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BAYES ESTIMATION OF NUMBER OF SIGNALS

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Abstract. Bayes estimation of the number of signals, q, based on a binomial prior distribution is studied. It is found that the Bayes estimate depends on the eigenvalues of the sample covariance matrix S for white-noise case and the eigenvalues of the matrix $S_2(S_1+A)^{-1}$ for the colored-noise case, where S_1 is the sample covariance matrix of observations consisting only noise, S_2 the sample covariance matrix of observations consisting both noise and signals and A is some positive definite matrix. Posterior distributions for both the cases are derived by expanding zonal polynomial in terms of monomial symmetric functions and using some of the important formulae of James (1964, *Ann. Math. Statist.*, **35**, 475–501).

Key words and phrases: Zonal polynomial, white-noise, colored-noise, Haar measure, partitions.

1. Introduction

In the area of signal processing, signals are observed at different time points from different sources to different sensors. However, due to atmospheric interference, the signals received by the sensors do not remain undistorted. A noise factor affects the signals in the receivers. In this area, a model often used is that the observed signal vector is the sum of a random noise vector and a linear transform of a random signal vector. Noise vector and signal vectors are assumed to be independently distributed as multivariate normal with zero mean vector. One of the important problems in this case is estimating the number of signals transmitted. In the classical case, this problem is equivalent to estimate the multiplicity of smallest eigenvalue of the covariance matrix of the observation vector. Anderson (1963), Krishnaiah (1976), Rao (1983) considered the problem of testing the hypothesis of the multiplicity of the smallest eigenvalue of the covariance matrix. Wax and Kailath (1985), Zhao *et al.* (1986*a*, 1986*b*) considered the problem of estimation of number of signals by information theoretic criteria proposed by Akaike (1972), Rissanen (1978) and Schwartz (1978).

In this paper, we will consider the problem of estimation of the number of signals from a Bayesian point of view. Bayes estimation of the binomial integer parameter n has been discussed by Hamedani and Walter (1988). In the similar fashion, we will estimate the integer parameter q, the number of signals transmitted from sources to the sensors. In Section 2, we discuss the model in signal processing and discuss different cases of the problem. In Section 3, we define some notations and derived some preliminary results useful for solving the problem. In Section 4, we discuss the Bayesian solution for estimating the number of signals for the white-noise case by choosing appropriate prior distributions for the parameters. A discussion for the colored-noise case is in Section 5.

2. Model in signal processing

The following model is used in signal processing

$$\boldsymbol{x}(t) = A\boldsymbol{s}(t) + \boldsymbol{n}(t), \tag{2.1}$$

where

$\boldsymbol{x}(t)$: $p \times 1$ observation vector at time t,

A: $p \times q$ matrix of unknown parameters associated with signals,

$\boldsymbol{s}(t)$: $q \times 1$ vector of unknown random signals,

$\boldsymbol{n}(t)$: $p \times 1$ vector of random noise.

The assumption for the model (2.1) is as follows

$$\begin{aligned} \boldsymbol{s}(t) &\sim N_q(\mathbf{0}, \Psi), \\ \boldsymbol{n}(t) &\sim N_p(\mathbf{0}, \Sigma_1) \end{aligned} \tag{2.2}$$

and $\boldsymbol{s}(t)$ and $\boldsymbol{n}(t)$ are independent. In (2.2), if $\Sigma_1 = \sigma^2 I$ then model (2.1) is called the white-noise model, otherwise it is called a colored-noise model. We will discuss the solution of the problem described in Section 1 for both cases. Under assumption (2.2), we can say from model (2.1) that

$$\boldsymbol{x}(t) \sim N_p(\mathbf{0}, A\Psi A' + \Sigma_1). \tag{2.3}$$

The number of signals transmitted is q ($< p$) which is the rank of $A\Psi A'$. So, here estimation of number of signals is equivalent to the estimation of the rank of $A\Psi A'$.

For the white-noise case, i.e. when $\Sigma_1 = \sigma^2 I$, we will assume that n independent observations on $\boldsymbol{x}(t)$ are available as $x(t_1), \dots, x(t_n)$. Let the sample covariance matrix be $n^{-1}S = (1/n)\sum_{i=1}^{n} \boldsymbol{x}(t_i)\boldsymbol{x}(t_i)'$. Then using (2.3) it is obvious that $S \sim W_p(\Sigma, n)$, where $\Sigma = \Gamma + \sigma^2 I$, Γ ($= A\Psi A'$) is non-negative of rank q ($< p$). We will discuss the problem of estimation of q from a Bayesian point of view, when σ^2 is known and when it is unknown.

For colored-noise case, i.e. when Σ_1 is arbitrary positive definite matrix, we will assume that $n_2^{-1} S_2$ based on the original data set $\boldsymbol{x}(t_1), \dots, \boldsymbol{x}(t_{n_2})$ estimates

Σ_2 $(= \Gamma + \Sigma_1)$ and $n_1^{-1} S_1$ based on a different data set $\boldsymbol{x}(t_{n_2+1}), \ldots, \boldsymbol{x}(t_n)$ independent of the original data set, where $n = n_1 + n_2$, estimates Σ_1. Then

$$
\begin{aligned}
S_2 &= \sum_{i=1}^{n_2} \boldsymbol{x}(t_i)\boldsymbol{x}(t_i)' \sim W_p(\Sigma_2, n_2), \\
S_1 &= \sum_{i=n_2+1}^{n} \boldsymbol{x}(t_i)\boldsymbol{x}(t_i)' \sim W_p(\Sigma_1, n_1)
\end{aligned} \tag{2.4}
$$

and S_1, S_2 are independent. We will discuss the problem of estimation of q in this case by Bayesian procedure.

