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TESTS FOR A GIVEN LINEAR STRUCTURE OF THE MEAN DIRECTION OF THE LANGEVIN DISTRIBUTION

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Abstract. This paper deals with Watson statistic T_W and likelihood ratio (LR) statistic T_L for testing hypothesis $H_{0s} : \boldsymbol{\mu} \in V$ (a given s-dimensional subspace) based on a sample of size n from a p-variate Langevin distribution $M_p(\boldsymbol{\mu}, \kappa)$. Asymptotic expansions of the null and non-null distributions of T_W and T_L are obtained when n is large. Asymptotic expressions of those powers are also obtained. It is shown that the powers of them are coincident up to the order n^{-1} when κ is unknown.

Key words and phrases: Asymptotic expansion, central limit theorem, Langevin distribution, likelihood ratio statistic, Watson statistic, power comparison.

1. Introduction

A random vector $\boldsymbol{x}$ in $\mathbb{R}^p$ of its length $\|\boldsymbol{x}\|$ unity is said to have a p-variate Langevin distribution $M_p(\boldsymbol{\mu}, \kappa)$ if its probability density function is given by

$$\{a_p(\kappa)\}^{-1} \exp(\kappa \boldsymbol{\mu}' \boldsymbol{x}) \tag{1.1}$$

on the $(p-1)$-dimensional unit sphere $\mathbb{S}^{p-1} = \{\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^p, \|\boldsymbol{x}\| = (\boldsymbol{x}'\boldsymbol{x})^{1/2} = 1\}$, where $\boldsymbol{\mu}'\boldsymbol{\mu} = 1$ and $\kappa > 0$. The normalizing constant is given by

$$a_p(\kappa) = (2\pi)^{p/2} I_{p/2-1}(\kappa) \kappa^{-p/2+1}, \tag{1.2}$$

where $I_\nu(\kappa)$ is the modified Bessel function of the first kind of order ν. The parameters $\boldsymbol{\mu}$ and κ are called the mean direction vector and the concentration parameter, respectively.

Some statistics have been proposed for testing hypotheses about $\boldsymbol{\mu}$ and κ. Watson (1983*b*, 1984) obtained asymptotic null and non-null distributions of these statistics in both of the situations where the sample size is large and κ is large. Chou (1986) obtained asymptotic expansions of the null and non-null distributions of the Watson statistic T_W for testing the mean direction

$$\tilde{H}_{01} : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{vs.} \quad \tilde{H}_{11} : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0. \tag{1.3}$$

Hayakawa (1990) obtained similar asymptotic expansions of the null and non-null distributions of the LR statistic T_L and other ones for the same problem, and made numerical comparisons of the powers of these statistics. In this paper we make some extension of their results for the case when the testing problem is

$$H_{0s} : \boldsymbol{\mu} \in V \quad \text{vs.} \quad H_{1s} : \boldsymbol{\mu} \notin V. \tag{1.4}$$

The hypothesis H_{0s} is a generalization of H_{01} in some sense, and can be used when H_{01} is rejected and we wish to examine whether the mean direction lies in a little wider range. In Section 3 we shall see that there is the case where $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ is rejected for some given $\boldsymbol{\mu}_0$ but $\boldsymbol{\mu} \in V$ is not rejected nevertheless $\boldsymbol{\mu}_0 \in V$. We obtain asymptotic expansions of the null and non-null distributions of T_W and T_L and power functions of them. Well, T_W does not change when $\boldsymbol{\mu}_0$ is replaced by $-\boldsymbol{\mu}_0$ and neither the asymptotic expansion of T_L does because the terms of odd order of $\boldsymbol{\mu}_0$ vanish in the derivation process. So it is noted that the asymptotic expansions of T_W and T_L under the hypothesis $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ are the same as those under $\boldsymbol{\mu} = -\boldsymbol{\mu}_0$. Thus we get asymptotic expansions of the distributions of the Watson statistic and the LR statistic for (1.3) if we put $s = 1$ in those for (1.4). This means our results contain a part of Chou and Hayakawa's results as special cases.

