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BOOTSTRAP METHOD AND EMPIRICAL PROCESS

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Abstract. In this paper we consider the sampling properties of the bootstrap process, that is, the empirical process obtained from a random sample of size n (with replacement) of a fixed sample of size n of a continuous distribution. The cumulants of the bootstrap process are given up to the order n^{-1} and their unbiased estimation is discussed. Furthermore, it is shown that the bootstrap process has an asymptotic minimax property for some class of distributions up to the order $n^{-1/2}$.

Key words and phrases: Bootstrap process, cumulants, unbiased estimators, asymptotic minimax property.

1. Introduction

The bootstrap method may be reviewed from different viewpoints. In this paper, we intend to consider the sampling properties of the bootstrap process, that is, the empirical process derived from the bootstrap sampling, i.e. that obtained from a random sample of size n (with replacement) of a fixed sample of size n of a certain distribution. Let $X_1, \dots, X_n$ be a sample of size n from a population with the distribution function $F(t)$ and let $X_1^*, \dots, X_n^*$ be a bootstrap sample of size n, that is, a random sample of size n from $X_1, \dots, X_n$. Let the empirical distribution functions obtained from $(X_1, \dots, X_n)$ and $(X_1^*, \dots, X_n^*)$ be denoted by $F_n(t)$ and $F_n^*(t)$, respectively. It is well known that $\sqrt{n}(F_n(t) - F(t))$ approaches a Gaussian process as $n \to \infty$, and, given $F_n(t)$, $\sqrt{n}(F_n^*(t) - F_n(t))$ conditionally approaches a Gaussian process with the same variance and covariance of $\sqrt{n}(F_n(t) - F(t))$ with $F(t)$ replaced by $F_n(t)$. Hence $\sqrt{n}(F_n^*(t) - F_n(t))$ can be considered to be the consistent estimator of $\sqrt{n}(F_n(t) - F(t))$. Note that $\sqrt{n}(F_n(t) - F(t))$ can not be usually observed since $F(t)$ is unknown, whereas the distribution of $\sqrt{n}(F_n^*(t) - F_n(t))$ can be completely computed from the sample. We further investigate how $\sqrt{n}(F_n^*(t) - F_n(t))$ will differ from $\sqrt{n}(F_n(t) - F(t))$ in higher order terms and we discuss possible improvements on $F_n^*(t)$.

In many problems of statistical inference, the procedures will depend on the distribution of a statistic T_n under an unknown distribution $F(t)$, which in many cases can be discussed in terms of $\sqrt{n}(F_n(t)-F(t))$, at least asymptotically. Hence $\sqrt{n}(F_n^*(t)-F_n(t))$ can be used instead of $\sqrt{n}(F_n(t)-F(t))$ in the derivative of the asymptotic distribution of the statistic. It will be shown that the method is in a sense asymptotically efficient in a nonparametric (or semiparametric) framework. A first order approximation was considered in the work of Efron (1979, 1982) and Beran (1982) considered a second order approximation from a different viewpoint. The purpose of this paper is to compute the cumulants up to the order n^{-1} and to show that the bootstrap process is in a sense, asymptotically, the best estimator of the empirical process up to the order $n^{-1/2}$, whereas in terms of the order n^{-1} there are many complications and although a slight improvement is possible over the usual bootstrap process, no uniformly optimal results seem to be obtainable. The bootstrap method is used to estimate the distribution of some statistic T_n under a general unknown population distribution and it is shown that it is asymptotically best up to the second order in the sense that the estimator of the asymptotic variance as well as that of the asymptotic distribution of T_n can not be uniformly improved if the class of possible population distributions is sufficiently wide.

2. Unbiased estimation of cumulants of the empirical process

In the framework of Section 1, we put $W_n(t)=\sqrt{n}(F_n(t)-F(t))$ and $W_n^*(t)=\sqrt{n}(F_n^*(t)-F_n(t))$. Consequently we have the following.

LEMMA 2.1. *The cumulants of $W_n(t)$ are given, up to the fourth order, as follows*:

$$
\begin{aligned}
&E[W_n(t)]=0,\\
&\mathrm{Cov}(W_n(t_1),\, W_n(t_2))=F(t_1)(1-F(t_2)) \qquad \textit{for} \quad t_1\le t_2,\\
&\kappa_3(W_n(t_1),\, W_n(t_2),\, W_n(t_3))\\
&\qquad =(1/\sqrt{n})F(t_1)(1-2F(t_2))(1-F(t_3)) \qquad \textit{for} \quad t_1\le t_2\le t_3,\\
&\kappa_4(W_n(t_1),\, W_n(t_2),\, W_n(t_3),\, W_n(t_4))\\
&\qquad =(1/n)F(t_1)(1-F(t_4))(1-4F(t_2)-2F(t_3)+6F(t_2)F(t_3))\\
&\qquad\qquad \textit{for} \quad t_1\le t_2\le t_3\le t_4.
\end{aligned}
$$

The proof is given in Section 4, but Lemma 2.1 may be also derived from Lemma 3.1 of Withers (1983). From Lemma 2.1 we have the following.

LEMMA 2.2. *Given $F_n(t)$, the conditional cumulants of $W_n^*(t)$ are given, up to the fourth order, as follows*:

$$
\begin{aligned}
&E[W_n^*(t)\mid F_n(t)]=0,\\
&\mathrm{Cov}(W_n^*(t_1),\, W_n^*(t_2)\mid F_n(t_1),\, F_n(t_2))=F_n(t_1)(1-F_n(t_2))\\
&\qquad\qquad \textit{for} \quad t_1\le t_2,\\
&\kappa_3(W_n^*(t_1),\, W_n^*(t_2),\, W_n^*(t_3)\mid F_n(t_1),\, F_n(t_2),\, F_n(t_3))\\
&\qquad =(1/\sqrt{n})F_n(t_1)(1-2F_n(t_2))(1-F_n(t_3)) \qquad \textit{for} \quad t_1\le t_2\le t_3,
\end{aligned}
$$

$$\kappa_4(W_n^*(t_1),\, W_n^*(t_2),\, W_n^*(t_3),\, W_n^*(t_4) \mid F_n(t_1),\, F_n(t_2),\, F_n(t_3),\, F_n(t_4))$$
$$= (1/n)F_n(t_1)(1 - F_n(t_4))(1 - 4F_n(t_2) - 2F_n(t_3) + 6F_n(t_2)F_n(t_3)) \qquad \text{for} \quad t_1 \le t_2 \le t_3 \le t_4.$$

The proof is straightforward from Lemma 2.1. On the other hand, we also have the following.