3. Notations and some preliminary results

Denote

$$\underset{p\times p}{\Lambda_{r+1}} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{r+1}), \tag{3.1}$$

$$\underset{p\times p}{\Lambda_{r1}} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r, 1, \ldots, 1), \tag{3.2}$$

$$\boldsymbol{\lambda}_r = (\lambda_1, \lambda_2, \ldots, \lambda_r)', \tag{3.3}$$

$$\mathcal{D}_r = \{\boldsymbol{\lambda}_r : 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{r-1} < \lambda_r < \infty\}, \tag{3.4}$$

$$\mathcal{D}_{r1} = \{\boldsymbol{\lambda}_r : 0 \le \lambda_1 \le \cdots \le \lambda_r < 1\}. \tag{3.5}$$

Let $P(j, m)$ be the set of all partitions $\tau = (t_1, t_2, \ldots)$ of j into no more than m parts such that $t_1 + t_2 + \cdots = j$, and for $\tau \in P(j, m)$, let $C_\tau(A)$ be the zonal polynomial formed from the eigenvalues of A. The zonal polynomial $C_\tau(A)$ can be expressed as a linear combination of monomial symmetric functions $M_\kappa(A)$ of eigenvalues of A, see James (1964) and Muirhead (1988). If A is a symmetric matrix of order $m \times m$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then

$$M_\kappa(A) = \sum \lambda_{i_1}^{k_1} \lambda_{i_2}^{k_2} \cdots \lambda_{i_p}^{k_p}, \tag{3.6}$$

where p is the number of non-zero parts in the partion $\kappa = (k_1, k_2, \ldots)$ and the summation is over the distinct permutations $(i_1, i_2, \ldots, i_p)$ of p different integers $1, 2, \ldots, m$.

$$C_\tau(A) = \sum_{\kappa \le \tau} c_{\tau,\kappa} M_\kappa(A), \tag{3.7}$$

where $c_{\tau,\kappa}$ are constants and $\kappa < \tau$ represents the lexicographical order of partitions κ and τ of the integer k. The coefficients $c_{\tau,\kappa}$ have been tabulated up to order $k = 12$ by Parkhurst and James (1974). In general, these coefficients have the following recurrence relation

$$c_{\tau,\kappa} = \sum_{\kappa < \mu \le \tau} \frac{[(k_i + t) - (k_j - t)]}{\rho_\tau - \rho_\kappa} c_{\tau,\mu}, \tag{3.8}$$

where $\kappa = (k_1, k_2, \ldots, k_m)$, $\mu = (k_1, \ldots, k_i + t, \ldots, k_j - t, \ldots, k_m)$ for $t = 1, 2, \ldots, k_j$ such that when the parts of the partition μ are arranged in descending order, μ is above κ and below or equal to τ in the lexicographical ordering, and $\rho_\kappa = \sum_1^m k_i(k_i - 1)$. Saw (1977) provides another procedure of computing these coefficients.

Another expansion of zonal polynomials of Kushner and Meisner (1984) can also be used here. However the coefficients of their expansion have not been yet tabulated.

If A and B are two symmetric matrices of order $r \times r$ and $s \times s$ respectively, $r + s = m$, then

$$C_\tau(A \oplus B) = \sum_{\rho,\sigma} a^\tau_{\rho\sigma} C_\rho(A) C_\sigma(B), \tag{3.9}$$

where $(A \oplus B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and where $a^\tau_{\rho\sigma}$ are the Hayakawa coefficients (see Hayakawa (1967)). The summation is over all partitions ρ of k_1 and σ of k_2 such that $k_1 + k_2 = k$, where k is the order of the partition τ. Hayakawa (1967) has tabulated these coefficients to the order $k = 4$. There is no general formula known for these coefficients, however they have a relation with Khatri and Pillai's g-coefficient $g^\tau_{\rho\sigma}$, that is

$$a^\tau_{\rho\sigma} = \binom{k}{k_1} g^\tau_{\rho\sigma} z_\tau / z_\rho z_\sigma, \tag{3.10}$$

where $z_\tau = C_\tau(I_m)/2^m(m/2)_\tau$ (see Davis (1979)). The coefficients $g^\tau_{\rho\sigma}$ up to order $k = 7$ have been tabulated by Khatri and Pillai (1968). Kushner (1988) provides a general formula for $g^\tau_{(m)\sigma}$.

LEMMA 3.1. *Let $U = \mathrm{Diag}(u_1, u_2, \ldots, u_r)$ and $g(U)$ be a symmetric function of $u_1, u_2, \ldots, u_r$, then*

$$\begin{aligned} \int_a^b \cdots \int_a^b C_\tau(U \oplus I_s) g(U) du_1 \cdots du_r \\ = \sum_{\rho,\sigma} a^\tau_{\rho\sigma} C_\sigma(I_s) \sum_{\kappa \leq \rho} c_{\rho,\kappa} \frac{r!}{(r-t)!} \int_a^b \cdots \int_a^b u_1^{k_1} \cdots u_t^{k_t} g(U) du_1 \cdots du_r, \end{aligned} \tag{3.11}$$

where $k_1, k_2, \ldots, k_t$ are the non-zero parts of the partition κ.

PROOF. From (3.9) and (3.7)

$$C_\tau(U \oplus I_s) = \sum_{\rho,\sigma} a^\tau_{\rho\sigma} C_\sigma(I_s) \sum_{\kappa \leq \rho} c_{\rho,\kappa} M_\kappa(U). \tag{3.12}$$

Since $g(U)$ is symmetric in u's, it can be seen from the representation (3.6) of $M_\kappa(U)$ that

$$\int_a^b \cdots \int_a^b M_\kappa(U) g(U) du_1 \cdots du_r = \frac{r!}{(r-t)!} \int_a^b \cdots \int_a^b u_1^{k_1} \cdots u_t^{k_t} g(U) du_1 \cdots du_r.$$

Now the lemma follows from (3.12).

