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LIMIT THEOREMS FOR THE MINIMUM INTERPOINT DISTANCE BETWEEN ANY PAIR OF I.I.D. RANDOM POINTS IN $\boldsymbol{R}^d$

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Abstract. The limit theorem for the minimum interpoint distance between any pair of i.i.d. random points in $\boldsymbol{R}^d$ with common density $f \in L^2$ was studied by a method which makes use of the convergence of point processes. Some one-dimensional examples with $f \notin L^2$ (including the cases Beta and Gamma distributions) were also considered.

Key words and phrases: Minimum interpoint distance, Poisson point process, compensator, Skorohod J_1-topology.

1. Introduction

Limit theorems for various statistics arising from spacing problems have been investigated by many authors (e.g. see Lévy (1939), Pyke (1965), Silverman and Brown (1978), Molchanov and Reznikova (1982), Aly *et al.* (1984), Onoyama *et al.* (1984), Rao Jammalamadaka and Janson (1986), Deheuvels *et al.* (1988)); a simple and typical example is the result of Lévy (1939) concerning the minimum and the maximum of respective lengths of the $n+1$ segments into which the unit interval $(0,1)$ is divided by i.i.d. random variables X_k, $1 \le k \le n$, with the uniform distribution. Molchanov and Reznikova (1982) discussed the same problem as above for a wider class of nonuniform densities. Similar problems in multidimensional spaces seem to be hard to deal with, but if the minimum length is regarded not from the view point of partitions but simply as the minimum of the interpoint distances between all pairs drawn from $\{X_k, 1 \le k \le n\}$, some parts of the results of Molchanov and Reznikova (1982) have been extended to a multidimensional case; for example, it is known (Silverman and Brown (1978), Onoyama *et al.* (1984), Rao Jammalamadaka and Janson (1986)) that the limit distribution of suitably normalized M_n, where M_n is the minimum of $|X_i - X_j|^d$, $1 \le i < j \le n$, is exponential if the X_i's are i.i.d. $\boldsymbol{R}^d$-valued random variables with common density $f \in L^2(\boldsymbol{R}^d)$.

In this paper we discuss again the limit theorem for M_n but with emphasis on a method which makes use of the convergence of point processes ($f \in L^2$ is assumed). Point processes were also used in Silverman and Brown (1978) but our way of using point processes is different from theirs. We also give some examples of results in one-dimensional cases where $f \notin L^2$. When $d \geq 2$ we assume $f \in L^2$ as in the earlier results; however, our point process method clarifies and improves some of the earlier results in the sense that the problem was formulated as the convergence of processes and the limiting process was obtained (Theorem 2.1). The limiting process is a Markovian decreasing process. We also consider one-dimensional examples, in which $f \notin L^2$ was emphasized and which include the cases of Beta and Gamma distributions.

2. Results in the case where $f \in L^2(\boldsymbol{R}^d)$

Given a sequence of $\boldsymbol{R}^d$-valued i.i.d. random variables $X_1, X_2, \dots$ with common probability density function f, we put

$$Y_n(t) = \begin{cases} n^2 \min\limits_{1 \leq i < j \leq [nt]} |X_i - X_j|^d, & t \geq 2/n, \\ \infty, & 0 < t < 2/n, \end{cases}$$

$$Z_n(t) = \begin{cases} X_k, & t \geq 2/n, \\ X_1, & 0 < t < 2/n, \end{cases}$$

where $k = k(t)$ is the unique integer with $2 \leq k \leq [nt]$ satisfying

$$n^2 \min_{1 \leq j < k} |X_j - X_k|^d = Y_n(t).$$

$Z_n(t)$ means roughly the place where the minimum distance is realized. In this section it is assumed that $f \in L^2(\boldsymbol{R}^d)$. Before stating our main theorem, we introduce a Poisson point process $\{p(t), t > 0\}$ on $(0, \infty) \times \boldsymbol{R}^d$ with compensator $c_d t dt d\xi f(x)^2 dx$ where c_d is the volume of a unit ball in $\boldsymbol{R}^d$ and write $p(t) = (q(t), r(t))$. Then $\{q(t), t > 0\}$ and $\{r(t), t > 0\}$ are Poisson point processes on $(0, \infty)$ and $\boldsymbol{R}^d$, respectively. For $t > 0$ we put