LEMMA 2.3.

$$E[F_n(t_1)(1 - F_n(t_2))] = \{1 - (1/n)\}F(t_1)(1 - F(t_2)) \qquad \text{for} \quad t_1 \le t_2,$$
$$E[F_n(t_1)(1 - 2F_n(t_2))(1 - F_n(t_3))]$$
$$= \{1 - (1/n)\}\{1 - (2/n)\}F(t_1)(1 - 2F(t_2))(1 - F(t_3)) \qquad \text{for} \quad t_1 \le t_2 \le t_3,$$
$$E[F_n(t_1)(1 - F_n(t_4))(1 - 4F_n(t_2) - 2F_n(t_3) + 6F_n(t_2)F_n(t_3))]$$
$$= \{1 - (1/n)\}\{1 - (2/n)\}\{1 - (3/n)\}F(t_1)(1 - F(t_4))$$
$$\cdot (1 - 4F(t_2) - 2F(t_3) + 6F(t_2)F(t_3))$$
$$- (1/n)\{1 - (1/n)\}F(t_1)(1 - F(t_4)) \qquad \text{for} \quad t_1 \le t_2 \le t_3 \le t_4.$$

The proof is given in Section 4. From Lemmas 2.2 and 2.3 it is seen that, given $F_n(t)$, the conditional cumulants of $W_n^*(t)$ are not unbiased estimators of the corresponding cumulants of $W_n(t)$.

LEMMA 2.4. *The (unconditional) cumulants of* $W_n^*(t)$ *are given, up to the fourth order, as follows:*

$$E[W_n^*(t)] = 0,$$
$$\mathrm{Cov}(W_n^*(t_1),\, W_n^*(t_2)) = \{1 - (1/n)\}F(t_1)(1 - F(t_2)) \qquad \text{for} \quad t_1 \le t_2,$$
$$\kappa_3(W_n^*(t_1),\, W_n^*(t_2),\, W_n^*(t_3))$$
$$= (1/\sqrt{n})\{1 - (1/n)\}\{1 - (2/n)\}F(t_1)(1 - 2F(t_2))(1 - F(t_3)) \qquad \text{for} \quad t_1 \le t_2 \le t_3,$$
$$\kappa_4(W_n^*(t_1),\, W_n^*(t_2),\, W_n^*(t_3),\, W_n^*(t_4))$$
$$= (1/n)\{1 - (1/n)\}\{1 - (2/n)\}\{1 - (3/n)\}F(t_1)(1 - F(t_4))$$
$$\cdot (1 - 4F(t_2) - 2F(t_3) + 6F(t_2)F(t_3))$$
$$- (1/n^2)\{1 - (1/n)\}F(t_1)(1 - F(t_4))$$
$$+ (1/n)F(t_1)(1 - F(t_4))(3 - 8F(t_2) - 4F(t_3) + 12F(t_2)F(t_3))$$
$$- (2/n^2)F(t_1)(1 - F(t_4))(3 - 10F(t_2) - 5F(t_3) + 15F(t_2)F(t_3))$$
$$+ (3/n^3)F(t_1)(1 - F(t_4))(1 - 4F(t_2) - 2F(t_3) + 6F(t_2)F(t_3)) + o(1/n^3)$$
$$\text{for} \quad t_1 \le t_2 \le t_3 \le t_4.$$

The proof is given in Section 4. From Lemmas 2.1, 2.2 and 2.3 we also have the following.

THEOREM 2.1. *The unbiased estimators of the covariance* $\mathrm{Cov}(W_n(t_1), W_n(t_2))$ *and the third order cumulant* $\kappa_3(W_n(t_1), W_n(t_2), W_n(t_3))$ *are given by*

$$\begin{aligned}
&\{n/(n-1)\}\,\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2)) \\
&\quad = \{1/(n-1)\}F_n(t_1)(1-F_n(t_2)) \qquad \textit{for} \quad t_1 \le t_2, \\
&\{n^2/(n-1)(n-2)\}\kappa_3(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3) \mid F_n(t_1), F_n(t_2), F_n(t_3)) \\
&\quad = \{n\sqrt{n}/(n-1)(n-2)\}F_n(t_1)(1-2F_n(t_2))(1-F_n(t_3)) \; \textit{for} \quad t_1 \le t_2 \le t_3,
\end{aligned}$$

respectively.

PROOF. From Lemmas 2.1, 2.2 and 2.3 we have for $t_1 \le t_2$

$$\begin{aligned}
E[\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2))] &= E[F_n(t_1)(1-F_n(t_2))] \\
&= \{1-(1/n)\}F(t_1)(1-F(t_2)) \\
&= \{(n-1)/n\}\,\mathrm{Cov}(W_n(t_1), W_n(t_2)),
\end{aligned}$$

hence

$$\{n/(n-1)\}\,\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2)) = \{n/(n-1)\}F_n(t_1)(1-F_n(t_2))$$

is an unbiased estimator of $\mathrm{Cov}(W_n(t_1), W_n(t_2))$. In a similar way we obtain for $t_1 \le t_2 \le t_3$

$$\begin{aligned}
&E[\kappa_3(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3) \mid F_n(t_1), F_n(t_2), F_n(t_3))] \\
&\quad = E[(1/\sqrt{n})F_n(t_1)(1-2F_n(t_2))(1-F_n(t_3))] \\
&\quad = (1/\sqrt{n})\{(n-1)(n-2)/n^2\}F(t_1)(1-2F(t_2))(1-F(t_3)) \\
&\quad = \{(n-1)(n-2)/n^2\}\kappa_3(W_n(t_1), W_n(t_2), W_n(t_3)),
\end{aligned}$$

hence

$$\begin{aligned}
&\{n^2/(n-1)(n-2)\}\kappa_3(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3) \mid F_n(t_1), F_n(t_2), F_n(t_3)) \\
&\quad = \{n\sqrt{n}/(n-1)(n-2)\}F_n(t_1)(1-2F_n(t_2))(1-F_n(t_3))
\end{aligned}$$

is an unbiased estimator of $\kappa_3(W_n(t_1), W_n(t_2), W_n(t_3))$. Thus we complete the proof.