We now prove the following lemmas which we shall use in the next sections.

LEMMA 3.2. *Let* $\tau \in P(j, p)$. *Let*

$$(3.13)\quad B_\tau^r(a, b, c) = r! \int_{\mathcal{D}_{r+1}} \prod_{i=1}^{r} \lambda_i^{a-1} \lambda_{r+1}^{b-1} \exp\left(-\sum_{i=1}^{r+1} \lambda_i / c\right) C_\tau(\Lambda_{r+1})/C_\tau(I) d\boldsymbol{\lambda}_{r+1}$$

for some $a, b, c > 0$. *Then*

$$(3.14)\quad B_\tau^r(a, b, c) = \begin{cases} c^{j+ra+b}\Gamma(j+ra+b) \sum_{\rho,\sigma} a_{\rho\sigma}^\tau \frac{C_\rho(I_{p-r})}{C_\tau(I_p)} & \\ \quad \cdot \sum_{\kappa \le \rho} c_{\rho,\kappa} \frac{r!}{(r-|\kappa|)!} b_\kappa(r, j), & \text{for} \quad r \neq 0, \\ c^{j+b}\Gamma(j+b), & \text{for} \quad r = 0 \end{cases}$$

where $|\kappa|$ *represents the number of non-zero parts in* κ *and*

$$(3.15)\quad b_\kappa(r, j) = \int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^r u_i^{k_i+a-1} du_1 \cdots du_r}{(1+\sum_{i=1}^r u_i)^{j+ra+b}}, \quad \text{for} \quad r \neq 0.$$

LEMMA 3.3. *Let* $\tau \in P(j, p)$. *Let*

$$(3.16)\quad D_\tau^r(a, b) = r! \int_{\mathcal{D}_{r1}} \prod_{i=1}^{r} \lambda_i^{a-1} (1-\lambda_i)^{b-1} C_\tau(\Lambda_{r1})/C_\tau(I) d\boldsymbol{\lambda}_r$$

for some $a, b > 0$. *Then*

$$(3.17)\quad D_\tau^r(a, b) = \begin{cases} \sum_{\rho,\sigma} a_{\rho\sigma}^\tau \frac{C_\sigma(I_{p-r})}{C_\tau(I_p)} \sum_{\kappa \le \rho} c_{\rho,\kappa} \frac{r!}{(r-|\kappa|!)} & \\ \quad \cdot \prod_{i=1}^{r} \frac{\Gamma(a+k_i)\Gamma(b)}{\Gamma(a+b+k_i)}, & \text{for} \quad r \neq 0 \\ 1, & \text{for} \quad r = 0. \end{cases}$$

LEMMA 3.4. (James (1960)) *If* A *and* B *are two symmetric* $k \times k$ *matrices, then*

$$(3.18)\quad \int_{O(k)} \exp\{\operatorname{tr}(BPAP')\} dH(P) = {}_0F_0(A, B),$$

where, $H(P)$ *is an invariant or Haar measure on the orthogonal group* $O(k)$, *and*

$$(3.19)\quad {}_0F_0(A, B) = \sum_{j=0}^{\infty} \sum_{\tau \in P(j,k)} \frac{C_\tau(A) C_\tau(B)}{j! C_\tau(I)}.$$

LEMMA 3.5. (James (1964)) *Let S, T, U be $k \times k$ positive definite matrices, then*

$$\frac{1}{\Gamma_k(a)} \int_{S>0} e^{-\operatorname{tr} S} |S|^{a-(k+1)/2} {}_0F_0(ST,\, U)dS = {}_1F_0(a;\; T,\, U), \tag{3.20}$$

where

$$\Gamma_k(a) = \int_{S>0} e^{-\operatorname{tr} S} |S|^{a-(k+1)/2} dS = \pi^{k(k-1)/4} \prod_{i=1}^{k} \Gamma\left(a - \frac{1}{2}(i-1)\right), \tag{3.21}$$

$${}_1F_0(a;\; T,\, U) = \sum_{j=0}^{\infty} \sum_{\tau \in P(j,k)} \frac{(a)_\tau C_\tau(T) C_\tau(U)}{j! C_\tau(I)}. \tag{3.22}$$

LEMMA 3.6. *Let $\tau \in P(j,\, p)$. Let*

$$E_\tau^r(a,\, b) = \int_{\mathcal{D}_{r1}} \prod_{1}^{r} \lambda_i^{a-1}(1-\lambda_i)^{b-1} \prod_{i>j}(\lambda_i - \lambda_j) \frac{C_\tau(\Lambda_{r1})}{C_\tau(I)} d\boldsymbol{\lambda}_r$$

for some a, $b > 0$. Then

$$E_\tau^r(a,\, b) = \sum_{\rho,\sigma} a_{\rho,\sigma}^\tau \frac{C_\sigma(I) C_\rho(I)}{C_\tau(I)} \cdot \frac{\Gamma_r\left(a + \frac{1}{2}(r-1),\, \rho\right) \Gamma_r\left(b + \frac{1}{2}(r-1)\right)}{\Gamma_r(a+b+r-1,\, \rho)} \frac{\Gamma_r\left(\frac{1}{2}r\right)}{\pi^{r^2/2}}, \tag{3.23}$$

where

$$\Gamma_r(t,\, \rho) = \pi^{r(r-1)/4} \prod_{i=1}^{r} \Gamma\left(t + \rho_i - \frac{1}{2}(i-1)\right)$$

if $(\rho_1,\, \rho_2, \ldots,\, \rho_r)$ is the partition corresponding to ρ.