$$Y(t) = \min_{s \leq t} q(s), \qquad Z(t) = r(t'),$$

where $t' \in (0, t]$ is determined by $q(t') = Y(t)$. The processes are seen that

$$P\{Y(t_1) > \xi_1, \dots, Y(t_m) > \xi_m\} = \prod_{i=1}^{m} \exp\left\{-\frac{1}{2} c_d \|f\|_2^2 (t_i^2 - t_{i-1}^2) \max_{i \leq j \leq m} \xi_j\right\},$$

$$P\{Z(t) \in dx\} = \{f(x)^2 / \|f\|_2^2\} dx,$$

for any $t > 0$, $0 < t_1 < t_2 < \cdots < t_m$ and $\xi_1, \xi_2, \dots, \xi_m > 0$, where $\|f\|_2$ is the L^2-norm of f. Moreover, for each fixed $t > 0$, $Y(t)$ and $Z(t)$ are independent.

THEOREM 2.1. *If $\|f\|_2 < \infty$, then the process $\{(Y_n(t), Z_n(t)), t > 0\}$ converges in law to $\{(Y(t), Z(t)), t > 0\}$ with repect to the Skorohod J_1-topology. In particular, for each fixed $t > 0$, $\xi > 0$ and an interval I in $\boldsymbol{R}^d$*

$$\lim_{n\to\infty} P\{Y_n(t) > \xi, Z_n(t) \in I\} = \exp\left\{-\frac{1}{2}c_d\|f\|_2^2t^2\xi\right\}\int_I \{f(x)^2/\|f\|_2^2\}dx. \tag{2.1}$$

Before proving the theorem we present some lemmas. Let $\mathcal{G}_k$ be the σ-field generated by $X_1, X_2, \ldots, X_k$ for each $k \geq 1$ and $\mathcal{F}_t^{(n)} = \mathcal{G}_{[nt]}$. Define an $\{\mathcal{F}_t^{(n)}\}$-adapted point process $p_n = (q_n, r_n)$ taking values in $(0, \infty) \times \boldsymbol{R}^d$ as follows: The domain of definition of p_n is $D_{p_n} = \{k/n : k = 2, 3, \ldots\}$ and

$$p_n(k/n) = (q_n(k/n), r_n(k/n)) = \left(n^2 \min_{1\leq j<k} |X_j - X_k|^d, X_k\right), \qquad k \geq 2. \tag{2.2}$$

Denote by N_{p_n} the counting measure corresponding to p_n, i.e.,

$$N_{p_n}((0,t] \times (0,\xi] \times A) = \sum_{2\leq k\leq [nt]} \mathbf{1}_{(0,\xi]\times A}(p_n(k/n)), \qquad \xi > 0,\ A \in \mathcal{B}(\boldsymbol{R}^d), \tag{2.3}$$

where $\mathbf{1}_{(0,\xi]\times A}(\cdot)$ denotes the indicator function of $(0, \xi] \times A$ and $\mathcal{B}(\boldsymbol{R}^d)$ is the σ-field of Borel subsets of $\boldsymbol{R}^d$. When $[nt] < 2$, the summation over $2 \leq k \leq [nt]$ (or over $k \leq [nt]$) is understood to be 0. The compensator of p_n is given by

$$\hat{N}_{p_n}((0,t] \times (0,\xi] \times A) = \sum_{2\leq k\leq [nt]} E[\mathbf{1}_{(0,\xi]\times A}(p_n(k/n)) \mid \mathcal{G}_{k-1}]. \tag{2.4}$$

LEMMA 2.1. (Onoyama *et al.* (1984)) *Under the assumption $f \in L^2$,*

$$\iiint_{\substack{|x_1-x_2|\leq\varepsilon\\|x_3-x_2|\leq\varepsilon}} f(x_1)f(x_2)f(x_3)dx_1dx_2dx_3 = o(\varepsilon^{3d/2}), \tag{2.5}$$

as $\varepsilon \downarrow 0$.