2. Main results

Let $\boldsymbol{x} \sim M(\boldsymbol{\mu}, \kappa)$. Then it is shown that the expectation vector and covariance matrix are given as follows, respectively.

$$E(\boldsymbol{x}) = A_p(\kappa)\boldsymbol{\mu}, \qquad \text{and}$$
$$D(\boldsymbol{x}) = A_p'(\kappa)\boldsymbol{\mu}\boldsymbol{\mu}' + \frac{A_p(\kappa)}{\kappa}(I_p - \boldsymbol{\mu}\boldsymbol{\mu}') = \Sigma, \qquad \text{say},$$

where $A_p(\kappa) = a_p'(\kappa)/a_p(\kappa)$ and $A_p'(\kappa) = (d/d\kappa)A_p(\kappa)$. It is noted that the determinant of Σ is explicitly positive when $0 < \kappa < \infty$. As far as it seems not to occur any confusion, we write $A_p(\kappa)$ as A for simplicity. We consider testing the hypotheses (1.4) based on a random sample $\boldsymbol{x}_j$ of size n from $M_p(\boldsymbol{\mu}, \kappa)$ in the both cases when κ is known and unknown. Without loss of generality, we may express V as

$$V = \{\boldsymbol{\mu} \mid \boldsymbol{\mu} = B_0\boldsymbol{\zeta}, \boldsymbol{\zeta}'\boldsymbol{\zeta} = 1\}, \tag{2.1}$$

where B_0 is a given $p \times s$ matrix such that $B_0'B_0 = I_s$. So we can rewrite H_{0s} as

$$H_{0s} : \boldsymbol{\mu} = B_0\boldsymbol{\zeta}, \qquad \|\boldsymbol{\zeta}\| = 1. \tag{2.2}$$

In a special case $B_0 = \boldsymbol{\mu}_0$, the hypothesis becomes $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ or $\boldsymbol{\mu} = -\boldsymbol{\mu}_0$, and so different from H_{01}, but asymptotic expansions of T_W and T_L are the same as we noted before. As a sequence of the alternatives, we consider

$$\begin{aligned} \boldsymbol{\mu} &= (\boldsymbol{\mu}_0 + n^{-1/2}\boldsymbol{\delta})\|\boldsymbol{\mu}_0 + n^{-1/2}\boldsymbol{\delta}\|^{-1} \\ &= (\boldsymbol{\mu}_0 + n^{-1/2}\boldsymbol{\delta})(1 + 2n^{-1}\lambda)^{-1/2}, \end{aligned} \tag{2.3}$$

where $\mu_0 = B_0\zeta$, $B_0'\delta = 0$ and $\lambda = \delta'\delta/2$. Let B_s be a $p \times (p-s)$ matrix such that $(B_0\ B_s) \in O(p)$ and $x. = \sum x_i$.

Now we consider the two statistics. Watson statistic T_W and LR statistic T_L are given as follows

$$T_{W1} = \frac{\kappa}{nA}\|(I_p - B_0B_0')x.\|^2,$$
$$T_{L1} = 2\kappa(\|x.\| - \|B_0'x.\|),$$

when κ is known and

$$T_{W2} = \frac{\hat{\kappa}}{nA_p(\hat{\kappa})}\|(I_p - B_0B_0')x.\|^2,$$
$$T_{L2} = 2\{n\log a_p(\tilde{\kappa}) - n\log a_p(\hat{\kappa}) - \tilde{\kappa}\|B_0'x.\| + \hat{\kappa}\|x.\|\},$$

when κ is unknown, respectively. The statistics T_W's have been proposed by Watson (1983*a*, 1983*b*), and $\hat{\kappa}$ denotes the maximum likelihood estimator (m.l.e.) of κ given by

$$\hat{\kappa} = A_p^{-1}(\|\bar{x}\|). \tag{2.4}$$

Further $\tilde{\kappa}$ satisfies $A_p(\tilde{\kappa}) = \|B_0'\bar{x}\|$, that is, $\tilde{\kappa}$ is given by

$$\tilde{\kappa} = A_p^{-1}(\|B_0'\bar{x}\|), \tag{2.5}$$

and this is the m.l.e. of κ under the hypothesis (2.2).