PROOF OF LEMMA 3.2. We first prove the lemma for $r \neq 0$. Since $C_\tau(A)$, for any smmetric matrix A, is symmetric in eigenvalues of A, it can be seen that, the integrand in (3.13) is symmetric in λ_1, $\lambda_2, \ldots,$ λ_r and therefore the left-hand side of (3.13) is

$$B_\tau^r(a,\, b,\, c) = \int_0^\infty \int_0^{\lambda_{r+1}} \cdots \int_0^{\lambda_{r+1}} \left[\prod_{1}^{r} \lambda_i^{a-1} \lambda_{r+1}^{b-1} \exp\left\{ -\sum_{i=1}^{r+1} \lambda_i / c \right\} \cdot \frac{C_\tau(\Lambda_{r+1})}{C_\tau(I)} \right] d\lambda_1 \cdots d\lambda_{r+1}.$$

Now using the transformation $u_i = \lambda_i/\lambda_{r+1}$, $i = 1, \ldots, r$, $\lambda_{r+1} = \lambda_{r+1}$, and since $C_\tau(hA) = h^j C_\tau(A)$ for any constant h, we get

$$\begin{aligned} B_\tau^r(a, b, c) &= \int_0^\infty \int_0^1 \cdots \int_0^1 \left[\prod_1^r u_i^{a-1} \lambda_{r+1}^{j+ra+b-1}\right] \\ &\quad \cdot \exp\left\{-\left(1+\sum_1^r u_i\right)\lambda_{r+1}/c\right\} \frac{C_\tau(U \oplus I_{p-r})}{C_\tau(I_p)} du_1 \cdots du_r d\lambda_{r+1} \\ &= c^{j+ra+b}\Gamma(j+ra+b) \\ &\quad \cdot \int_0^1 \cdots \int_0^1 \frac{\prod_1^r u_i^{a-1}}{(1+\sum_1^r u_i)^{j+ra+b}} \frac{C_\tau(U \oplus I_{p-r})}{C_\tau(I_p)} du_1 \cdots du_r, \end{aligned}$$

where $U = \mathrm{Diag}(u_1, \ldots, u_r)$. Now the result, for $r \neq 0$, follows from Lemma 3.1.
For $r = 0$, since $C_\tau(\lambda_1 I) = \lambda_1^j C_\tau(I)$,

$$\begin{aligned} B_\tau^0(a, b, c) &= \int_0^\infty \lambda_1^{b-1} \exp\{-\lambda_1/c\} C_\tau(\lambda_1 I)/C_\tau(I) d\lambda_1 \\ &= \int_0^\infty \lambda_1^{j+b-1} \exp\{-\lambda_1/c\} d\lambda_1 = c^{j+b}\Gamma(j+b). \end{aligned}$$

Hence, the lemma.

PROOF OF LEMMA 3.3. First, let $r \neq 0$, then, by the same argument of symmetry as in the proof of Lemma 3.2,

$$\begin{aligned} D_\tau^r(a, b) &= r! \int_{\mathcal{D}_{r1}} \prod_{i=1}^r \lambda_i^{a-1}(1-\lambda_i)^{b-1} C_\tau(\Lambda_{r1})/C_\tau(I) d\boldsymbol{\lambda}_r \\ &= \int_0^1 \cdots \int_0^1 \prod_1^r \lambda_i^{a-1}(1-\lambda_i)^{b-1} C_\tau(\Lambda \oplus I_{p-r})/C_\tau(I) d\boldsymbol{\lambda}_r, \end{aligned}$$

where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r)$. Now the result, for $r \neq 0$, follows from Lemma 3.1.
For $r = 0$, clearly, $D_\tau^0(a, b) = 1$. This proves the lemma.

PROOF OF LEMMA 3.6. From (3.9)

$$C_\tau(\Lambda_{r1}) = \sum_{\rho,\sigma} a_{\rho\sigma}^\tau C_\rho(\Lambda) C_\sigma(I_{p-r}), \tag{3.24}$$

where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r)$. Hence from (3.23) and (3.24), it is enough to show

$$\begin{aligned} &\int_{\mathcal{D}_{r1}} \prod_1^r \lambda_i^{a-1}(1-\lambda_i)^{b-1} \prod_{i>j}(\lambda_i - \lambda_j) \frac{C_\rho(\Lambda)}{C_\rho(I)} d\boldsymbol{\lambda}_r \\ &\quad = \frac{\Gamma_r\left(a+\frac{1}{2}(r-1), \rho\right)\Gamma_r\left(b+\frac{1}{2}(r-1)\right)}{\Gamma_r(a+b+r-1, \rho)} \frac{\Gamma_r\left(\frac{1}{2}r\right)}{\pi^{r^2/2}}. \end{aligned} \tag{3.25}$$

Now, using equation (22) of Constantine (1963) and substituting $R = I_r$, $m = r$, $\kappa = \rho$, we have

$$\int_0^{I_r} |S|^{t-(r+1)/2}|I - S|^{u-(r+1)/2} \frac{C_\rho(S)}{C_\rho(I_r)} dS = \frac{\Gamma_r(t,\, \rho)\Gamma_r(u)}{\Gamma_r(t+u,\, \rho)}. \tag{3.26}$$

Using the transformation $S \to P\Lambda P'$, where P is orthogonal of order r and the first column of P is non-negative and using the result (Anderson (1984))

$$\int_{O(r)} J(S \to P,\, \Lambda) dP = \frac{\pi^{r^2/2} \prod_{i>j}(\lambda_i - \lambda_j)}{\Gamma_r\left(\frac{1}{2}r\right)}$$

we get, from (3.26),

$$\int_{\mathcal{D}_{r1}} \prod_1^r \lambda_i^{t-(r+1)/2}(1-\lambda_i)^{u-(r+1)/2} \prod_{i>j}(\lambda_i - \lambda_j) \frac{C_\rho(\Lambda)}{C_\rho(I)} d\lambda_r = \frac{\Gamma_r(t,\, \rho)\Gamma_r(u)}{\Gamma_r(t+u,\, \rho)} \frac{\Gamma_r(r/2)}{\pi^{r^2/2}}$$

and this proves (3.25).

