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POSITIVE DEPENDENCE ORDERINGS AND STOPPING TIMES

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Abstract. We study the positive dependence of pairs of stochastic processes and examine its relation with the properties of certain stopping times. Some special cases, such as dependent random walks, Gaussian processes and exchangeable sequences of elliptically contoured random variables, are taken into account.

Key words and phrases: Stochastic orderings, bivariate random walks, bivariate Gaussian processes, exchangeable sequences.

1. Introduction

The concept of positive dependence has been widely studied in the literature. The reader is referred to Lehmann (1966) for the original contribution, to Kimeldorf and Sampson (1987, 1989) for comprehensive surveys, and to Block *et al.* (1990) for a collection of recent contributions on the subject.

In particular, an ordering of positive dependence for pairs of random variables has been defined and characterized in different ways. In this paper we provide a characterization of the positive dependence ordering in terms of stopping times of stochastic processes composed of independent copies of the original pairs of random variables. In this vein, we try to shed some light on the idea of positive dependence of bivariate stochastic processes by relating it to the behavior of suitably chosen stopping times.

To the best of our knowledge, this idea has not been developed in the literature. It should be pointed out, though, that the link between correlation and hitting times is at the core of Slepian's inequality (see, e.g., Adler (1990)). In a different direction, a dynamic approach to association has been proposed by Arjas and Norros (1984).

In Section 2 a characterization of the positive dependence ordering in terms of stopping times is presented. In Section 3 the issue of dependence of bivariate processes is dealt with. First, the case of random walks is analyzed; a family of stopping times is suggested, and it is shown that their Laplace ordering is equivalent to a condition on the bivariate random walks which can be seen as a natural dependence comparison. Then, Gaussian processes are studied. Under a weak dependence condition, an inequality among expected values of suprema of difference processes is obtained. A stronger dependence condition yields stochastic order of certain hitting times. For a family of Brownian motions, it is shown that the stochastic ordering of these hitting times is equivalent to a natural dependence index.

The latter is indeed one of the few cases in which we were able to establish necessary and sufficient conditions for the ordering of stopping times which are meaningful in terms of dependence. Another such case is almost trivial: convex combinations of processes. There are two main sources of difficulties in establishing such results in general frameworks: one is that they become rapidly very hard to prove, from a technical point of view, as soon as the context becomes less than very well-known; a second source is that there is no guarantee that the same type of ordering among stopping times and the same dependence conditions are to be taken into account in different situations.

Thus, the fact that we deal, throughout the paper, with a collection of special cases is certainly due to our lack of capability to prove more general results, but it is presumably due also to the fact that very general results which apply in all setups need not exist.

Section 4 is devoted to comparing two infinite exchangeable elliptically contoured sequences. Suitable stopping times are defined, the stochastic ordering of which is shown to be equivalent to the ordering of the correlation coefficients among the variables within each sequence.

2. Positive dependence ordering

The positive dependence ordering $\leq_{\mathrm{PD}}$ for pairs of random variables can be characterized in different ways. We say that the pair (X_1, X_2) is less positively dependent than (Y_1, Y_2), and write $(X_1, X_2) \leq_{\mathrm{PD}} (Y_1, Y_2)$, if $\mathrm{E}\phi(X_1, X_2) \leq \mathrm{E}\phi(Y_1, Y_2)$ for every 2-monotone function ϕ (recall that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is 2-monotone if $\phi(x_1, x_2) + \phi(y_1, y_2) \geq \phi(x_1, y_2) + \phi(y_1, x_2)$, $\forall x_1 \leq y_1$, $x_2 \leq y_2$).

An equivalent characterization in terms of distribution functions is: $(X_1, X_2) \leq_{\mathrm{PD}} (Y_1, Y_2)$ if $F_{X_1} = F_{Y_1}$, $F_{X_2} = F_{Y_2}$ and $F_{X_1,X_2}(s,t) \leq F_{Y_1,Y_2}(s,t)$ $\forall s, t \in \mathbb{R}^2$.

We shall give a different characterization in terms of stopping times. For this purpose we need a preliminary result relating set orderings and stopping times.