THEOREM 2.1. *Under a sequence of the alternatives* (2.3) *the distribution function of T_{W1} and T_{L1} can be asymptotically expanded, respectively as*

$$P(T_{W1} \le x) = P(\chi^2_{p-s}(A\kappa\lambda) \le x) + \frac{1}{4n}\sum_{j=0}^{4} d_jP(\chi^2_{p-s+2j}(A\kappa\lambda) \le x) + O(n^{-3/2}), \tag{2.6}$$

and

$$P(T_{L1} \le x) = P(\chi^2_{p-s}(A\kappa\lambda) \le x) + \frac{1}{4n}\sum_{j=0}^{3} h_jP(\chi^2_{p-s+2j}(A\kappa\lambda) \le x) + O(n^{-3/2}), \tag{2.7}$$

where

$$
\begin{aligned}
d_0 &= 2(A'\kappa^2+3A\kappa)\lambda^2+2(p-s)\left(\frac{A'\kappa}{A}-1\right)\lambda \\
&\quad +\frac{1}{2}(p-s)(p-s+2)\left(\frac{A'}{A^2}-\frac{1}{A\kappa}\right), \\
d_1 &= -8A\kappa\lambda^2-2(p-s)\left(\frac{A'\kappa}{A}-1\right)\lambda \\
&\quad -(p-s)(p-s+2)\left(\frac{A'}{A^2}-\frac{1}{A\kappa}\right), \\
d_2 &= -4(A'\kappa^2-A\kappa)\lambda^2-2(p-s+2)\left(\frac{A'\kappa}{A}-1\right)\lambda \\
&\quad +\frac{1}{2}(p-s)(p-s+2)\left(\frac{A'}{A^2}-\frac{1}{A\kappa}\right), \\
d_3 &= 2(p-s+2)\left(\frac{A'\kappa}{A}-1\right)\lambda, \qquad d_4=2(A'\kappa^2-A\kappa)\lambda^2,
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
h_0 &= 2(A'\kappa^2+3A\kappa)\lambda^2-2(p-s)\lambda+\frac{1}{2A\kappa}(p-s)(p+s-4), \\
h_1 &= -4(A'\kappa^2+2A\kappa)\lambda^2+2(2p-s-1)\lambda-\frac{1}{2A\kappa}(p-s)(p+s-4), \\
h_2 &= 2(A'\kappa^2+2A\kappa)\lambda^2-2(p-1)\lambda, \qquad h_3=-2A\kappa\lambda^2.
\end{aligned} \tag{2.9}
$$

Here $\chi^2_f(A\kappa\lambda)$ denotes a noncentral χ^2-variate with f degrees of freedom and noncentrality parameter $A\kappa\lambda$.

Letting $\boldsymbol{\delta}=\mathbf{0}$ in (2.7), we obtain asymptotic expansions of the null distributions of T_{L1} and T_{W1}. This result implies that $\tilde{T}_{L1}=\{1+(p+s-4)/4nA\kappa\}T_{L1}$ gives a better χ^2-approximation, since

$$P(\tilde{T}_{L1}\le x)=P(\chi^2_{p-s}\le x)+O(n^{-3/2}).$$

When κ is unknown similar expansions can be derived in the same way.

THEOREM 2.2. *Under a sequence of the alternatives (2.3) the distribution functions of T_{W2} and T_{L2} can be asymptotically expanded, respectively as*

$$
\begin{aligned}
P(T_{W2}\le x) &= P(\chi^2_{p-s}(A\kappa\lambda)\le x) \\
&\quad +\frac{1}{4n}\sum_{j=0}^{4}d_j^* P(\chi^2_{p-s+2j}(A\kappa\lambda)\le x)+O(n^{-3/2}),
\end{aligned} \tag{2.10}
$$

and

$$(2.11)\quad P(T_{L2} \le x) = P(\chi^2_{p-s}(A\kappa\lambda) \le x) + \frac{1}{4n}\sum_{j=0}^{3} h_j^* P(\chi^2_{p-s+2j}(A\kappa\lambda) \le x) + O(n^{-3/2}),$$