Remark 2.1. Let $X_1', \dots, X_{n-1}'$ be a bootstrap sample of size $n-1$, that is, a random sample of size $n-1$ from $X_1, \dots, X_n$. We put $\tilde{W}_{n-1}^*(t) = \sqrt{n}(F_{n-1}^*(t) - F_n(t))$ with the empirical distribution $F_{n-1}^*(t)$ of $X_1', \dots, X_{n-1}'$. Then it follows from Lemmas 2.1, 2.2, 2.3 and Theorem 2.1 that

$$\mathrm{Cov}(\tilde{W}_{n-1}^*(t_1), \tilde{W}_{n-1}^*(t_2) \mid F_n(t_1), F_n(t_2)) = \{n/(n-1)\}F_n(t_1)(1-F_n(t_2))$$

is also an unbiased estimator of $\mathrm{Cov}(W_n(t_1), W_n(t_2))$, but $\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2))$ is not unbiased for it. Hence it is desirable to use the bootstrap sample of size $n-1$ in place of size n. And also the biases of higher order cumulants become smaller.

3. Minimax property of the bootstrap estimator

In this section we consider the estimation problem based on the i.i.d. sample $X_1, \ldots, X_n$ on some real parameter θ which can be defined as a functional $\theta = \Psi(F)$ of a continuous distribution F. Then the natural estimator is $\hat{\theta}_n = \Psi(F_n)$, where F_n is the empirical distribution function. We shall show that the bootstrap estimator of the distribution of $\hat{\theta}_n$ has a minimax property for some parametric family of distributions. We assume the following condition.

(A.1) The functional Ψ is Fréchet differentiable up to the third order, that is, there are functions $\partial\Psi/\partial F$, $\partial^2\Psi/\partial F\partial F$ and $\partial^3\Psi/\partial F\partial F\partial F$ such that

$$
\begin{aligned}
(3.1)\quad \Psi(G) - \Psi(F) = &\int_{-\infty}^{\infty} (\partial\Psi/\partial F) d(G - F) \\
&+ (1/2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\partial^2\Psi/\partial F\partial F) d(G - F) d(G - F) \\
&+ (1/6) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\partial^3\Psi/\partial F\partial F\partial F) \\
&\cdot dF(G - F) d(G - F) d(G - F) \\
&+ o(\|G - F\|^3),
\end{aligned}
$$

where $\|G - F\| = \sup_x |G(x) - F(x)|$.

Putting $W_n(x) = \sqrt{n}(F_n(x) - F(x))$, we have from (3.1)

$$
\begin{aligned}
(3.2)\quad \sqrt{n}(\hat{\theta}_n - \theta) = &\int_{-\infty}^{\infty} \phi_1(x) dW_n(x) \\
&+ (1/2\sqrt{n}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_2(x, y) dW_n(x) dW_n(y) \\
&+ (1/6n) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_3(x, y, z) dW_n(x) dW_n(y) dW_n(z) \\
&+ o_p(1/n),
\end{aligned}
$$

where $\phi_1(x) = (\partial\Psi/\partial F)(x)$, $\phi_2(x, y) = (\partial^2\Psi/\partial F\partial F)(x, y)$ and $\phi_3(x, y, z) = (\partial\Psi^3/\partial F\partial F\partial F)(x, y, z)$. We also assume that the following holds.

$$
\begin{aligned}
(\text{A.2})\quad &\int \phi_1(x) dF(x) = 0, \qquad \int \phi_2(x, t) dF(t) = \int \phi_2(s, y) dF(s) = 0, \\
&\int \phi_3(x, y, u) dF(u) = \int \phi_3(x, t, z) dF(t) = \int \phi_3(s, y, z) dF(s) = 0,
\end{aligned}
$$

and the functions $\phi_2(x, y)$ and $\phi_3(x, y, z)$ are symmetric in (x, y) and (x, y, z), respectively.

Furthermore, using $T_n = \sqrt{n}(\hat{\theta}_n - \theta)$, we assume the following condition.

$$
(\text{A.3})\qquad E(T_n^4) < \infty.
$$

LEMMA 3.1. *Assume that the conditions* (A.1), (A.2) *and* (A.3) *hold. Then the asymptotic cumulants of* T_n *are given as follows.*