LEMMA 2.2. *For any $t > 0$, $\xi > 0$ and $A \in \mathcal{B}(\boldsymbol{R}^d)$,*

$$\hat{N}_{p_n}((0,t] \times (0,\xi] \times A) \xrightarrow{P} \frac{1}{2}c_dt^2\xi\int_A f(x)^2dx, \tag{2.6}$$

as $n \to \infty$, where "$\xrightarrow{P}$" means convergence in probability.

PROOF. For $t > 0$ and $\xi > 0$,

$$\begin{aligned}
&\hat{N}_{p_n}((0,t] \times (0,\xi] \times A) \\
&= \sum_{k\leq[nt]} E\left[\mathbf{1}_{(0,\xi]}\left(n^2 \min_{1\leq j<k}|X_j - X_k|^d\right)\mathbf{1}_A(X_k) \mid \mathcal{G}_{k-1}\right] \\
&= \sum_{k\leq[nt]} \int_{\cup_{1\leq j<k}B(X_j,(\xi/n^2)^{1/d})} \mathbf{1}_A(x)f(x)dx,
\end{aligned} \tag{2.7}$$

where $B(x,r) = \{y \in \boldsymbol{R}^d : |y-x| < r\}$. Put

$$I_k = \int_{\cup_{1\le j<k} B(X_j,(\xi/n^2)^{1/d})} \mathbf{1}_A(x) f(x)dx,$$
$$I'_k = \sum_{1\le j<k} \int_{B(X_j,(\xi/n^2)^{1/d})} \mathbf{1}_A(x) f(x)dx,$$
$$I''_k = \sum_{1\le j<m<k} \int_{B(X_j,(\xi/n^2)^{1/d})\cap B(X_m,(\xi/n^2)^{1/d})} \mathbf{1}_A(x) f(x)dx.$$

Then it is clear that $I'_k - I''_k \le I_k \le I'_k$. Making use of Lemma 2.1, we have that $E\big[\sum_{k\le[nt]} I''_k\big]$ converges to zero as $n\to\infty$. Therefore

$$\sum_{k\le[nt]} I''_k \xrightarrow{p} 0, \qquad n\to\infty. \tag{2.8}$$

Next we show that

$$\sum_{k\le[nt]} I'_k - \sum_{k\le[nt]}\sum_{1\le j\le k-1} c_d(\xi/n^2)\mathbf{1}_A(X_j)f(X_j) \xrightarrow{p} 0, \tag{2.9}$$

as $n\to\infty$. Put

$$h_n(x) = (c_d\xi/n^2)^{-1}\int_{B(x,(\xi/n^2)^{1/d})} \mathbf{1}_A(y)f(y)dy = (h*\phi_n)(x),$$

where $h(x) = \mathbf{1}_A(x)f(x)$ and $\phi_n(x) = (c_d\xi/n^2)^{-1}\mathbf{1}_{B(0,(\xi/n^2)^{1/d})}(x)$. Since

$$\|h_n\|_2 = \|h*\phi_n\|_2 \le \|f\|_2 \quad \text{and} \quad \|h_n - h\|_2 \to 0, \qquad (n\to\infty),$$