Let X and Y be two random variables with values in a Polish space E with laws P and Q respectively. Let $\hat{X} = \{\hat{X}(n) \mid n \in \mathbb{N}\}$ and $\hat{Y} = \{\hat{Y}(n) \mid n \in \mathbb{N}\}$ be two sequences of independent copies of X and Y. For a Borel set H define $T_H^X = \inf\{n \in \mathbb{N} \mid \hat{X}(n) \in H\}$.

LEMMA 2.1. *$P(H) \leq Q(H)$ if and only if $T_H^X \geq_{\mathrm{st}} T_H^Y$, where $\geq_{\mathrm{st}}$ is the usual stochastic ordering.*

PROOF. From the independence assumption it follows easily that $\mathrm{Prob}(T_H^X > k) = [1 - P(H)]^k \geq [1 - Q(H)]^k = \mathrm{Prob}(T_H^Y > k)$. □

The ordering of positive dependence can now be characterized resorting to the above setup.

PROPOSITION 2.1. *Let* $F_{X_1} = F_{Y_1}$ *and* $F_{X_2} = F_{Y_2}$. *Then* $(X_1, X_2) \leq_{\rm PD} (Y_1, Y_2)$ *if and only if* $T_H^{(X_1,X_2)} \geq_{\rm st} T_H^{(Y_1,Y_2)}$ *for every set* H *of the form* $(-\infty, x_1) \times (-\infty, x_2)$.

PROOF. Let P be the law of (X_1, X_2) and let Q be the law of (Y_1, Y_2). Let also $\mathcal{A} = \{(-\infty, x_1) \times (-\infty, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. By definition, $(X_1, X_2) \leq_{\rm PD} (Y_1, Y_2)$ if and only if $P(H) \leq Q(H) \forall H \in \mathcal{A}$, and the lemma above ensures that this is in turn equivalent to $T_H^{(X_1,X_2)} \geq_{\rm st} T_H^{(Y_1,Y_2)}$, $\forall H \in \mathcal{A}$. □

3. Dependence of bivariate processes

In this section we study the positive dependence of pairs of special types of stochastic processes.

The setup is as follows: we take four real valued stochastic processes, A^1, A^2, B^1 and B^2, each with the same law, and consider the two pairs $\boldsymbol{A} = (A^1, A^2)$ and $\boldsymbol{B} = (B^1, B^2)$, which may display a different dependence structure. We let the four processes start at zero, and consider the random variables

$$T_\ell^A = \inf\{t \geq 0 \mid |D_t^A| \geq \ell\} \qquad \ell > 0, \tag{3.1}$$

where $D^A = A^1 - A^2$. When there is no ambiguity, we shall write T^A instead of T_ℓ^A. D^B and T^B are defined similarly. We shall refer to T^A and T^B as *hitting times*.

Intuitively, in case of perfect positive dependence the difference process D^A is constant at zero, and $T_\ell^A = \infty \forall \ell > 0$. Thus, it seems reasonable to try and quantify the assertion: *stochastically greater hitting times correspond to more positive dependent pairs of processes.*

Unfortunately, we are able to provide necessary and sufficient conditions for the relation $T_\ell^A \geq_{\rm st} T_\ell^B$ only in the trivial case of convex combinations of processes. For a pair $\boldsymbol{X} = (X^1, X^2)$ of stochastic processes, we denote with U^X the covariance of the increments, namely

$$U^X(s,t) = \mathrm{Cov}(X_t^1 - X_s^1, X_t^2 - X_s^2) \qquad s, t \in \mathbb{R}_+.$$

Let $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ be independent, real valued, zero mean stochastic processes on $\mathbb{R}_+$ with continuous paths, and let $A^j = k^A A^{(0)} + (1 - k^A) A^{(j)}$ $0 < k^A < 1$; $j = 1, 2$. Let $B^{(0)}$, $B^{(j)}$ and B^j be defined similarly. Suppose that $A^{(0)}$ and $B^{(0)}$ have the same probability law and that $A^{(1)}$, $A^{(2)}$, $B^{(1)}$ and $B^{(2)}$ have the same probability law. Then $k^A \leq k^B \Leftrightarrow T_\ell^A \leq_{\rm st} T_\ell^B$ for every $\ell > 0$. In fact

$$\mathrm{Prob}(T_\ell^A \leq t) = \mathrm{Prob}\left(\sup_{s \leq t} |D_s^A| \geq \ell\right) = \mathrm{Prob}\left(\sup_{s \leq t} |A_s^{(1)} - A_s^{(2)}| \geq \frac{\ell}{1 - k^A}\right)$$

so that $\mathrm{Prob}(T_\ell^A \le t)$ is decreasing in k^A.