$$
\begin{aligned}
E(T_n) = {} & (1/2\sqrt{n})\left\{\int_{-\infty}^{\infty}\phi_2(x,\,x)dF(x) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_2(x,\,y)dF(x)dF(y)\right\} \\
& + o(1/n) \\
= {} & (1/\sqrt{n})b_1 + o(1/n) \qquad (say), \\
V(T_n) = {} & \int_{-\infty}^{\infty}\phi_1^2(x)dF(x) - \left\{\int_{-\infty}^{\infty}\phi_1(x)dF(x)\right\}^2 \\
& + (1/n)\left\{\int_{-\infty}^{\infty}\phi_1(x)\phi_2(x,\,x)dF(x)\right. \\
& \qquad - 3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(x)\phi_2(x,\,y)dF(x)dF(y) \\
& \qquad \left. + 2\left(\int_{-\infty}^{\infty}\phi_1(x)dF(x)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_2(x,\,y)dF(x)dF(y)\right\} \\
& + (1/2n)\left\{\int_{-\infty}^{\infty}\phi_2(x,\,x)dF(x) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_2(x,\,y)dF(x)dF(y)\right\}^2 \\
& + (1/n)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(x)\phi_3(x,\,y,\,y)dF(x)dF(y)\right. \\
& \qquad - 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(x)\phi_3(x,\,y,\,z)dF(x)dF(y)dF(z) \\
& \qquad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(x)\phi_3(y,\,z,\,u) \\
& \qquad \left. \cdot\, dF(x)dF(y)dF(z)dF(u)\right\} + o(1/n) \\
= {} & v_0 + (1/n)v_1 + o(1/n) \qquad (say), \\
\kappa_3(T_n) = {} & E[\{T_n - E(T_n)\}^3] \\
= {} & (1/\sqrt{n})\left\{\int_{-\infty}^{\infty}\phi_1^3(x)dF(x) - 3\left(\int_{-\infty}^{\infty}\phi_1^2(x)dF(x)\right)\left(\int_{-\infty}^{\infty}\phi_1(x)dF(x)\right)\right. \\
& \qquad + 2\left(\int_{-\infty}^{\infty}\phi_1(x)dF(x)\right)^3 \\
& \qquad + 3\left(\int_{-\infty}^{\infty}\phi_1^2(x)dF(x)\right)\int_{-\infty}^{\infty}\phi_2(x,\,x)dF(x) \\
& \qquad - 3\left(\int_{-\infty}^{\infty}\phi_1^2(x)dF(x)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_2(x,\,y)dF(x)dF(y) \\
& \qquad + 3\left(\int_{-\infty}^{\infty}\phi_1(x)dF(x)\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_2(x,\,y)dF(x)dF(y) \\
& \qquad \left. + (3/2)\left(\int_{-\infty}^{\infty}\phi_1(x)dF(x)\right)^2\int_{-\infty}^{\infty}\phi_2(x,\,x)dF(x)\right\} \\
& + o(1/n), \\
= {} & (1/\sqrt{n})\beta_3 + o(1/n) \qquad (say), \\
\kappa_4(T_n) = {} & E[\{T_n - E(T_n)\}^4] - 3\{V(T_n)\}^2 = O(1/n).
\end{aligned}
$$

The proof is omitted since Lemma 3.1 is similar to Theorem 3.1 of Withers (1983).

Remark 3.1. With the condition (A.3) and from the fact that there exists a finite positive constant c such that

$$P\left\{\sup_x \sqrt{n}|F_n(x) - F(x)| > r\right\} < ce^{-2r^2} \quad \text{(Dvoretzky } et\ al. \text{ (1956))},$$

holds for all $r \geq 0$ and all positive integers n, it follows that the above expansion of the remainder term is valid.

For an estimator $\hat{\theta}_n^*$ based on the bootstrap sample $X_1^*, \ldots, X_n^*$ of size n, we put $T_n^* = \sqrt{n}(\hat{\theta}_n^* - \theta)$.

LEMMA 3.2. *Assume that the conditions* (A.1), (A.2) *and* (A.3) *hold. Then the conditional asymptotic cumulants of* T_n^*, *given the empirical distribution function* F_n, *have the following form.*

$$\begin{aligned}
E[T_n^* \mid F_n] &= (1/\sqrt{n})b_1 + (1/n)\xi_1 + o_p(1/n), \\
V(T_n^* \mid F_n) &= v_0 + (1/\sqrt{n})\xi_2 + (1/n)v_1 + o_p(1/n), \\
\kappa_3(T_n^* \mid F_n) &= (1/\sqrt{n})\beta_3 + (1/n)\xi_3 + o_p(1/n), \\
\kappa_4(T_n^* \mid F_n) &= \kappa_4(T_n) + o_p(1/n),
\end{aligned}$$

where $\xi_1 = O_p(1)$, $\xi_2 = O_p(1)$, $\xi_3 = O_p(1)$, *and* b_1, v_0, v_1 *and* β_3 *are constants given in Lemma* 3.1.

The proof is given in Section 4.

Remark 3.2. In order to evaluate the bootstrap estimator $\hat{\theta}_n^*$, it is seen from Lemmas 3.1 and 3.2 that the variance of $\xi_2 = \sqrt{n}(V(T_n^* \mid F_n) - V(T_n)) + o_p(1/\sqrt{n})$ plays an important part.

LEMMA 3.3. *Under the conditions* (A.1), (A.2) *and* (A.3), *the variance of* ξ_2 *is given by*

$$V(\xi_2) = \int_{-\infty}^{\infty} \phi_1^2(x)\{\phi_1(x) - 2m\}^2 dF(x) - \left\{\int_{-\infty}^{\infty} \phi_1^2(x)dF(x) - 2m^2\right\}^2,$$

where $m = \int_{-\infty}^{\infty} \phi_1(x)dF(x)$.

The proof is given in Section 4. Now we consider a parametric family $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ of distribution functions, where Θ is an open set of R^1 involving the origin. Take F_{θ_0} as the previous distribution function F. We assume that, for each $\theta \in \Theta$, the distribution function F_θ is absolutely continuous with respect to a σ-finite measure μ, and denote $dF_\theta(x)/d\mu(x)$ by $f_\theta(x)$. For each $\theta \in \Theta$, we put

$$v_\theta = \int_{-\infty}^{\infty} \phi_1^2(x) f_\theta(x) d\mu - \left\{\int_{-\infty}^{\infty} \phi_1(x) f_\theta(x) d\mu\right\}^2.$$

Since $\hat{\theta}_n = \Psi(F_n)$ is an asymptotically unbiased estimator of θ, we have by Taylor's expansion of v_θ around $\theta = \theta_0$

$$v_{\hat{\theta}_n} = v_{\theta_0} + [\partial v_\theta/\partial\theta]_{\theta=\theta_0}(\hat{\theta}_n - \theta_0) + o_p(1/\sqrt{n}),$$

hence the variance of $v_{\hat{\theta}_n}$ is given by

$$V_{\theta_0}(v_{\hat{\theta}_n}) = ([\partial v_\theta/\partial\theta]_{\theta=\theta_0})^2 V_{\theta_0}(\hat{\theta}_n) + o(1/n).$$

Assume that the Fisher information amount $I(\theta)$ exists, i.e.