where

$$(2.12)\quad \begin{aligned}
d_0^* &= 2(A'\kappa^2 + 3A\kappa)\lambda^2 + \frac{1}{2}(p-s)(p+3s-6)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) \\
&\quad + (p-s)\frac{A''}{A'^2\kappa}, \\
d_1^* &= -4(A'\kappa^2 + A\kappa)\lambda^2 + 4(s-2)\left(1 - \frac{A}{A'\kappa}\right)\lambda + \frac{2AA''}{A'^2}\lambda \\
&\quad - 2(p-s)(s-2)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) - (p-s)\frac{A''}{A'^2\kappa}, \\
d_2^* &= 2\left(A'\kappa^2 - 2A\kappa + \frac{A^2}{A'}\right)\lambda^2 + 2(p-3s+6)\left(1 - \frac{A}{A'\kappa}\right)\lambda \\
&\quad - \frac{2AA''}{A'^2}\lambda - \frac{1}{2}(p-s)(p-s+2)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right), \\
d_3^* &= 4\left(A\kappa - \frac{A^2}{A'}\right)\lambda^2 - 2(p-s+2)\left(1 - \frac{A}{A'\kappa}\right)\lambda, \\
d_4^* &= -2\left(A\kappa - \frac{A^2}{A'}\right)\lambda^2,
\end{aligned}$$

and

$$(2.13)\quad \begin{aligned}
h_0^* &= 2(A'\kappa^2 + 3A\kappa)\lambda^2 + \frac{1}{2}(p-s)(p+s-4)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) \\
&\quad + (p-s)\frac{A''}{A'^2\kappa}, \\
h_1^* &= -4(A'\kappa^2 + A\kappa)\lambda^2 + 2(s-3)\left(1 - \frac{A}{A'\kappa}\right)\lambda + \frac{2AA''}{A'^2}\lambda \\
&\quad - \frac{1}{2}(p-s)(p+s-4)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) - (p-s)\frac{A''}{A'^2\kappa}, \\
h_2^* &= 2\left(A'\kappa^2 - 2A\kappa + \frac{A^2}{A'}\right)\lambda^2 - 2(s-3)\left(1 - \frac{A}{A'\kappa}\right)\lambda - \frac{2AA''}{A'^2}\lambda, \\
h_3^* &= 2\left(A\kappa - \frac{A^2}{A'}\right)\lambda^2.
\end{aligned}$$

Letting $\boldsymbol{\delta} = \mathbf{0}$ in (2.10) and (2.11), we obtain asymptotic expansions of the null distributions of T_{W2} and T_{L2}, respectively. This implies that the Bartlett correction factor for T_{L2} is given by

$$1 + \frac{1}{4n}(p+s-4)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) + \frac{A''}{2nA'^2\kappa}.$$

Next we consider the powers of these statistics. Let β_{Wj} and β_{Lj} be the powers of T_{Wj} and T_{Lj} with a level of significance α for $j = 1, 2$. Then from Theorems 2.1 and 2.2 it is possible to obtain asymptotic expansions for β_{Wj} and β_{Lj}. A useful expression for such powers has been obtained by Fujikoshi (1988). Applying his result to previous theorems, we obtain the following theorem.

THEOREM 2.3. *Under a sequence of the alternatives* (2.3) *the powers* β_{W1} *and* β_{L1} *of* T_{W1} *and* T_{L1} *with a level of significance* α *are given by*

$$(2.14)\quad \begin{aligned}\beta_{W1} &= P(\chi^2_{p-s}(A\kappa\lambda) \geq x_\alpha)\\ &+ \frac{1}{n}\left[\left\{-(A'\kappa^2 + 3A\kappa)\lambda^2 - (p-s)\left(\frac{A'\kappa}{A} - 1\right)\lambda\right\}\right.\\ &\qquad \cdot g_{p-s+2}(x_\alpha; A\kappa\lambda)\\ &\qquad + \left\{-(A'\kappa^2 - A\kappa)\lambda^2 + \frac{1}{2}(p-s+2)\left(\frac{A'\kappa}{A} - 1\right)\lambda\right\}\\ &\qquad\quad \cdot g_{p-s+4}(x_\alpha; A\kappa\lambda)\\ &\qquad \left. + (A'\kappa^2 - A\kappa)\lambda^2 g_{p-s+6}(x_\alpha; A\kappa\lambda)\right] + O(n^{-3/2}),\end{aligned}$$

and

$$(2.15)\quad \begin{aligned}\beta_{L1} &= P(\chi^2_{p-s}(A\kappa\lambda) \geq x_\alpha)\\ &+ \frac{1}{n}\left[\left\{-(A'\kappa^2 + 3A\kappa)\lambda^2 + (p-s)\lambda\right\} g_{p-s+2}(x_\alpha; A\kappa\lambda)\right.\\ &\qquad + \left\{(A'\kappa^2 + A\kappa)\lambda^2 - \frac{1}{2}(p-s+2)\lambda\right\} g_{p-s+4}(x_\alpha; A\kappa\lambda)\\ &\qquad \left. - A\kappa\lambda^2 g_{p-s+6}(x_\alpha; A\kappa\lambda)\right] + O(n^{-3/2}).\end{aligned}$$