It is a simple matter to check that $U^A(s,t) = (k^A)^2 \operatorname{Var}(A_t^{(0)} - A_s^{(0)})$. Hence $U^A < U^B \Leftrightarrow k^A < k^B$, and this shows that the index U agrees well with k, which is the natural positive dependence index in this case. The function U will prove to be a suitable measure of dependence also in the sequel.

The next two results will show, for two particular classes of processes, that if the pair $\boldsymbol{A}$ is less positively dependent than $\boldsymbol{B}$, then some sort of comparison can be established for the hitting times.

3.1 *Dependent random walks*

The processes under consideration here are pairs of dependent random walks having the same marginal distributions but different dependence structure.

In this case the appropriate positive dependence index is the covariance of the increments, or, equivalently, the probability of simultaneous forward jumps. The result we are able to prove is in terms of the Laplace transform order $\le_{\mathrm{L}}$. Recall that, given two positive random variables X and Y, we write $X \le_{\mathrm{L}} Y$ when

$$Ee^{-\alpha X} \ge Ee^{-\alpha Y} \qquad \forall \alpha > 0.$$

For $j = 1, 2$, let A^j be a random walk starting at 0, let ΔA^j be its one-step increment, namely

$$\Delta A_n^j = A_{n+1}^j - A_n^j \qquad n \in \mathbb{N}$$

and let the joint distribution of ΔA^1 and ΔA^2 be determined by the following relations:

$$\begin{aligned} &\mathrm{Prob}(\Delta A_n^j = 1) = 1 - \mathrm{Prob}(\Delta A_n^j = -1) = p \qquad n \in \mathbb{N}, \\ &\mathrm{Prob}(\Delta A_n^1 = \Delta A_n^2 = 1) = p^A \qquad n \in \mathbb{N}, \end{aligned}$$

where $(2p-1)^+ \le p^A \le p$. Let B^1, B^2 etc. be defined similarly.

PROPOSITION 3.1. *The following are equivalent*:
(i) $U^A \le U^B$,
(ii) $p^A \le p^B$,
(iii) $T_\ell^A \le_{\mathrm{L}} T_\ell^B$ *for every positive even integer* ℓ.

PROOF. For notational simplicity, we omit the subscript n where appropriate. Obviously,

$$\begin{aligned} &\mathrm{Prob}(\Delta D^A = 2) = \mathrm{Prob}(\Delta A^1 = 1, \Delta A^2 = -1) = p - p^A, \\ &\mathrm{Prob}(\Delta D^A = -2) = \mathrm{Prob}(\Delta A^1 = -1, \Delta A^2 = 1) = p - p^A, \\ &\mathrm{Prob}(\Delta D^A = 0) \\ &\quad = \mathrm{Prob}(\Delta A^1 = -1, \Delta A^2 = -1) + \mathrm{Prob}(\Delta A^1 = 1, \Delta A^2 = 1) \\ &\quad = 1 - 2(p - p^A). \end{aligned} \tag{3.2}$$

Thus $\mathrm{Cov}(\Delta A^1, \Delta A^2) = 4(p^A - p^2)$. Now, since the increments are independent, we have

$$\begin{aligned} U^A(n, n+k) &= \mathrm{Cov}(A^1_{n+k} - A^1_{n+k-1} + A^1_{n+k-1} - \cdots - A^1_n, A^2_{n+k} - \cdots - A^2_n) \\ &= k\,\mathrm{Cov}(\Delta A^1, \Delta A^2) = 4k(p^A - p^2) \end{aligned}$$

so that (i)⇔(ii).