$$0 < I(\theta) = \int_{-\infty}^{\infty} \{\partial \log f_\theta(x)/\partial\theta\}^2 f_\theta(x) d\mu < \infty,$$

then we have by Cramér-Rao's inequality that

$$nV_{\theta_0}(v_{\hat{\theta}_n}) \geq ([\partial v_\theta/\partial\theta]_{\theta=\theta_0})^2/I(\theta) + o(1), \tag{3.3}$$

provided that the differentiation under the integral sign is allowed. We further restrict our attention to a family of subclasses $\mathcal{F}_\psi = \{F_\theta : dF_\theta(x)/d\mu = f_\theta(x)$ with the form $\log(f_\theta(x)/f_{\theta_0}(x)) = c(\theta) + \theta\psi(x)$ a.e. $[\mu]$ with $c(0) = 0\}$ of $\mathcal{F}$, where $\psi(x)$ is a function with finite variance at f_{θ_0}. Then we have the following.

THEOREM 3.1. *Assume that the conditions* (A.1), (A.2) *and* (A.3) *hold. Then the bootstrap estimator* $\hat{\theta}_n^*$ *has a minimax property in the above family, i.e.*

$$\max_{\mathcal{F}_\psi} \min_{\hat{\theta}_n} nV_{\theta_0}(v_{\hat{\theta}_n}) = nV_{\theta_0}(v_{\hat{\theta}_n^*}) + o(1),$$

provided that the differentiation under the integral sign is allowed.

The proof is given in Section 4.

Remark 3.3. From Theorem 3.1 we see that the maximum of relative efficiency of the bootstrap estimator $\hat{\theta}_n^*$ is equal to $1 + o(1)$, i.e.

$$\max_{\mathcal{F}_\psi} \left[\left\{ \min_{\hat{\theta}_n} nV_{\theta_0}(v_{\hat{\theta}_n}) \right\} \Big/ nV_{\theta_0}(v_{\hat{\theta}_n^*}) \right] = 1 + o(1).$$

It also follows from Theorem 3.1 that in a semiparametric situation where the class of distributions is sufficiently wide to include $\mathcal{F}_\psi$, it is impossible to get an estimator with a smaller asymptotic variance than $v_{\hat{\theta}_n^*}$.

4. Proofs

In this section the proofs of lemmas and theorems are given. In order to prove Lemma 2.1 we have the following.

LEMMA 4.1. *Let Z be a real random variable. Assume that, for each $i =$ 1, 2, 3, 4, $Y_i = 1$ for $Z \le c_i$, $Y_i = 0$ for $Z > c_i$, where $c_1 \le c_2 \le c_3 \le c_4$. Then*

$$\kappa_3(Y_1, Y_2, Y_3) = E[(Y_1 - p_1)(Y_2 - p_2)(Y_3 - p_3)] = p_1(1 - 2p_2)(1 - p_3),$$

$$\begin{aligned}\kappa_4(Y_1, Y_2, Y_3, Y_4) &= E[(Y_1 - p_1)(Y_2 - p_2)(Y_3 - p_3)(Y_4 - p_4)] \\ &\quad - \mathrm{Cov}(Y_1, Y_2)\,\mathrm{Cov}(Y_3, Y_4) - \mathrm{Cov}(Y_1, Y_3)\,\mathrm{Cov}(Y_2, Y_4) \\ &\quad - \mathrm{Cov}(Y_1, Y_4)\,\mathrm{Cov}(Y_2, Y_3) \\ &= p_1(1 - p_4)(1 - 4p_2 - 2p_3 + 6p_2p_3),\end{aligned}$$

where for each $i = 1, 2, 3, 4$, $p_i = P\{Z \le c_i\}$ and $\mathrm{Cov}(\cdot, \cdot)$ denotes the covariance.

PROOF. It is seen that $p_1 \le p_2 \le p_3 \le p_4$. Since $E(Y_i) = p_i$ $(i = 1, 2, 3)$, $E(Y_1Y_2) = E(Y_1) = p_1$, $E(Y_2Y_3) = E(Y_2) = p_2$, $E(Y_1Y_2Y_3) = E(Y_1) = p_1$, it follows that

$$\begin{aligned}\kappa_3(Y_1, Y_2, Y_3) &= E[(Y_1 - p_1)(Y_2 - p_2)(Y_3 - p_3)] \\ &= E(Y_1Y_2Y_3) - p_1E(Y_2Y_3) - p_2E(Y_1Y_3) - p_3E(Y_1Y_2) + 2p_1p_2p_3 \\ &= p_1(1 - 2p_2)(1 - p_3).\end{aligned}$$

In a similar way, we have

$$\begin{aligned}&E[(Y_1 - p_1)(Y_2 - p_2)(Y_3 - p_3)(Y_4 - p_4)] \\ &\qquad = \{p_1(1 - 2p_2)(1 - p_3) + p_1p_2p_3\}(1 - p_4), \\ &\mathrm{Cov}(Y_i, Y_j) = p_i(1 - p_j) \qquad (1 \le i \le j \le 4).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\kappa_4(Y_1, Y_2, Y_3, Y_4) &= E[(Y_1 - p_1)(Y_2 - p_2)(Y_3 - p_3)(Y_4 - p_4)] \\ &\quad - \mathrm{Cov}(Y_1, Y_2)\,\mathrm{Cov}(Y_3, Y_4) - \mathrm{Cov}(Y_1, Y_3)\,\mathrm{Cov}(Y_2, Y_4) \\ &\quad - \mathrm{Cov}(Y_1, Y_4)\,\mathrm{Cov}(Y_2, Y_3) \\ &= p_1(1 - p_4)(1 - 4p_2 - 2p_3 + 6p_2p_3).\end{aligned}$$

Thus we complete the proof.