it follows that

$$\begin{aligned} E&\left[\left|\sum_{k\le[nt]} I'_k - \sum_{k\le[nt]}\sum_{1\le j<k} c_d(\xi/n^2)h(X_j)\right|\right] \\ &\le \sum_{k\le[nt]}\sum_{1\le j<k} E[|c_d(\xi/n^2)\{h_n(X_j)-h(X_j)\}|] \\ &\le \sum_{k\le[nt]}\sum_{1\le j<k} c_d(\xi/n^2)\|h_n-h\|_2\|f\|_2 \to 0, \end{aligned}$$

as $n\to\infty$. Thus (2.9) holds. Finally an application of the law of large numbers yields

$$\sum_{k\le[nt]}\sum_{1\le j<k} c_d(\xi/n^2)h(X_j) \xrightarrow{p} \frac{1}{2}c_d t^2\xi\int_A f(x)^2dx, \tag{2.10}$$

as $n \to \infty$. Thus (2.6) follows from (2.7)-(2.10). □

We are now able to prove Theorem 2.1 as follows. Since Lemma 2.2 holds, an application of Theorem 3.1 of Durrett and Resnick (1978) (see also Theorem 5.1 of Kasahara and Watanabe (1986)) implies that

$$N_{p_n} \xrightarrow{d} N_p, \qquad n \to \infty, \tag{2.11}$$

where N_p is the counting measure of the Poisson point process p and "$\xrightarrow{d}$" means the convergence in law of random variables taking values in the space $\mathcal{M}$ of non-negative Radon measures on $(0,\infty) \times (0,\infty) \times \boldsymbol{R}^d$, which was equipped with the vague topology (see also Theorem 5.1 of Kasahara and Watanabe (1986)). Since $Y_n(t) = \min_{s \le t} q_n(s)$ and $Z_n(t) = r_n(t')$ where $t' \in (0,t]$ was determined by $q_n(t') = Y_n(t)$, any finite dimensional distribution of $\{(Y_n(t), Z_n(t)), t > 0\}$ converges to that of $\{(Y(t), Z(t)), t > 0\}$. Moreover, in the present case we can show that the convergence of finite dimensional distributions implies the convergence in law with respect to the Skorohod J_1-topology by using a method similar to that in p. 211 of Jagers (1974) (see also Lindvall (1973) and Serfozo (1982) for the Skorohod J_1-topology on $D(0,\infty)$ and related results which are useful for the present proof).

Remark 2.1. $Y(t)$ is a temporally inhomogeneous Markov process with state space $(0,\infty)$. However, $X(t) = Y(t^{1/2})$ is a temporally homogeneous Markov process with the generator L defined by

$$(L\phi)(x) = \frac{1}{2} c_d \|f\|_2^2 \int_{-x}^{0} \{\phi(x+y) - \phi(x)\} dy.$$

Furthermore, it is easy to see that the process $M(t) = (X(t))^{-1}$ is an extremal process generated by a distribution function $F(x) = \exp\{-(1/2)c_d\|f\|_2^2/x\}$, $x \ge 0$. For the detail of the extremal process see Chapter IV in Resnick (1987).

Remark 2.2. Let f be the density function of the arcsine law, i.e.

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{x(1-x)}} & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

In this case $f \notin L^2(\boldsymbol{R})$; however, the above method can also be applied to this case; that is, if we define a point process $p_n(t)$ on $(0,\infty)$ by

$$p_n(k/n) = (n^2 \log n) \min_{1 \le j < k} |X_j - X_k|,$$

then we can prove that its compensator $\hat{N}_{p_n}((0,t] \times (0,\xi])$ converges in probability to $4\pi^{-2}t^2\xi$ as $n \to \infty$. Therefore, we have

$$P\left\{(n^2 \log n) \min_{1 \le j < k \le n} |X_j - X_k| > \xi\right\} \to \exp\{-4\pi^{-2}\xi\}, \qquad n \to \infty. \tag{2.12}$$

Another method for proving (2.12) will be presented in Section 3 (see Example 3).