Let us prove now (ii)⇔(iii). For a random variable Y, and real α, write $\phi_Y(\alpha) = E(\exp\{-\alpha Y\})$, provided that the integrals involved exist. Then, from (3.2),

$$\begin{aligned} \phi_{\Delta D^A}(\alpha) &= (p - p^A)(e^{-2\alpha} + e^{2\alpha}) + 1 + 2(p^A - p) \\ &= 2(p - p^A)[\cosh(2\alpha) - 1] + 1 \end{aligned}$$

and

$$\phi_{D^A_n}(\alpha) = [\phi_{\Delta D^A}(\alpha)]^n.$$

Incidentally, notice that $1 \leq \phi_{\Delta D^A} < \infty$. For notational simplicity, let us write D for D^A, ϕ for $\phi_{\Delta D^A}$ and T for T^A_ℓ. It is clear that

$$M_n = \frac{\exp\{-\alpha D_n\}}{[\phi(\alpha)]^n}$$

is a martingale for each α, and the optional stopping theorem applies. Thus

$$\begin{aligned} 1 = EM_T &= E\left[\frac{\exp\{-\alpha D_T\}}{(\phi(\alpha))^T}\right] \\ &= E[\exp\{-\alpha D_T - T\log\phi(\alpha)\}]. \end{aligned} \tag{3.3}$$

Now, since the Laplace order involves only positive values of α, we may consider the restrictions of cosh and ϕ to $\mathbb{R}_+$, so that the inverses are well defined:

$$\phi^{-1}(t) = \frac{1}{2}\cosh^{-1}\left(\frac{t-1}{2(p - p^A)} + 1\right) \qquad t > 1.$$

The independence of T and D_T allows us to rephrase the equality (3.3) as

$$E[\exp\{-T\log\phi(\alpha)\}] = (E[\exp\{-\alpha D_T\}])^{-1} \qquad \alpha > 0.$$

Observing that $\log\circ\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a bijection, and letting $\beta = \log\phi(\alpha)$, we get

$$\phi_T(\beta) = E[\exp\{-\beta T\}] = (E[\exp\{-\phi^{-1}(e^\beta)D_T\}])^{-1} \qquad \beta > 0.$$

Now, since the law of D_T is symmetric and since, by (3.1), D_T assumes values $+\ell$ and $-\ell$ (where ℓ is a positive even integer), we obtain

$$\begin{aligned} E[\exp\{-\phi^{-1}(e^\beta)D_T\}] &= \frac{\exp\{-\ell\phi^{-1}(e^\beta)\} + \exp\{\ell\phi^{-1}(e^\beta)\}}{2} \\ &= \cosh(\ell\phi^{-1}(e^\beta)) \\ &= \cosh\left(\frac{\ell}{2}\cosh^{-1}\left(\frac{e^\beta - 1}{2(p - p^A)} + 1\right)\right) \end{aligned}$$

and this increases with p^A, for each $\beta > 0$, so that $\phi_T(\beta)$ decreases with p^A. Thus

$$p^A \leq p^B \Leftrightarrow \phi_{T^A}(\beta) \geq \phi_{T^B}(\beta) \forall \beta > 0 \Leftrightarrow T^A \leq_{\mathrm{L}} T^B. \qquad \square$$

3.2 *Gaussian processes*

Here, we assume that the processes A^1, A^2, B^1 and B^2 are bounded, centered, real valued Gaussian processes with continuous paths. A relation among expected values of suprema of difference processes emerges, and is put in correspondence with hitting times by

$$\left\{\sup_{s\leq t} D_s^A \geq \ell\right\} = \{S_\ell^A \leq t\} \tag{3.4}$$

where

$$S_\ell^A = \inf\{t > 0 \mid D_t^A \geq \ell\}.$$

We consider hitting times of the process D^A, instead of $|D^A|$, in order to be able to draw from the existing literature about suprema distributions for Gaussian processes. Notice, though, that the fact that D^A is centered implies

$$\mathrm{Prob}(S_\ell^A \leq t) = \mathrm{Prob}\left(\sup_{s\leq t} D_s^A \geq \ell\right) = \mathrm{Prob}\left(\inf_{s\leq t} D_s^A \leq -\ell\right) = \mathrm{Prob}(\tilde{S}_\ell^A \leq t)$$

where

$$\tilde{S}_\ell^A = \inf\{s > 0 \mid D_s^A \leq -\ell\}.$$

Analogously to what was said after the definition of T_ℓ^A, in case of perfect positive dependence $D^A \equiv 0$, so that S_ℓ^A and $\tilde{S}_\ell^A$ are a.s. infinite, for all $\ell > 0$. Thus, we may conjecture that stochastically greater stopping times S_ℓ^A correspond to more positive dependent pairs of processes.