PROOF OF LEMMA 2.1. Since $W_n(t) = \sqrt{n}(F_n(t) - F(t))$, it is easily seen that $E[W_n(t)] = 0$ and $\mathrm{Cov}(W_n(t_1), W_n(t_2)) = F(t_1)(1 - F(t_2))$ for $t_1 \le t_2$. From Lemma 4.1 we have

$$\kappa_3(W_n(t_1), W_n(t_2), W_n(t_3)) = (1/\sqrt{n})F(t_1)(1 - 2F(t_2))(1 - F(t_3)) \qquad \text{for} \quad t_1 \le t_2 \le t_3,$$

$$\begin{aligned}&\kappa_4(W_n(t_1), W_n(t_2), W_n(t_3), W_n(t_4)) \\ &\qquad = (1/n)F(t_1)(1 - F(t_4))(1 - 4F(t_2) - 2F(t_3) + 6F(t_2)F(t_3)) \\ &\qquad\qquad \text{for} \quad t_1 \le t_2 \le t_3 \le t_4.\end{aligned}$$

This completes the proof.

In order to prove Lemma 2.3 we have the following.

LEMMA 4.2. *Suppose that, for each* $i = 1, 2, 3, 4$, Y_i *is a real random variable with mean* $E(Y_i) = m_i$. *Then*

$$\begin{aligned}
&E[Y_1(1-Y_2)] = m_1(1-m_2) - \sigma_{12},\\
&E[Y_1(1-2Y_2)(1-Y_3)] = m_1 - m_1m_3 - 2m_1m_2 + 2m_1m_2m_3\\
&\qquad\qquad - \sigma_{13} - 2\sigma_{12} + 2m_1\sigma_{23} + 2m_2\sigma_{13} + 2m_3\sigma_{12} + 2\kappa_{123},\\
&E[Y_1(1-Y_4)(1-4Y_2-2Y_3+6Y_2Y_3)]\\
&\quad = m_1 - 4(\sigma_{12}+m_1m_2) - 2(\sigma_{13}+m_1m_3) - (\sigma_{14}+m_1m_4)\\
&\qquad + 6(\kappa_{123} + m_1\sigma_{23} + m_2\sigma_{13} + m_3\sigma_{12} + m_1m_2m_3)\\
&\qquad + 4(\kappa_{124} + m_1\sigma_{24} + m_2\sigma_{14} + m_4\sigma_{12} + m_1m_2m_4)\\
&\qquad + 2(\kappa_{134} + m_1\sigma_{34} + m_3\sigma_{14} + m_4\sigma_{13} + m_1m_3m_4)\\
&\qquad - 6(\kappa_{1234} + m_4\kappa_{123} + m_3\kappa_{124} + m_2\kappa_{134} + m_1\kappa_{234}\\
&\qquad\quad + m_1m_4\sigma_{23} + m_2m_4\sigma_{13} + m_3m_4\sigma_{12} + m_1m_3\sigma_{24} + m_2m_3\sigma_{14}\\
&\qquad\quad + m_1m_2\sigma_{34} + m_1m_2m_3m_4 + \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}),
\end{aligned}$$

where, for $1 \le i \le j \le k \le r \le 4$, $\sigma_{ij} = \mathrm{Cov}(Y_i, Y_j)$, $\kappa_{ijk} = \kappa_3(Y_i, Y_j, Y_k)$ *and* $\kappa_{ijkr} = \kappa_4(Y_i, Y_j, Y_k, Y_r)$.

PROOF. The first one is easily derived. Since

$$E(Y_1Y_2Y_3) = \kappa_{123} + m_1\sigma_{23} + m_2\sigma_{13} + m_3\sigma_{12} + m_1m_2m_3,$$

it follows that

$$\begin{aligned}
E[Y_1(1-2Y_2)(1-Y_3)] &= E(Y_1) - E(Y_1Y_3) - 2E(Y_1Y_2) + 2E(Y_1Y_2Y_3)\\
&= m_1 - m_1m_3 - 2m_1m_2 + 2m_1m_2m_3\\
&\quad - \sigma_{13} - 2\sigma_{12} + 2m_1\sigma_{23} + 2m_2\sigma_{13} + 2m_3\sigma_{12} + 2\kappa_{123}.
\end{aligned}$$

Since

$$\begin{aligned}
&E[Y_1Y_2Y_3Y_4]\\
&\quad = \kappa_{1234} + \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23} + m_4\kappa_{123} + m_3\kappa_{124} + m_2\kappa_{134} + m_1\kappa_{234}\\
&\qquad + m_1m_4\sigma_{23} + m_2m_4\sigma_{13} + m_3m_4\sigma_{12} + m_1m_3\sigma_{24} + m_2m_3\sigma_{14} + m_1m_2\sigma_{34}\\
&\qquad + m_1m_2m_3m_4,
\end{aligned}$$

it follows that

$$\begin{aligned}
&E[Y_1(1-Y_4)(1-4Y_2-2Y_3+6Y_2Y_3)]\\
&\quad = m_1 - 4E(Y_1Y_2) - 2E(Y_1Y_3) - E(Y_1Y_4) + 6E(Y_1Y_2Y_3) + 4E(Y_1Y_2Y_4)\\
&\qquad + 2E(Y_1Y_3Y_4) - 6E(Y_1Y_2Y_3Y_4),
\end{aligned}$$

hence the desired result follows.

PROOF OF LEMMA 2.3. For $i = 1, 2, 3, 4$, we put $Y_i = F_n(t_i)$ and $m_i = F(t_i) = E[F_n(t_i)]$. Then we have $\sigma_{ij} = (1/n)m_i(1-m_j)$, $\kappa_{ijk} = (1/n\sqrt{n})m_i(1-2m_j)(1-m_k)$, $\kappa_{ijkr} = (1/n^2)m_i(1-m_r)(1-4m_j-2m_k+6m_jm_k)$ for $1 \le i \le j \le k \le r \le 4$. From Lemmas 4.1 and 4.2 we have the conclusion of Lemma 2.3.