We next consider the minimum distance between any pair of points of a configuration in $\boldsymbol{R}^d$ distributed according to a Poisson distribution. By definition a locally finite configuration in $\boldsymbol{R}^d$ is a subset ω of $\boldsymbol{R}^d$ such that $\omega \cap A$ is a finite set for any bounded subset A of $\boldsymbol{R}^d$. Let Ω be the set of locally finite configurations in $\boldsymbol{R}^d$ and for $\omega \in \Omega$ and $A \subset \boldsymbol{R}^d$ put

$$N(A) = N(A, \omega) = \text{the number of points in } \omega \cap A.$$

Given $\lambda > 0$ and a nonnegative function $f \in L^1_{loc}(\boldsymbol{R}^d)$ we denote by P_λ the Poisson distribution on Ω with intensity measure $\lambda f(x)dx$, i.e. the probability measure on Ω such that

i) for each $A \in \mathcal{B}(\boldsymbol{R}^d)$ with $\lambda_A = \lambda \int_A f(x)dx < \infty$, $N(A)$ was distributed according to the Poisson distribution with mean λ_A;

ii) for any disjoint $A_1, \ldots, A_n \in \mathcal{B}(\boldsymbol{R}^d)$ with $\int_{A_k} f(x)dx < \infty$ $(1 \leq k \leq n)$, $N(A_1), \ldots, N(A_n)$ are independent.

For $A \in \mathcal{B}(\boldsymbol{R}^d)$ we put

$$M(A) = M(A, \omega) = \begin{cases} \inf\limits_{\substack{x,y \in \omega \cap A \\ x \neq y}} |x - y|^d & \text{if } N(A, \omega) \geq 2, \\ \infty & \text{if } N(A, \omega) \leq 1. \end{cases} \tag{2.13}$$

Notice that $M(A_1), \ldots, M(A_n)$ are independent (w.r.t. P_λ) provided that $A_1, \ldots,$ A_n $(\in \mathcal{B}(\boldsymbol{R}^d))$ are disjoint. The following theorem can be proved by using Theorem 2.1 or Theorem 3.1 of Silverman and Brown (1978).

THEOREM 2.2. *If $\int_A f(x)dx < \infty$ and $\int_A f(x)^2 dx < \infty$, then for any $x > 0$*

$$\lim_{\lambda \to \infty} P_\lambda\{\lambda^2 M(A) > x/c_d\} = e^{-\mu(A)x}, \tag{2.14}$$

where $\mu(A) = \int_A f(x)^2 dx$.

3. One-dimensional examples with $f \notin L^2(\boldsymbol{R})$

We consider one-dimensional examples in which the underlying random variables have a common probability density function f not belonging to $L^2(\boldsymbol{R})$. The first example covers the case of Gamma distributions.

3.1 *Example* 1

Let $X_1, X_2, \dots$ be i.i.d. real random variables and assume that X_k has a probability density function f such that

$$f(x) = 0 \qquad \text{on } (-\infty, 0], \tag{3.1a}$$

$$f(x) \sim ax^{\mu-1} \qquad \text{as } x \downarrow 0, \tag{3.1b}$$

$$\int_r^\infty f(x)^2 dx < \infty \qquad \text{for any } r > 0, \tag{3.1c}$$

where a and μ are positive constants and $f \sim g$ means that f/g tends to 1. It is also assumed that $0 < \mu < 1/2$ and hence

$$\int_0^r f(x)^2 dx = \infty \qquad \text{for any } r > 0. \tag{3.2}$$

We are going to discuss a limit theorem for M_n. Let $\tau_1, \tau_2, \dots$ be i.i.d. random variables with $P\{\tau_k > t\} = e^{-t}$, $t > 0$, and put

$$\xi_n = \tau_1 + \cdots + \tau_n, \qquad \xi = \inf_{n \geq 1}(\xi_{n+1}^{1/\mu} - \xi_n^{1/\mu}). \tag{3.3}$$

It is easy to see that ξ is a strictly positive random variable.