Further, β_{W2} *and* β_{L2} *are coincident up to the order* n^{-1} *and given by*

$$\begin{aligned}\beta_{W2} &= P(\chi^2_{p-s}(A\kappa\lambda) \geq x_\alpha)\\ &+ \frac{1}{n}\left[-(A'\kappa^2 + 3A\kappa)\lambda^2 g_{p-s+2}(x_\alpha; A\kappa\lambda)\right.\\ &\qquad + \left\{(A'\kappa^2 - A\kappa)\lambda^2 + \frac{1}{2}(p-s+2)\left(1 - \frac{A}{A'\kappa}\right)\lambda\right\}\\ &\qquad\quad \cdot g_{p-s+4}(x_\alpha; A\kappa\lambda)\\ &\qquad \left. + \left(A\kappa - \frac{A^2}{A'}\right)\lambda^2 g_{p-s+6}(x_\alpha; A\kappa\lambda)\right] + O(n^{-3/2}),\end{aligned}$$

where x_α *is the upper* α *point of* χ^2_{p-s} *and* $g_f(x_\alpha; A\kappa\lambda)$ *is the probability density function of* $\chi^2_f(A\kappa\lambda)$.

Then taking the difference between (2.14) and (2.15),

$$(2.16)\quad \beta_{W1} = \beta_{L1} + \frac{1}{n}\Bigg[- (p-s)\frac{A'\kappa}{A}\lambda g_{p-s+2}(x_\alpha; A\kappa\lambda)$$

$$+\left\{-2A'\kappa^2\lambda^2+\frac{1}{2}(p-s+2)\frac{A'\kappa}{A}\lambda\right\}g_{p-s+4}(x_\alpha;A\kappa\lambda)$$

$$\left.+A'\kappa^2\lambda^2 g_{p-s+6}(x_\alpha;A\kappa\lambda)\right]+O(n^{-3/2}).$$

In Fig. 1 some graphs are given for the $1/n$ terms of the difference between β_{W1} and β_{L1} in some special cases that $\alpha = 0.01$, 0.05, $\kappa = 1$, 5, 10 and $(p, s) = (3, 2)$, $(3, 1)$, $(2, 1)$. These show that for small κ, $\beta_{W1} > \beta_{L1}$ at first, i.e., as λ is small, next $\beta_{W1} < \beta_{L1}$ as λ becomes larger and finally it goes to zero. So when κ is quite small T_{W1} is better for small λ but for a moderate λ, T_{L1} is preferable. The larger the κ is, the smaller the difference is and for larger κ that is very small. Theorem 2.3 also shows that the differences between the powers of T_{W2} and T_{L2} are very small when n is large.

As mentioned in Section 1, when we take $s = 1$ in each theorem, we get the results of testing $\tilde{H}_{01}$.

3. Numerical example

We consider a data set "Measurements of magnetic remanence in specimens of Palaeozoic red-beds from Argentina" (Fisher *et al.* (1987, p. 279), Embleton (1970)). First we test

$$\tilde{H}_{01}:\boldsymbol{\mu}=(\cos 55^\circ,\cos 150^\circ\sin 55^\circ,\sin 150^\circ\sin 55^\circ)=\boldsymbol{\mu}_0,\qquad \text{say}.$$