4. Bayes estimation of q under white-noise

Let

$$S \sim W_p(\Sigma,\, n) \qquad (n > p),$$

where $\Sigma = \Gamma + \sigma^2 I$, Γ is of rank q. We consider two cases, when σ^2 unknown and when $\sigma^2 = 1$.

Case 1. *σ^2 unknown:* Since Γ is of rank q, Σ can be reparametrized into an orthogonal matrix P and Λ_{q+1} defined as in Section 3, such that

$$\Sigma^{-1} = P'\Lambda_{q+1}P,$$

where Λ_{q+1} is defined in Section 3. Then the parameter space

$$\Theta = \{(P,\, \boldsymbol{\lambda}_{q+1},\, q) \colon P \text{ is orthogonal}, \quad 0 < \lambda_1 \le \cdots \le \lambda_{q+1} < \infty, \\ q \in (0,\, 1, \ldots,\, p-1)\}.$$

We consider the following prior on Θ:

$$h(\boldsymbol{\lambda}_{r+1}) = \frac{(r+1)!}{[\Gamma(\alpha)\beta^\alpha]^{(r+1)}} \prod_1^{r+1} \lambda_i^{\alpha-1} \exp\left\{-\sum_1^{r+1} \lambda_i/\beta\right\}, \\ 0 < \lambda_1 \le \cdots \le \lambda_{r+1} < \infty, \tag{4.1}$$

for some α, $\beta > 0$;

$$p(r) = P(q = r) = \binom{p-1}{r} \phi^r (1-\phi)^{p-r-1}, \qquad r = 0,\, 1, \ldots,\, p-1, \tag{4.2}$$

for some $0 \leq \phi \leq 1$, and let the prior on P be $H(P)$ an invariant or Haar measure on the orthogonal group $O(p)$. In addition, we also assume that $(\boldsymbol{\lambda}_{q+1}, q)$ and P are independent and $\boldsymbol{\lambda}_{r+1}$ and q are independent for all r.

Then the Bayes estimate of q under squared loss is

$$\hat{q} = E(q \mid S) = \sum_{r=0}^{p-1} r p(q = r \mid S), \tag{4.3}$$

where $p(q = r \mid S) = f(q = r, S)/f(S)$ is the posterior prior. The density of S is given by

$$f(S \mid \Theta) = C(p, n)|S|^{(n-p-1)/2} \exp\left\{-\frac{1}{2} \operatorname{tr} P'\Lambda_{q+1}PS\right\} |\Lambda_{q+1}|^{n/2},$$

where

$$C(p, n) = \left[2^{np/2}\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{n-i+1}{2}\right)\right]^{-1}.$$

Thus

$$\begin{aligned} f(q = r, S) = p(r) \int_{\mathcal{D}_{r+1}} \int_{O(p)} & C(p, n)|S|^{(n-p-1)/2} \\ & \cdot \exp\left\{-\frac{1}{2} \operatorname{tr} P'\Lambda_{r+1}PS\right\} |\Lambda_{r+1}|^{n/2} h(\boldsymbol{\lambda}_{r+1}) dH(P) d\boldsymbol{\lambda}_{r+1}. \end{aligned}$$

By Lemma 3.4,

$$\begin{aligned} f(q = r, S) = {} & p(r)C(p, n)|S|^{(n-p-1)/2} \\ & \cdot \int_{\mathcal{D}_{r+1}} |\Lambda_{r+1}|^{n/2} h(\boldsymbol{\lambda}_{r+1}) {}_0F_0\left(\Lambda_{r+1}, -\frac{1}{2}S\right) d\boldsymbol{\lambda}_{r+1}. \end{aligned}$$

Now, from (3.19) and (4.1),

$$\begin{aligned} f(q = r, S) = {} & p(r)C(p, n)|S|^{(n-p-1)/2} \frac{(r+1)!}{[\Gamma(\alpha)\beta^{\alpha}]^{(r+1)}} \\ & \cdot \sum_{j=0}^{\infty} \sum_{\tau \in P(j, p)} \frac{1}{j!} C_{\tau}\left(-\frac{1}{2}S\right) \int_{\mathcal{D}_{r+1}} \prod_{i=1}^{r} \lambda_i^{\alpha+n/2-1} \lambda_{r+1}^{\alpha+n(p-r)/2-1} \\ & \cdot \exp\left\{-\sum_{1}^{r+1} \lambda_i/\beta\right\} C_{\tau}(\Lambda_{r+1})/C_{\tau}(I) d\boldsymbol{\lambda}_{r+1}. \end{aligned} \tag{4.4}$$

Now, by Lemma 3.2,

$$\begin{aligned} f(q = r, S) = {} & p(r)C(p, n)|S|^{(n-p-1)/2} \frac{(r+1)!}{r![\Gamma(\alpha)\beta^{\alpha}]^{r+1}} \\ & \cdot \sum_{j=0}^{\infty} \sum_{\tau \in P(j, p)} \frac{1}{j!} C_{\tau}\left(-\frac{1}{2}S\right) B_{\tau}^{(r)}\left(\alpha + \frac{n}{2}, \alpha + n(p-r)/2, \beta\right), \end{aligned}$$

where $B_\tau^{(r)}(a, b, c)$ are defined as in Lemma 3.2. Thus the posterior prior

$$p(q = r \mid S) = \frac{\frac{p(r)(r+1)}{[\Gamma(\alpha)\beta^\alpha]^{(r+1)}} \sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) B_\tau^{(r)}\left(\alpha+\frac{n}{2}, \alpha+\frac{n(p-r)}{2}, \beta\right)}{\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-1} \frac{p(r)(r+1)}{[\Gamma(\alpha)\beta^\alpha]^{(r+1)}} B_\tau^{(r)}\left(\alpha+\frac{n}{2}, \alpha+\frac{n(p-r)}{2}, \beta\right)}$$