Put $\xi_{a,\mu} = (\mu/a)^{1/\mu}\xi$.

THEOREM 3.1. *For any* $x > 0$

$$\lim_{n\to\infty} P\{n^{1/\mu}M_n > x\} = P\{\xi_{a,\mu} > x\}. \tag{3.4}$$

PROOF. We first show that

$$\lim_{\lambda\to\infty} P_\lambda\{\lambda^{1/\mu}M([0,\infty)) > x\} = P\{\xi_{a,\mu} > x\}, \tag{3.5}$$

where P_λ is the Poisson distribution on the space of configurations in $(0,\infty)$ with intensity measure $\lambda f(x)dx$ and $M(\cdot)$ is defined by (2.13). We treat only the special case $f(x) = ax^{\mu-1}$, $0 < x < r_0$, for some $r_0 > 0$. It is not difficult to prove the general case. Let $\eta_n = (\mu/a)^{1/\mu}\lambda^{-1/\mu}\xi_n$, $n \geq 1$, and $N_\lambda = \max\{n : \eta_n \leq r_0\}$. Then we have

$$\begin{aligned}(3.6)\quad &\lim_{\lambda\to\infty} P\Big\{\lambda^{1/\mu} \min_{1\leq n<N_\lambda}(\eta_{n+1} - \eta_n) > x\Big\} \\ &= \lim_{\lambda\to\infty} P\Big\{(\mu/a)^{1/\mu} \min_{1\leq n<N_\lambda}(\xi_{n+1}^{1/\mu} - \xi_n^{1/\mu}) > x\Big\} = P\{\xi_{a,\mu} > x\}.\end{aligned}$$

On the other hand it is easy to see that the configuration $\omega = \{\eta_n, n \geq 1\}$ has the Poisson distribution with intensity measure $\lambda a x^{\mu-1}\mathbf{1}_{(0,\infty)}(x)dx$. Furthermore we note that

$$\lim_{\lambda\to\infty} P_\lambda\{M([0,r]) = M([0,r']) = M([0,\infty))\} = 1. \tag{3.7}$$

Therefore

$$\{M([0,r_0]),P_\lambda\} \stackrel{d}{=} \left\{\min_{1\le n<N_\lambda}(\eta_{n+1}-\eta_n),P\right\},$$

where "$\stackrel{d}{=}$" means the equivalence in law and consequently (3.6) and (3.7) imply (3.5).

Now we prove (3.4). Take a small $\varepsilon > 0$ and let N_n^+ and N_n^- be Poisson random variables with means $n(1+\varepsilon)$ and $n(1-\varepsilon)$, respectively. We assume that $\{X_n, n\ge 1\}$ and $\{N_n^+,N_n^-\}$ are independent. We put

$$M_n^+ = \min_{1\le i<j\le N_n^+}|X_i-X_j|, \qquad M_n^- = \min_{1\le i<j\le N_n^-}|X_i-X_j|.$$

Then

(i) $M_n^+\le M_n\le M_n^-$ if $N_n^-\le n\le N_n^+$.

(ii) $\lim_{n\to\infty} P\{N_n^-\le n\le N_n^+\}=1$.

(iii) $\{M_n^\pm,P\}\stackrel{d}{=}\{M([0,\infty)),P_{n(1\pm\varepsilon)}\}$.

(iv) $\lim_{n\to\infty} P\{n^{1/\mu}(1\pm\varepsilon)^{1/\mu}M_n^\pm > x\} = P\{\xi_{a,\mu}>x\}$.