PROOF OF LEMMA 2.4. From Lemmas 2.2 and 2.3 it follows that

$$\begin{aligned}
&E[W_n^*(t)] = E[E[W_n^*(t) \mid F_n(t)]] = 0,\\
&\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2)) = E[\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2))]\\
&\qquad\qquad = \{1-(1/n)\}F(t_1)(1-F(t_2)) \qquad \text{for} \quad t_1 \le t_2,\\
&\kappa_3(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3))\\
&\quad = E[\kappa_3(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3) \mid F_n(t_1), F_n(t_2), F_n(t_3))]\\
&\quad = (1/\sqrt{n})\{1-(1/n)\}\{1-(2/n)\}F(t_1)(1-2F(t_2))(1-F(t_3))\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t_1 \le t_2 \le t_3.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
&\kappa_4(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3), W_n^*(t_4))\\
&\quad = E[\kappa_4(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3), W_n^*(t_4) \mid F_n(t_1), F_n(t_2), F_n(t_3), F_n(t_4))]\\
&\qquad + \mathrm{Cov}(\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2)),\\
&\qquad\qquad \mathrm{Cov}(W_n^*(t_3), W_n^*(t_4) \mid F_n(t_3), F_n(t_4)))\\
&\qquad + \mathrm{Cov}(\mathrm{Cov}(W_n^*(t_1), W_n^*(t_3) \mid F_n(t_1), F_n(t_3)),\\
&\qquad\qquad \mathrm{Cov}(W_n^*(t_2), W_n^*(t_4) \mid F_n(t_2), F_n(t_4)))\\
&\qquad + \mathrm{Cov}(\mathrm{Cov}(W_n^*(t_1), W_n^*(t_4) \mid F_n(t_1), F_n(t_4)),\\
&\qquad\qquad \mathrm{Cov}(W_n^*(t_2), W_n^*(t_3) \mid F_n(t_2), F_n(t_3)))\\
&\quad = E[\kappa_4(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3), W_n^*(t_4) \mid F_n(t_1), F_n(t_2), F_n(t_3), F_n(t_4))]\\
&\qquad + \gamma_4 \quad \text{(say)}.
\end{aligned}$$

Since, by Lemma 2.2,

$$\mathrm{Cov}(W_n^*(t_i), W_n^*(t_j) \mid F_n(t_i), F_n(t_j)) = F_n(t_i)(1-F_n(t_j)) \quad \text{for} \quad t_i \le t_j,$$

it follows that

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{Cov}(W_n^*(t_1), W_n^*(t_2) \mid F_n(t_1), F_n(t_2)),\\
&\quad \mathrm{Cov}(W_n^*(t_3), W_n^*(t_4) \mid F_n(t_3), F_n(t_4)))\\
&\quad = \mathrm{Cov}(F_n(t_1)(1-F_n(t_2)), F_n(t_3)(1-F_n(t_4)))\\
&\quad = E[F_n(t_1)(1-F_n(t_2))F_n(t_3)(1-F_n(t_4))]\\
&\qquad - E[F_n(t_1)(1-F_n(t_2))]E[F_n(t_3)(1-F_n(t_4))].
\end{aligned}$$

We put $Y_i = F_n(t_i)$ and $m_i = F(t_i) = E[F_n(t_i)]$ for $i = 1, 2, 3, 4$. Then we have from Lemma 4.2

$$\begin{aligned}\gamma_4 &= E[Y_1(1-Y_2)Y_3(1-Y_4)] - E[Y_1(1-Y_2)]E[Y_3(1-Y_4)] \\ &\quad + E[Y_1(1-Y_3)Y_2(1-Y_4)] - E[Y_1(1-Y_3)]E[Y_2(1-Y_4)] \\ &\quad + E[Y_1(1-Y_4)Y_2(1-Y_3)] - E[Y_1(1-Y_4)]E[Y_2(1-Y_3)] \\ &= (1/n)m_1(1-m_4)(3-8m_2-4m_3+12m_2m_3) \\ &\quad - (2/n^2)m_1(1-m_4)(3-10m_2-5m_3+15m_2m_3) \\ &\quad + (3/n^3)m_1(1-m_4)(1-4m_2-2m_3+6m_2m_3) + o(1/n^2).\end{aligned}$$

Hence we obtain from Lemmas 2.2 and 2.3

$$\begin{aligned}&\kappa_4(W_n^*(t_1), W_n^*(t_2), W_n^*(t_3), W_n^*(t_4)) \\ &\quad = (1/n)\{1-(1/n)\}\{1-(2/n)\}\{1-(3/n)\}F(t_1)(1-F(t_4)) \\ &\qquad \cdot (1-4F(t_2)-2F(t_3)+6F(t_2)F(t_3)) \\ &\qquad - (1/n^2)\{1-(1/n)\}F(t_1)(1-F(t_4)) \\ &\qquad + (1/n)F(t_1)(1-F(t_4))(3-8F(t_2)-4F(t_3)+12F(t_2)F(t_3)) \\ &\qquad - (2/n^2)F(t_1)(1-F(t_4))(3-10F(t_2)-5F(t_3)+15F(t_2)F(t_3)) \\ &\qquad + (3/n^3)F(t_1)(1-F(t_4))(1-4F(t_2)-2F(t_3)+6F(t_2)F(t_3)) + o(1/n^3).\end{aligned}$$

Thus we complete the proof.

PROOF OF LEMMA 3.2. From Lemma 3.1, it follows that the conditional cumulants of T_n^*, given $F_n(t)$, have the form of

$$\begin{aligned}&E(T_n^* \mid F_n) = (1/\sqrt{n})b_1^* + o_p(1/n) = (1/\sqrt{n})b_1 + (1/n)\xi_1 + o_p(1/n), \\ &V(T_n^* \mid F_n) = v_0^* + (1/n)v_1^* + o_p(1/n) = v_0 + (1/\sqrt{n})\xi_2 + (1/n)v_1 + o_p(1/n), \\ &\kappa_3(T_n^* \mid F_n) = (1/\sqrt{n})\beta_3^* + o_p(1/n) = (1/\sqrt{n})\beta_3 + (1/n)\xi_3 + o_p(1/n), \\ &\kappa_4(T_n^* \mid F_n) = (1/n)\beta_4^* + o_p(1/n) = (1/n)\beta_4 + o_p(1/n).\end{aligned}$$

This completes the proof.