Define C^A by

$$C^A(s,t) = \mathrm{Cov}(A_s^1, A_t^2) \qquad s,t \in \mathbb{R}_+$$

and define C^B analogously.

PROPOSITION 3.2. *Let A^i and B^i $(i = 1, 2)$ be a.s. bounded, centered, real valued Gaussian processes with continuous paths, each with covariance function R.*

(a) *If*

$$U^A(s,t) \leq U^B(s,t) \qquad \forall s,t \in \mathbb{R}_+ \tag{3.5}$$

then

$$E\left(\sup_{s\leq t} D_s^A\right) \geq E\left(\sup_{s\leq t} D_s^B\right). \tag{3.6}$$

(b) *If*

$$C^A(t,t) = C^B(t,t) \quad \textit{and} \quad C^A(s,t) \geq C^B(s,t) \quad \forall s,t \in \mathbb{R}_+ \tag{3.7}$$

then

$$S_\ell^A \leq_{\text{st}} S_\ell^B, \quad \forall \ell > 0. \tag{3.8}$$

PROOF. (a) The Sudakov-Fernique inequality (see, e.g., Adler (1990)), in our case, reads as follows: (3.6) is implied by

$$E(D_t^A - D_s^A)^2 \geq E(D_t^B - D_s^B)^2 \quad \forall s,t \in \mathbb{R}_+. \tag{3.9}$$

But

$$\begin{aligned} E(D_t^A - D_s^A)^2 &= E(A_t^1 - A_t^2 - A_s^1 + A_s^2)^2 \\ &= E(A_t^1 - A_s^1)^2 + E(A_t^2 - A_s^2)^2 - 2U^A(t,s) \\ &= 2[R(t,t) + R(s,s) - 2R(t,s) - U^A(t,s)], \end{aligned}$$

so that $U^A(t,s) \mapsto E(D_t^A - D_s^A)^2$ is a decreasing function. Hence (3.5) implies (3.9) and (3.6).

(b) If (3.7) holds, then

$$\begin{aligned} \text{Cov}(D_t^A, D_s^A) &= E((A_t^1 - A_t^2)(A_s^1 - A_s^2)) \\ &= 2R(t,s) - [C^A(t,s) + C^A(s,t)] \leq \text{Cov}(D_t^B, D_s^B). \end{aligned}$$

Furthermore

$$\text{Var}(D_t^A) = 2[R(t,t) - C^A(t,t)] = \text{Var}(D_t^B).$$

Thus, Slepian's Lemma applies and one gets

$$\text{Prob}\left(\sup_{s\leq t} D_s^A \geq \ell\right) \geq \text{Prob}\left(\sup_{s\leq t} D_s^B \geq \ell\right) \quad \forall t,\ell \in \mathbb{R}_+$$

i.e.

$$\text{Prob}(S_\ell^A \leq t) \geq \text{Prob}(S_\ell^B \leq t) \quad \forall t,\ell \in \mathbb{R}_+$$

and hence (3.8) follows. □

Remark 1. Notice that, in view of (3.4), (3.6) is equivalent to

$$\int_0^\infty \text{Prob}(S_\ell^A \leq t)d\ell \geq \int_0^\infty \text{Prob}(S_\ell^B \leq t)d\ell,$$

which is a necessary condition for (3.8).

Furthermore, (3.7) implies (3.5). In fact

$$\begin{aligned} U^A(s,t) &= C^A(t,t) - C^A(t,s) - C^A(s,t) + C^A(s,s) \\ &\leq C^B(t,t) - C^B(t,s) - C^B(s,t) + C^B(s,s) = U^B(s,t). \end{aligned}$$

Remark 2. The following example shows that it is indeed positive dependence we should deal with, and not dependence *tout court*. Let A^1, A^2, B^1 and B^2 be standard Brownian motions such that $A^1 = -A^2$ while B^1 and B^2 are independent. Then $D_t^A = 2A_t^1$ and D_t^B are centered normal random variables with variance $4t$ and $2t$ respectively, and

$$\text{Prob}(S_\ell^A \le t) = 2\,\text{Prob}(D_t^A \ge \ell) \ge 2\,\text{Prob}(D_t^B \ge \ell) = \text{Prob}(S_\ell^B \le t)$$

so that $S_\ell^A \le_{\text{st}} S_\ell^B$.