and thus, from (4.3),

$$\hat{q} = \frac{\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-1} \frac{p(r)r(r+1)}{[\Gamma(\alpha)\beta^\alpha]^{(r+1)}} B_\tau^{(r)}\left(\alpha+\frac{n}{2}, \alpha+\frac{n(p-r)}{2}, \beta\right)}{\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-1} \frac{p(r)(r+1)}{[\Gamma(\alpha)\beta^\alpha]^{(r+1)}} B_\tau^{(r)}\left(\alpha+\frac{n}{2}, \alpha+\frac{n(p-r)}{2}, \beta\right)}.$$

Now, from (4.2), it can be seen that

(4.5)
$$\hat{q} = \frac{(p-1)\phi \sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-2} A_2(r) B_\tau^{(r+1)}\left(\alpha+\frac{n}{2}, \alpha+\frac{n(p-r-1)}{2}, \beta\right)}{\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-1} A_1(r) B_\tau^{(r)}\left(\alpha+\frac{n}{2}, \alpha+\frac{n(p-r)}{2}, \beta\right)},$$

where, for $i = 1, 2$,

$$A_i(r) = \frac{(r+i)}{[\Gamma(\alpha)\beta^\alpha]^{r+i}} \binom{p-i}{r} \phi^r (1-\phi)^{p-r-i}.$$

A difficult question remains about the convergence of the series in $\hat{q}$. It is difficult to find a radius of convergence but we shall show that the series converge if the parameter β of the prior is small.

From (4.4), the series converge absolutely if the following series converges:

$$\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(\frac{1}{2}S\right) \cdot \int_{\mathcal{D}_{r+1}} \prod_{i=1}^r \lambda_i^{\alpha+n/2-1} \lambda_{r+1}^{\alpha+n(p-r)/2-1} \exp\left\{-\sum_1^{r+1} \lambda_i/\beta\right\} \frac{C_\tau(\Lambda_{r+1})}{C_\tau(I)} d\boldsymbol{\lambda}_{r+1}.$$

Now since $C_\tau(\Lambda_{r+1})/C_\tau(I) \leq \lambda_{r+1}^j$, where j is the degree of the partition τ, the above series is less than or equal to

$$\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(\frac{1}{2}S\right) \int_0^\infty \cdots \int_0^\infty \prod_1^r \lambda_i^{\alpha+n/2-1} \lambda_{r+1}^{\alpha+n(p-r)/2+j-1} \cdot \exp\left\{-\sum_1^{r+1} \lambda_i/\beta\right\} d\boldsymbol{\lambda}_{r+1}$$
$$\leq \left[\Gamma\left(\alpha+\frac{n}{2}\right)\right]^r \beta^{\alpha(r+1)+np/2} \sum_{j=0}^\infty \sum_\tau \frac{\Gamma(\alpha+n(p-r)/2+j)}{j!} \beta^j C_\tau\left(\frac{1}{2}S\right)$$

and since $\sum_\tau C_\tau(S/2) = (\mathrm{tr}(S/2))^j$, it can be seen using ratio test that the above series converges if $(1/2)\,\mathrm{tr}(S) < \beta^{-1}$. Therefore a β, small enough, can be choosen such that all the series in (4.5) converge.

Case 2. *σ^2 known:* We assume without loss of generality that $\sigma^2 = 1$. Thus $\Sigma = \Gamma + I$. Since Γ is of rank q, Σ can be reparametrize into an orthogonal matrix P and Λ_{q1} such that

$$\Sigma^{-1} = P'\Lambda_{q1}P,$$

where Λ_{q1} is defined in Section 3. Then the parameter space

$$\Theta = \{(P, \boldsymbol{\lambda}_q, q) : P \text{ is orthogonal}, \quad 0 < \lambda_1 \le \cdots \le \lambda_q < 1, \\ q \in (0, 1, \ldots, p-1)\}.$$

We consider the same prior on Θ as in Case 1 except the prior on $\boldsymbol{\lambda}_r$ as

$$(4.6) \quad g(\boldsymbol{\lambda}_r) = r! \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^r \prod_1^r \lambda_i^{\alpha-1}(1-\lambda_i)^{\beta-1}, \qquad 0 \le \lambda_1 \le \cdots \le \lambda_r < 1,$$

for some α, $\beta > 0$.

The density of S is given by

$$f(S \mid \Theta) = C(p, n)|S|^{(n-p-1)/2} \exp\left\{-\frac{1}{2}\,\mathrm{tr}\, P'\Lambda_{r1}PS\right\} |\Lambda_{r1}|^{n/2}.$$

Then, by Lemma 3.4,

$$\begin{aligned} f(q=r, S) &= p(r) \int_{\mathcal{D}_{r1}} \int_{O(p)} C(p, n)|S|^{(n-p-1)/2} \\ &\quad \cdot \exp\left\{-\frac{1}{2}\,\mathrm{tr}\, P'\Lambda_{r1}PS\right\} |\Lambda_{r1}|^{n/2} g(\boldsymbol{\lambda}_r)\, dH(P)\, d\boldsymbol{\lambda}_r \\ &= r!\,p(r)C(p, n)|S|^{(n-p-1)/2} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^r \\ &\quad \cdot \int_{\mathcal{D}_{r1}} \prod_1^r \lambda_i^{n/2+\alpha-1}(1-\lambda_i)^{\beta-1}{}_0F_0\left(\Lambda_{r1}, -\frac{1}{2}S\right) d\boldsymbol{\lambda}_r. \end{aligned}$$