Therefore

$$P\{n^{1/\mu}M_n^+ > x\}\le P\{n^{1/\mu}M_n > x\}+o(1)\le P\{n^{1/\mu}M_n^- > x\}+o(1),$$

where $o(1)$ converges to 0. Making n tend to ∞ we obtain

$$\begin{aligned} P\{\xi_{a,\mu} > (1+\varepsilon)^{-1/\mu}x\} &\le \varliminf_{n\to\infty} P\{n^{1/\mu}M_n > x\} \\ &\le \varlimsup_{n\to\infty} P\{n^{1/\mu}M_n > x\}\le P\{\xi_{a,\mu} > (1-\varepsilon)^{-1/\mu}x\}. \end{aligned}$$

Since $\varepsilon > 0$ can be made arbitrarily small, we obtain (3.4). □

3.2 *Example* 2

We consider the case where the common probability density f of i.i.d. X_k's is given by

$$f(x):\begin{cases} =0 & \text{for } x<0,\\ \sim a_0x^{\mu-1} & \text{as } x\downarrow 0,\\ \in L^2 & \text{on } [\varepsilon,1-\varepsilon] \text{ for any } 0<\varepsilon<1/2,\\ \sim a_-(1-x)^{\mu-1} & \text{as } x\uparrow 1,\\ \sim a_+(x-1)^{\mu-1} & \text{as } x\downarrow 1,\\ \in L^2 & \text{on } [1+\varepsilon,\infty) \text{ for any } \varepsilon>0, \end{cases}$$

where a_0, a_-, a_+ are positive constants and $0<\mu<1/2$. To state the result we need some auxiliary random variables. Let

$$\tau_1^0,\tau_2^0,\ldots;\ \tau_1^+,\tau_2^+,\ldots;\ \tau_1^-,\tau_2^-,\ldots$$

be i.i.d. random variables whose common distribution is the exponential one with mean 1 and put

$$
\begin{aligned}
&\xi_n^0 = \tau_1^0 + \cdots + \tau_n^0, \\
&\xi_n^+ = \tau_1^+ + \cdots + \tau_n^+, \qquad \xi_n^- = \tau_1^- + \cdots + \tau_n^-, \\
&\xi_0 = \inf_{n\geq 1}\{(\xi_{n+1}^0)^{1/\mu} - (\xi_n^0)^{1/\mu}\}, \\
&\xi_+ = \inf_{n\geq 1}\{(\xi_{n+1}^+)^{1/\mu} - (\xi_n^+)^{1/\mu}\}, \qquad \xi_- = \inf_{n\geq 1}\{(\xi_{n+1}^-)^{1/\mu} - (\xi_n^-)^{1/\mu}\}, \\
&\eta = ((\mu/a_0)^{1/\mu}\xi_0) \wedge ((\mu/a_+)^{1/\mu}\xi_+) \wedge ((\mu/a_-)^{1/\mu}\xi_-) \\
&\qquad \wedge \{(\mu/a_+)^{1/\mu}(\xi_1^+)^{1/\mu} + (\mu/a_-)^{1/\mu}(\xi_1^-)^{1/\mu}\}.
\end{aligned}
$$

THEOREM 3.2. *For any $x > 0$,*

$$
\lim_{n\to\infty} P\{n^{1/\mu}M_n > x\} = P\{\eta > x\}. \tag{3.8}
$$

3.3 *Example* 3

This is the case where $\mu = 1/2$ in Example 1. We use the notation in Example 1. We treat only the special case where $f(x) = ax^{-1/2}$, $0 < x \leq r_0$, for some $r_0 > 0$; the general case where $f(x) \sim ax^{-1/2}$ can be treated easily. Let $x > 0$ and $0 < \varepsilon < 1$ and consider the events

$$
\begin{aligned}
\Lambda_\lambda &= \left\{\min_{1\leq n<N_\lambda}(\eta_{n+1} - \eta_n) > \frac{x}{\lambda^2\log\lambda}\right\}, \\
\Lambda_{\lambda,k} &= \left\{\min_{k\leq n<N_\lambda}(\eta_{n+1} - \eta_n) > \frac{x}{\lambda^2\log\lambda}\right\} \cap \{n_-(\lambda) \leq N_\lambda \leq n_+(\lambda)\},
\end{aligned}
$$