Thus, the independent pair "meets later" than the (negatively dependent) mirror-image pair. This is consistent with the idea underlying this note, and indeed one has $U^A \le U^B$ also in this case.

Remark 3. The example in Remark 2 can be generalized. Let W_1 and W_2 be two independent standard Brownian motions, and let, for $0 < \alpha < 1$,

$$\begin{aligned} A^1 &= \frac{1}{\sqrt{2(1+\alpha^2)}}[(1+\alpha)W_1 + (1-\alpha)W_2], \\ A^2 &= \frac{1}{\sqrt{2(1+\alpha^2)}}[(1-\alpha)W_1 + (1+\alpha)W_2]. \end{aligned}$$

Let B_1, B_2 be defined similarly, with β replacing α. Then A^1 and A^2 are again standard Brownian motions, with

$$\text{Cov}(A_t^1, A_t^2) = \frac{1-\alpha^2}{1+\alpha^2}t,$$

so that the correlation between the two processes decreases with α. The difference process D^A is also a Brownian motion, with $\text{Var}(D_t^A) = (4\alpha^2/1+\alpha^2)t$. Thus

$$\text{Prob}(S_\ell^A \le t) = 2\,\text{Prob}(D_t^A \ge \ell) = 2\left[1 - \Phi\left(\frac{\ell\sqrt{1+\alpha^2}}{2\alpha\sqrt{t}}\right)\right]$$

where Φ denotes the cumulative distribution function of a standard normal random variable. It is now easy to see that $\text{Prob}(S_\ell^A \le t)$ increases with α, for each $t > 0$ and $\ell > 0$, so that

$$\alpha \ge \beta \Leftrightarrow \text{Cov}(A_t^1, A_t^2) \le \text{Cov}(B_t^1, B_t^2) \Leftrightarrow U^A \le U^B \Leftrightarrow S_\ell^A \le_{\text{st}} S_\ell^B.$$

4. Exchangeable elliptically contoured sequences

Stopping times can be related also to the correlation of an infinite exchangeable sequence. We recall first that the law of a random vector $\boldsymbol{W}$ is elliptically contoured if for some vector $\boldsymbol{m}$ and some non-negative definite matrix $\boldsymbol{\Sigma}$, the characteristic function of $\boldsymbol{W} - \boldsymbol{m}$ depends on its argument $\boldsymbol{t}$ only through the quadratic form $\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}$, namely

$$\mathrm{E}(\exp\{i\boldsymbol{t}'\boldsymbol{W}\}) = \exp\{i\boldsymbol{t}'\boldsymbol{m}\}\psi(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}) \tag{4.1}$$

for some function ψ.

It is well known that if $\boldsymbol{W}$ admits moments of the first two orders, then its mean vector is $\boldsymbol{m}$ and its covariance matrix is proportional to $\boldsymbol{\Sigma}$. See, for example, Cambanis *et al.* (1981).

Now, let $W = \{W_n \mid n \in \mathbb{N}\}$ be a sequence of random variables with elliptically contoured finite dimensional laws as in (4.1), such that, for every $n, m \in \mathbb{N}$, $\mathrm{E}W_n = \mu$, $\mathrm{Var}(W_n) = \sigma^2$ and $\mathrm{Cov}(W_n, W_m) = \rho\sigma^2$. Indicate this by $W \sim \mathrm{EC}(\mu, \sigma^2, \rho, \psi)$. This clearly implies that the sequence is exchangeable.

We define a stopping time in terms of the first pair (W_{2n-1}, W_{2n}) of random variables sufficiently close to each other, namely

$$T_\epsilon^W = \inf\{n \in \mathbb{N} \mid |W_{2n} - W_{2n-1}| < \epsilon\} \qquad \epsilon > 0.$$

The next proposition will show that T_ϵ^W is stochastically decreasing with respect to ρ.