It can be seen now as in Case 1, using Lemma 3.3, that

$$\begin{aligned} (4.7) \quad \hat{q} &= E(q \mid S) \\ &= \frac{(p-1)\phi \sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-2} G_2(r) D_\tau^{(r+1)}\left(\alpha + \frac{n}{2}, \beta\right)}{\sum_{j=0}^\infty \sum_\tau \frac{1}{j!} C_\tau\left(-\frac{1}{2}S\right) \sum_{r=0}^{p-1} G_1(r) D_\tau^{(r)}\left(\alpha + \frac{n}{2}, \beta\right)} \end{aligned}$$

where, for $i = 1, 2$

$$G_i(r) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^{r+i-1} \binom{p-i}{r} \phi^r (1-\phi)^{p-r-i}.$$

Regarding the convergence of series in (4.7), it can be seen using the same argument as before and since $C_\tau(\Lambda_{r1})/C_\tau(I) \le 1$ that the series in (4.7) converge if

$$\sum_{j=0}^{\infty} \frac{1}{j!}\left(\operatorname{tr}\left(\frac{1}{2}S\right)\right)^j < \infty,$$

and clearly this is true for all S. Therefore all the series in (4.7) converge for all S.

To illustrate the computation of $\hat{q}$, we consider a special case: $p = 2$. Suppose there are two sensors in the system and the object is to detect if there is any signal, from a source, present in the atmosphere or not. This problem is equivalent to estimation of q ($= 0$ or 1). The formula (4.5), when σ is unknown, and formula (4.7), when σ is known, can be used to estimate q. The coefficients $B_\tau^1(\alpha + n/2,\, \alpha + n/2,\, \beta)$ and $D_\tau^1(\alpha + n/2,\, \beta)$, needed to compute $\hat{q}$, have been tabulated up to order $k = 4$ in the Appendix. These coefficients are computed using Table 1 of Hayakawa (1967) and tables of Parkhurst and James (1974). Although $\hat{q}$ can not be computed exactly because (4.5) and (4.7) involve infinite series, but a close form of $\hat{q}$ can be obtained by taking first few terms of the series.

We generate a random sample of size $N = 6$ from a normal distribution with $q = 0$ and $\sigma^2 = 1$. The sample gives

$$S = \begin{pmatrix} 3.21 & -1.63 \\ -1.63 & 4.22 \end{pmatrix}.$$

When σ is unknown, we consider the priors (4.1) and (4.2) with $\alpha = 1$, $\beta = 1/2$ and $\phi = 1/2$. By taking first five terms of the infinite series in (4.5), we get $\hat{q} \approx 0.1711$. Thus we conclude that $\hat{q} = 0$. When σ is known, we consider the priors (4.2) and (4.6) with $\alpha = 1$, $\beta = 1$ and $\phi = 1/2$. By taking first five terms of the infinite series in (4.7), we get $\hat{q} \approx 0.1586$. Thus we conclude that $\hat{q} = 0$.

5. Bayes estimation of q under colored-noise

Let

$$S_1 \sim W_p(\Sigma_1,\, n_1), \qquad S_2 \sim W_p(\Sigma_2,\, n_2),$$

where $\Sigma_2 = \Gamma + \Sigma_1$, and Γ is of rank q. We assume that S_1 and S_2 are independent. Then the Bayes estimate of q is given by

$$\hat{q} = E(q \mid S_1,\, S_2).$$

We decompose the parameters of Σ_1 and Σ_2 into an orthogonal matrix Q, Λ_{q1} and Σ_1^{-1} such that

$$\Sigma_1^{1/2}\Sigma_2^{-1}\Sigma_1^{1/2} = Q'\Lambda_{q1}Q,$$

where Λ_{q1} is as defined in Section 3 and $\Sigma_1^{1/2}$ is the symmetric square root of Σ_1. Then the parameter space

$$\Theta = \{(Q,\, \Lambda_{q1},\, q,\, \Sigma_1^{-1})\colon Q \text{ is orthogonal}, \quad 0 < \lambda_1 \le \cdots \le \lambda_q < 1, \quad \Sigma_1^{-1} > 0 \quad \text{and} \quad q \in (0,\, 1, \ldots,\, p-1)\}.$$

We consider the following prior on Θ:

$$(5.1)\quad g(\boldsymbol{\lambda}_r) = r!\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^r \prod_1^r \lambda_i^{\alpha-1}(1-\lambda_i)^{\beta-1},$$

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r < 1,$$

$$(5.2)\quad p(r) = P(q=r) = \binom{p-1}{r}\phi^r(1-\phi)^{p-r-1}, \qquad r = 0, 1, \ldots, p-1,$$

$$(5.3)\quad f_1(\Sigma_1^{-1}) = C(p, m)|A|^{-m/2}|\Sigma_1^{-1}|^{(m-p-1)/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_1^{-1}A\right\},$$

for some $A > 0$, and let the prior on Q be $H(Q)$. We also assume that $(\boldsymbol{\lambda}_q, q)$, Q and Σ_1^{-1} are independent, and $\boldsymbol{\lambda}_r$ and q are independent for all r.