PROOF OF LEMMA 3.3. Since $W_n(x) = \sqrt{n}(F_n(x) - F(x))$, it follows that

$$\begin{aligned}\xi_2 &= \int_{-\infty}^{\infty} \phi_1^2(x)dW_n(x) - (1/\sqrt{n})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(x)\phi_1(y)dW_n(x)dW_n(y) \\ &\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(x)\phi_1(y)dW_n(x)dF(y) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(x)\phi_1(y)dF(x)dW_n(y) \\ &= \int_{-\infty}^{\infty} \phi_1^2(x)dW_n(x) - 2m\int_{-\infty}^{\infty} \phi_1(x)dW_n(x) \\ &\quad - (1/\sqrt{n})\left(\int_{-\infty}^{\infty} \phi_1(x)dW_n(x)\right)^2,\end{aligned}$$

where $m = \int_{-\infty}^{\infty} \phi_1(x)dF(x)$. Then we have

$$\begin{aligned} E(\xi_2) &= -(1/\sqrt{n})E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(x)\phi_1(y)dW_n(x)dW_n(y)\right] \\ &= -(1/\sqrt{n})\left\{\int_{-\infty}^{\infty} \phi_1^2(x)dF(x) - \left(\int_{-\infty}^{\infty} \phi_1(x)dF(x)\right)^2\right\} \\ &= O(1/\sqrt{n}). \end{aligned}$$

We also obtain

$$\begin{aligned} E(\xi_2^2) = E&\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(x)\phi_1(y)\{\phi_1(x)\phi_1(y) - 4m\phi_1(x) + 4m^2\}dW_n(x)dW_n(y)\right] \\ &+ o(1/\sqrt{n}) \\ = &\int_{-\infty}^{\infty} \phi_1^2(x)\{\phi_1^2(x) - 4m\phi_1(x) + 4m^2\}dF(x) \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(x)\phi_1(y)\{\phi_1(x)\phi_1(y) - 4m\phi_1(x) + 4m^2\}dF(x)dF(y) \\ &+ o(1/\sqrt{n}) \\ = &\int_{-\infty}^{\infty} \phi_1^2(x)\{\phi_1(x) - 2m\}^2 dF(x) - \left\{\int_{-\infty}^{\infty} \phi_1^2(x)dF(x) - 2m^2\right\}^2 \\ &+ o(1/\sqrt{n}). \end{aligned}$$

Since $V(\xi_2) = E(\xi_2^2) + o(1/\sqrt{n})$, we have the desired result.

PROOF OF THEOREM 3.1. Since the scaling of θ is arbitrary, without loss of generality, we assume that $\theta_0 = 0$ and $I(0) = 1$. It is enough to obtain $\psi(x)$ which maximizes $[\partial v_\theta/\partial\theta]_{\theta=0}$ under the condition

$$1 = \int_{-\infty}^{\infty} \{[(\partial/\partial\theta)\log f_\theta(x)]_{\theta=0}\}^2 f_0(x)d\mu - \int_{-\infty}^{\infty} \{c'(0) + \psi(x)\}^2 f_0(x)d\mu,$$

that is, to get $\psi(x)$ which minimizes

$$\int_{-\infty}^{\infty} \{c'(0) + \psi(x)\}^2 f_0(x)d\mu$$

under the condition

$$\int_{-\infty}^{\infty} \phi_1(x)\{\phi_1(x) - 2m\}\{c'(0) + \psi(x)\}f_0(x)d\mu = 1.$$

We put $h(x) = \phi_1(x)\{\phi_1(x) - 2m\}$. With the Lagrange multipliers λ_0 and λ_1, we have $\psi(x) + c'(0) = \lambda_0 h(x) + \lambda_1$ and it follows that

$$\begin{aligned} &\lambda_0\int_{-\infty}^{\infty} h^2(x)f_0(x)d\mu + \lambda_1\int_{-\infty}^{\infty} h(x)f_0(x)d\mu = 1, \\ &\lambda_0\int_{-\infty}^{\infty} h(x)f_0(x)d\mu + \lambda_1 = 0, \end{aligned}$$

hence

$$\lambda_0 = 1 \Big/ \left[\int_{-\infty}^{\infty} h^2(x) f_0(x) d\mu - \left\{ \int_{-\infty}^{\infty} h(x) f_0(x) d\mu \right\}^2 \right],$$

$$\lambda_1 = - \left\{ \int_{-\infty}^{\infty} h(x) f_0(x) d\mu \right\} \Big/ \left[\int_{-\infty}^{\infty} h^2(x) f_0(x) d\mu - \left\{ \int_{-\infty}^{\infty} h(x) f_0(x) d\mu \right\}^2 \right].$$

From Lemma 3.3 we have

$$\int_{-\infty}^{\infty} \{c'(0) + \psi(x)\}^2 h(x) d\mu = 1 \Big/ \left[\int_{-\infty}^{\infty} h^2(x) f_0(x) d\mu - \left\{ \int_{-\infty}^{\infty} h(x) f_0(x) d\mu \right\}^2 \right]$$
$$= 1/\{V(\xi_2) + o(1)\},$$

hence, by (3.3),

$$\max_{\mathcal{F}_\psi} \min_{\hat{\theta}_n} nV_{\theta_0}(v_{\hat{\theta}_n}) = V(\xi_2) + o(1) = nV_{\theta_0}(v_{\hat{\theta}_n^*}) + o(1).$$

This completes the proof.

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References

Beran, R. (1982). Estimated sampling distributions: bootstrap and competitors, *Ann. Statist.*, **10**, 212–225.

Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1956). Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator, *Ann. Math. Statist.*, **27**, 642–669.

Efron, B. (1979). Bootstrap methods: another look at the jackknife, *Ann. Statist.*, **7**, 1–26.

Efron, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans*, CBMS Regional Conference Series in Applied Mathematics 38, SIAM, Philadelphia.

Withers, C. S. (1983). Expansions for the distribution and quantiles of regular functional of the empirical distribution with applications to nonparametric confidence intervals, *Ann. Statist.*, **11**, 577–587.