{\prod_{i=1}^{k-1}(q_1p_2p_1^{i-1})^{n_i}\right\}p_2p_1^{k-1}\right]$$

where the summand is of the form (2.30) in Mohanty (1966*b*) since

$$q_2+\sum_{i=1}^{k-1}(q_1p_2p_1^{i-1})+p_2p_1^{k-1}=1.$$

Similarly from (2.23) we have

$$(4.2)\qquad G_1(1)=p_1^k+\sum_{i=0}^{k-1}\frac{p_2p_1^{k+i-1}q_1}{p_2p_1^{k-1}}=1.$$

Therefore we have the following:

THEOREM 4.1. *$\{P(X_2=x)\}$ given in Proposition 2.1 and $\{P(X_1=x)\}$ given in Proposition 2.3 are proper distributions.*

The distributions in (2.20) and $\{(2.24a), (2.24b), (2.24c)\}$ are called *geometric type* distributions of order k.

In a ballot type general case, consider (3.8) and get

$$\tilde{G}_2(r,1) = \left[\sum_{n_0=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{r}{r+\sum_{i=0}^{k-1}(\mu_i+1)n_i} \binom{r+\sum_{i=0}^{k-1}(\mu_2+1)n_i}{n_0,\ldots,n_k} \cdot q_2^{n_0} \left\{\prod_{i=1}^{k-1}(q_1 p_2 p_1^{i-1})^{n_i}\right\} (q_1 p_2 p_1^{k-1})^{\sum_{i=0}^{k-1}\mu_i n_i + r}\right].$$

In the last factor replace $q_1p_2p_1^{k-1}$ by $p_2p_1^{k-1}$. In other words, in addition to the transition probability $P(s \mid f) = p_2$ we further assume that $P(s \mid$ there is a run of $k-1$ successes each with probability $p_1) = p_2$. Then by Mohanty (1966*b*) we prove the following:

THEOREM 4.2. *A new random variable $X_2^*(r)$ where $P(X_2^*(r)=x)$ is given by*

$$(4.3)\qquad P(X_2^*(r)=x) = \left[\sum_{\{(n_0,\ldots,n_{k-1}):\sum_{i=0}^{k-1}(i+1)n_i+kn_k=x\}} \frac{r}{r+\sum_{i=0}^{k-1}(\mu_i+1)n_i}\binom{r+\sum_{i=0}^{k-1}(\mu_i+1)n_i}{n_0,\ldots,n_k} \cdot q_2^{n_0}\left\{\prod_{i=1}^{k-1}(q_1p_2p_1^{i-1})^{n_i}\right\}(p_2p_1^{k-1})^{n_k}\right],$$
$$x = kr, kr+1, \ldots$$

has a proper distribution.

The distribution of $X_2^*(r)$ is called a *ballot type* of distribution of order k. When $\mu_i = 0$ for all i (remember that $n_k = \sum_{i=0}^{k-1}\mu_i n_i + r$), the distribution is called a *negative binomial type* distribution of order k. Let $X(r)$ be the random variable having the negative binomial type distribution as suggested above. Then we have

COROLLARY 4.1.

$$(4.4)\quad P(X(r)=x) = \left[\sum_{\{(n_0,\ldots,n_{k-1}):\sum_{i=0}^{k-1}(i+1)n_i+kr=x\}}\binom{n_0+\cdots+n_{k-1}+r-1}{n_0,\ldots,n_{k-1},r-1} \cdot q_2^{n_0}\left\{\prod_{i=1}^{k-1}(q_1p_2p_1^{i-1})^{n_i}\right\}(p_2p_1^{k-1})^{r}\right],$$
$$x = kr, kr+1, \ldots.$$

In (4.4) let q_1 and $q_2 \to 0$ such that $rq_1 \to \lambda_1$ and $rq_2 \to \lambda_2$. The limiting probability turns out to define a distribution. The corresponding random variable denoted by $X(\lambda_1, \lambda_2)$ is said to have a *Poisson type* distribution of order k.

COROLLARY 4.2.

$$(4.5) \qquad P(X(\lambda_1, \lambda_2) = x) = \sum_{\{(n_0,\dots,n_{k-1}):\sum_{i=0}^{k-1}(i+1)n_i = x\}} \frac{e^{-\lambda_2}\lambda_2^{n_0}}{n_0!} \cdot \frac{e^{-(k-1)\lambda_1}\lambda_1^{n_1+\cdots+n_{k-1}}}{n_1!\cdots n_{k-1}!}, \qquad x = 0, 1, \dots.$$

When $p_1 = p_2 = p$, $q_1 = q_2 = q$ in (4.4) and $\lambda_1 = \lambda_2 = \lambda$ in (4.5), these reduce to the respective distributions of order k for independent trials, as can be verified in Philippou *et al.* (1983).

Finally, the logarithmic series distribution of order k in Aki *et al.* (1984) is extended to the case of Markov dependent trials. Take the limit of the conditional distribution of $X(r)$ given $X(r) > kr$, as $r \to 0$ (use (4.4)). The limit turns out to provide a distribution which may be termed as a *logarithmic series type* distribution of order k (see Johnson and Kotz (1977), Chapter 7). Denoting by X the corresponding random variable, we have

COROLLARY 4.3.

$$(4.6) \qquad P(X = x) = \sum_{\{(n_0,\dots,n_{k-1}):\sum_{i=0}^{k-1}(i+1)n_i = x\}} \frac{-(n_0 + \cdots + n_{k-1} - 1)!}{(\log(p_2 p_1^{k-1}))\prod_{i=1}^{k-1} n_i!} \cdot q_2^{n_0}\prod_{i=1}^{k-1}(q_1 p_2 p_1^{i-1})^{n_i}, \qquad x = 1, 2, \dots.$$

In developing distributions of order k we have dealt with games starting with coin 2. So far as games starting with coin 1 are concerned, it is observed in (2.14) that the emerging geometric type distribution is obtained by compounding the geometric type distribution arising out of games starting with coin 2. Similar compounding will result in a new set of distribution of order k. Because of the length the new analogous distributions are not presented.

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