The sample mean direction is given as $(0.54, -0.66, 0.49)$ so that the resultant length is 0.99. We need to calculate the m.l.e. $\hat{\kappa}$ of κ which satisfies (2.4). From the fact that $A_p(\kappa)$ is an increasing function of κ taking its value in $(0, 1)$ for $\kappa \in (0, \infty)$, it is easily seen that $\hat{\kappa}$ is quite large. It is noted that this example is for $p = 3$ so

$$a_3(\kappa)=\frac{4\pi\sinh\kappa}{\kappa}\qquad\text{and}\qquad A_3(\kappa)=\frac{\cosh\kappa}{\sinh\kappa}-\frac{1}{\kappa},$$

see Watson (1980). Thus $\hat{\kappa}$ and $\tilde{\kappa}$ are obtained by numerical calculations precisely. Then $\hat{\kappa}$ is 73.28 and $\tilde{\kappa}$ is equal to 55.95, where $\tilde{\kappa}$ satisfies now $A_3(\tilde{\kappa}) = \boldsymbol{\mu}_0'\bar{\boldsymbol{x}}$. Two statistics are given as follows: $T_W = 16.08$, $T_L = 14.04$. These are quite large and we reject the hypothesis. Next consider the hypothesis

$$H_{02}:\boldsymbol{\mu}=B_0\boldsymbol{\zeta},\qquad \|\boldsymbol{\zeta}\|=1,$$

where

$$B_0=\begin{pmatrix}\cos 55^\circ & 0\\ \cos 150^\circ\sin 55^\circ & \sin 150^\circ\\ \sin 150^\circ\sin 55^\circ & -\cos 150^\circ\end{pmatrix}.$$

It is obvious that the subspace $V = \{\boldsymbol{\mu} = B_0\boldsymbol{\zeta}, \|\boldsymbol{\zeta}\| = 1\}$ contains $\boldsymbol{\mu}_0$ with $\boldsymbol{\zeta}' = (1, 0)$. Now two statistics become $T_W' = 1.06$, $T_L' = 1.05$, and these are not significant at 5% level.