where $n_-(\lambda) = 2\lambda a r_0^{1/2}(1-\varepsilon)$, $n_+(\lambda) = 2\lambda a r_0^{1/2}(1+\varepsilon)$ and $1 < k < n_-(\lambda)$. Next we put

$$
\begin{aligned}
\overline{P}_{\lambda,k} &= P\left\{\min_{k\leq n<n_-(\lambda)}(\eta_{n+1} - \eta_n) > \frac{x}{\lambda^2\log\lambda}\right\}, \\
\underline{P}_{\lambda,k} &= P\left\{\min_{k\leq n<n_+(\lambda)}(\eta_{n+1} - \eta_n) > \frac{x}{\lambda^2\log\lambda}\right\}.
\end{aligned}
$$

Then we can prove that

$$
\begin{aligned}
\exp\{-2a^2x(1+\varepsilon)^{-2}\} &\leq \lim_{k\to\infty}\varliminf_{\lambda\to\infty} \underline{P}_{\lambda,k} \leq \lim_{k\to\infty}\varliminf_{\lambda\to\infty} P(\Lambda_{\lambda,k}) \leq \varliminf_{\lambda\to\infty} P(\Lambda_\lambda) \\
&\leq \varlimsup_{\lambda\to\infty} P(\Lambda_\lambda) \leq \lim_{k\to\infty}\varlimsup_{\lambda\to\infty} P(\Lambda_{\lambda,k}) \leq \lim_{k\to\infty}\varlimsup_{\lambda\to\infty} \overline{P}_{\lambda,k} \\
&\leq \exp\{-2a^2x(1-\varepsilon)^{-2}\}.
\end{aligned}
$$

Since $\varepsilon > 0$ can be made arbitrarily small we obtain

$$
\lim_{\lambda\to\infty} P(\Lambda_\lambda) = \exp\{-2a^2x\}. \tag{3.9}
$$

Once the result (3.9) is obtained we can proceed as in the proof of Theorem 3.1 to obtain the following results:

$$\text{(3.10)} \quad \lim_{\lambda\to\infty} P_\lambda\{\lambda^2(\log\lambda)M([0,r]) > x\} = \exp\{-2a^2x\}, \qquad r>0, \quad x>0,$$

$$\text{(3.11)} \quad \lim_{n\to\infty} P\{n^2(\log n)M_n > x\} = \exp\{-2a^2x\}, \qquad x>0.$$

The case of the arcsine law can be discussed by a method similar to that of Example 2. The limit distribution is exponential.

3.4 *Example* 4

Finally we treat the case where the common distribution of i.i.d. X_k's is the Beta distribution with density

$$f(x) = \begin{cases} \dfrac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)} x^{\mu-1}(1-x)^{\nu-1} & \text{for } x \in (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

where μ and ν are positive constants. The result in this case is a special case of Theorem 3.1 and Theorem 3.2. Since the limit distribution we are interested in depends symmetrically on μ and ν, it is enough to consider the case where $\mu \geq \nu$. Let η_+ and η_- be independent copies of $(\mu\Gamma(\mu)^2/\Gamma(2\mu))^{1/\mu} \cdot \xi$ where ξ is defined by (3.4) and denote by Φ_μ and Ψ_μ the probability distributions of η_+ and $\min\{\eta_+, \eta_-\}$, respectively. Then the result is summarized in the following table.

Table 1.

Case	Normalized random variable	Limit distribution
$\frac{1}{2} < \mu \leq \nu$	$n^2 M_n$	Exponential distribution
$\frac{1}{2} = \mu \leq \nu$	$n^2(\log n)M_n$	Exponential distribution
$0 < \mu < \frac{1}{2}$, $\mu < \nu$	$n^{1/\mu}M_n$	Φ_μ
$0 < \mu = \nu < \frac{1}{2}$	$n^{1/\mu}M_n$	Ψ_μ

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