PROPOSITION 4.1. *Let $X \sim \mathrm{EC}(\mu_X, \sigma^2, \rho_X, \psi)$ and $Y \sim \mathrm{EC}(\mu_Y, \sigma^2, \rho_Y, \psi)$. Then $T_\epsilon^X \leq_{\mathrm{st}} T_\epsilon^Y$ if and only if $\rho_X \geq \rho_Y$.*

PROOF. Let $Z^X = \{Z_n^X \mid n \in \mathbb{N}\}$ be given by $Z_n^X = X_{2n} - X_{2n-1}$, and let Z^Y be defined similarly. Standard results about elliptically contoured distributions imply that $Z^X \sim \mathrm{EC}(0, 2\sigma^2(1-\rho_X), 0, \psi)$. Let $Z^* \sim \mathrm{EC}(0, 1, 0, \psi)$. Then

$$\begin{aligned}
\mathrm{Prob}(T_\epsilon^X > k) &= \mathrm{Prob}\left(\bigcap_{j=1}^{k}\{|Z_j^X| > \epsilon\}\right) \\
&= \mathrm{Prob}\left(\bigcap_{j=1}^{k}\left\{|Z_j^*| > \frac{\epsilon}{\sigma\sqrt{2(1-\rho_X)}}\right\}\right) \\
&\leq \mathrm{Prob}\left(\bigcap_{j=1}^{k}\left\{|Z_j^*| > \frac{\epsilon}{\sigma\sqrt{2(1-\rho_Y)}}\right\}\right) \\
&= \mathrm{Prob}\left(\bigcap_{j=1}^{k}\{|Z_j^Y| > \epsilon\}\right) \\
&= \mathrm{Prob}(T_\epsilon^Y > k).
\end{aligned}$$

□

Remark 4. Notice that the above proof shows also that T_ϵ^W is stochastically increasing in σ.

5. Discussion and conclusions

We have tried to establish a connection between the concept of positive dependence ordering and the behavior of some suitable stopping times. Given two bivariate processes such that all the marginal processes have the same law, we consider, for each process, the difference between the two components. Heuristically, if the components of one process are more positively dependent, then the difference process will tend to be smaller, and therefore the first time its absolute value crosses some threshold will tend to be larger.

It is hard to translate this heuristics into general results. In fact, depending on the nature of the processes, different types of orderings can be established among the threshold crossing times. Furthermore, in some cases we were able to find necessary and sufficient conditions for these orderings, while in some other cases we found only sufficient conditions.

Even when we obtained necessary and sufficient conditions, the results are not completely satisfactory. In fact, for the case of random walks we obtained only Laplace ordering for the stopping times T_ℓ^A and T_ℓ^B, whereas stochastic ordering was obtained only for the trivial case of convex combinations of processes. The other results concern the unilateral stopping times S_ℓ^A and S_ℓ^B. The problem of extending these results to the bilateral stopping times is by no means trivial: to realize that, consider that the existing literature on distribution for suprema of Gaussian processes deals mainly with the unilateral case (Sudakov-Fernique inequality, Slepian's inequality, etc.).

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References

Adler, R. J. (1990). *An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes*, IMS Lecture Notes—Monograph Series, **12**, Hayward, California.

Arjas, E. and Norros, I. (1984). Life lengths and association: A dynamic approach, *Math. Oper. Res.*, **9**, 151–158.

Block, H. W., Sampson, A. R. and Savits, T. H. (editors) (1990). *Topics in Statistical Dependence*, IMS Lecture Notes—Monograph Series, **16**, Hayward, California.

Cambanis, S., Huang, S. and Simons, G. (1981). On the theory of elliptically contoured distributions, *J. Multivariate Anal.*, **11**, 368–385.

Kimeldorf, G. and Sampson, A. R. (1987). Positive dependence orderings, *Ann. Inst. Statist. Math.*, **39**, 113–128.

Kimeldorf, G. and Sampson, A. R. (1989). A framework for positive dependence, *Ann. Inst. Statist. Math.*, **41**, 31–45.

Lehmann, E. (1966). Some concepts of dependence, *Ann. Math. Statist.*, **73**, 1137–1153.