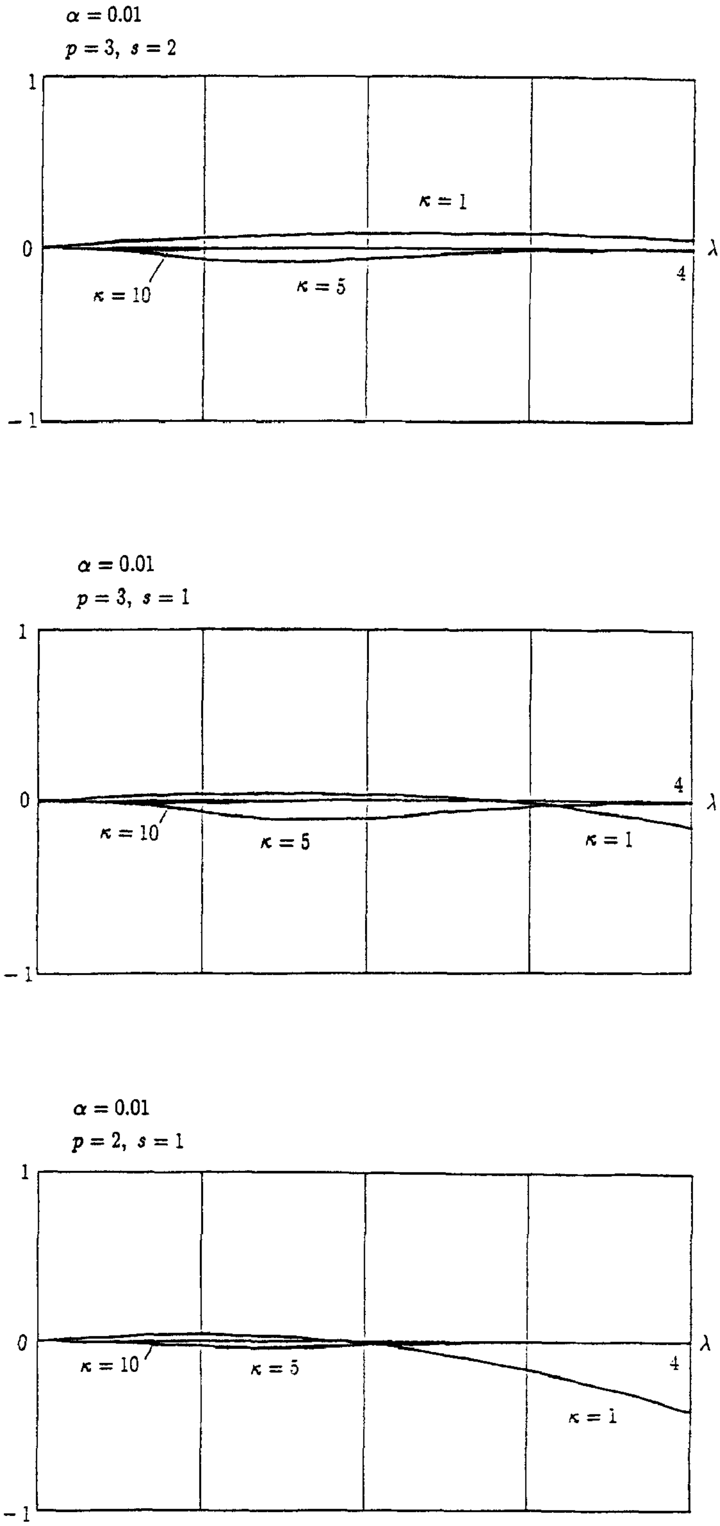
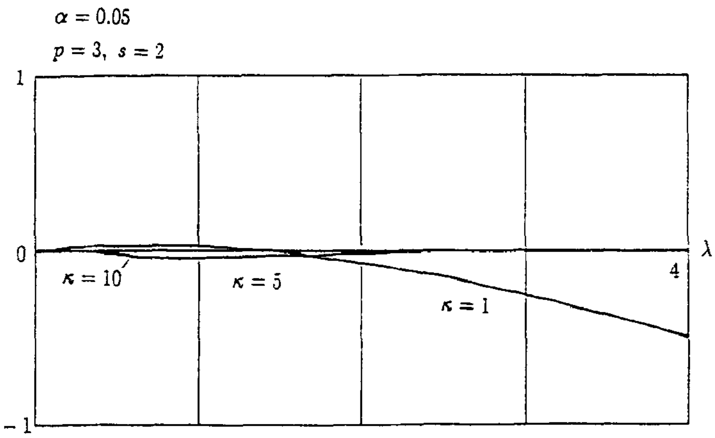
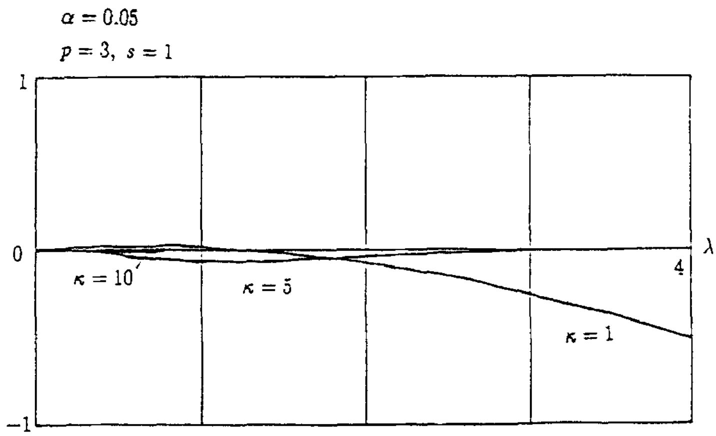


Fig. 1. The n^{-1} terms of the difference between β_{W1} and β_{L1}.





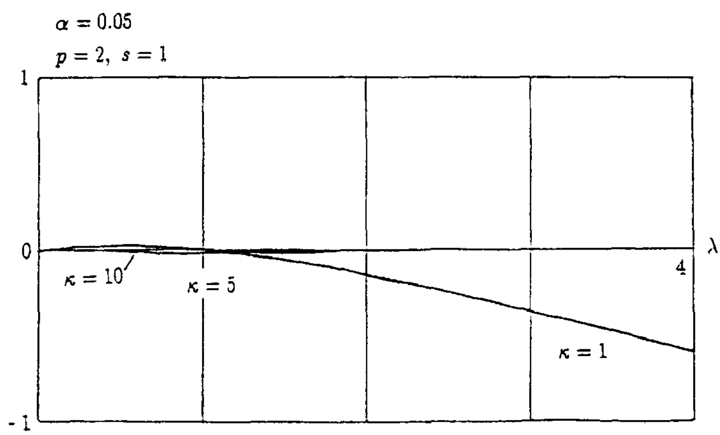


Fig. 1. (continued)

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