The joint density of S_1 and S_2 is given by

$$\begin{aligned} &f(S_1, S_2 \mid \Theta) \\ &\quad = C(p, n_1)C(p, n_2)|\Sigma_1^{-1}|^{(n_1+n_2)/2}|\Lambda_{q1}|^{n_2/2}|S_1|^{(n_1-p-1)/2}|S_2|^{(n_2-p-1)/2} \\ &\quad\quad \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_1^{-1}S_1 - \frac{1}{2}\operatorname{tr}Q'\Lambda_{r1}Q\Sigma_1^{-1/2}S_2\Sigma_1^{-1/2}\right\}. \end{aligned}$$

Thus

$$\begin{aligned} (5.4)\qquad & f(q=r, S_1, S_2) \\ &\quad = C_*(S_1, S_2)p(r) \\ &\quad\quad \cdot \int_{\mathcal{D}_{r1}}\int_{\Sigma_1^{-1}>0}\int_{O(p)} \prod_1^r \lambda_i^{n_2/2}|\Sigma_1^{-1}|^{n/2}f_1(\Sigma_1^{-1})g(\boldsymbol{\lambda}_r) \\ &\quad\quad \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_1^{-1}S_1 - \frac{1}{2}\operatorname{tr}Q'\Lambda_{r1}Q\Sigma_1^{-1/2}S_2\Sigma_1^{-1/2}\right\} \\ &\quad\quad \cdot dH(Q)d\Sigma_1^{-1}d\boldsymbol{\lambda}_r, \end{aligned}$$

where $C_*(S_1, S_2)$, is some constant depending only on S_1 and S_2 and $n = n_1 + n_2$. By Lemma 3.4, from (5.4),

$$\begin{aligned} f(q=r, S_1, S_2) &= C_*(S_1, S_2)p(r) \\ &\quad \cdot \int_{\mathcal{D}_{r1}}\int_{\Sigma_1^{-1}>0} \prod_1^r \lambda_i^{n_2/2}|\Sigma_1^{-1}|^{(m+n-p-1)/2}|A|^{-m/2}g(\boldsymbol{\lambda}_r) \\ &\quad \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_1^{-1}(S_1+A)\right\} {}_0F_0\left(-\frac{1}{2}\Sigma_1^{-1}S_2, \Lambda_{r1}\right) d\Sigma_1^{-1}d\boldsymbol{\lambda}_r. \end{aligned}$$

Now by using the transformation $\Sigma_1^{-1} \to (S_1+A)^{1/2}\Sigma_1^{-1}(S_1+A)^{1/2}$ which has the Jacobian $|S_1+A|^{-(p+1)/2}$, and by using Lemma 3.5,

$$\begin{aligned} &f(q=r, S_1, S_2) \\ &\quad = C_*(S_1, S_2)|A|^{-m/2}|S_1+A|^{-(p+1)/2}\Gamma_p\left(\frac{m+n}{2}\right) \\ &\quad\quad \cdot \int_{\mathcal{D}_{r1}} \prod_1^r \lambda_i^{n_2/2} g(\boldsymbol{\lambda}_r)\,{}_1F_0\left(\frac{m+n}{2}, -\frac{1}{2}(S_1+A)^{-1}S_2, \Lambda_{r1}\right) d\boldsymbol{\lambda}_r. \end{aligned}$$

From (3.22), (5.1), and using Lemma 3.3,

(5.5)
$$
\begin{aligned}
f(q = r,\, S_1,\, S_2)
&= C_{**}(S_1,\, S_2) p(r) \sum_{j=0}^{\infty} \sum_{\tau \in P(j,p)} \frac{1}{j!} \left(\frac{m+n}{2}\right)_\tau C_\tau\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right) \\
&\quad \cdot r! \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^r \int_{\mathcal{D}_{r1}} \prod_1^r \lambda_i^{\alpha+n_2/2-1}(1-\lambda_i)^{\beta-1} C_\tau(\Lambda_{r1})/C_\tau(I)\, d\boldsymbol{\lambda}_r \\
&= C_{**}(S_1,\, S_2) \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^r p(r) \sum_{j=0}^{\infty} \sum_{\tau \in P(j,p)} \left[\frac{1}{j!} \left(\frac{m+n}{2}\right)_\tau \right. \\
&\quad \left. \cdot C_\tau\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right) D_\tau^{(r)}\left(\alpha + \frac{n_2}{2},\, \beta\right)\right],
\end{aligned}
$$

where $C_{**}(S_1,\, S_2)$ is some constant.

It can be seen now, that

$$
\begin{aligned}
\hat{q} = E(q \mid S) &= \sum_{r=0}^{p-1} r f(r,\, S) \Bigg/ \sum_{r=0}^{p-1} f(r,\, S) \\
&= \frac{\begin{array}{c}(p-1)\phi \sum_{j=0}^{\infty} \sum_\tau \frac{1}{j!} \left(\frac{m+n}{2}\right)_\tau C_\tau\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right) \\ \cdot \sum_{r=0}^{p-2} G_2(r) D_\tau^{(r+1)}\left(\alpha + \frac{n_2}{2},\, \beta\right)\end{array}}{\begin{array}{c}\sum_{j=0}^{\infty} \sum_\tau \frac{1}{j!} \left(\frac{m+n}{2}\right)_\tau C_\tau\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right) \\ \cdot \sum_{r=0}^{p-1} G_1(r) D_\tau^{(r)}\left(\alpha + \frac{n_2}{2},\, \beta\right)\end{array}}.
\end{aligned}
$$

Again here it can be seen from (5.5) that the series in $\hat{q}$ converge if

$$
\sum_{j=0}^{\infty} \sum_\tau \frac{1}{j!} \left(\frac{m+n}{2}\right)_\tau C_\tau\left(\frac{1}{2}(S_1+A)^{-1}S_2\right) < \infty.
$$

From Theorem (6.3) of Gross and Richards (1987), the above is true if $\lambda_{\max}((1/2) \cdot (S_1+A)^{-1}S_2) < 1$. Therefore for the large enough parameter A, all the series in $\hat{q}$ converge.

If we choose the prior for $\boldsymbol{\lambda}_r$ as

$$
g_2(\boldsymbol{\lambda}_r) = C \prod_{i=1}^r \lambda_i^{(\nu_1-p-1)/2} (1-\lambda_i)^{(\nu_2-p-1)/2} \prod_{i>j}^r (\lambda_i - \lambda_j), \qquad 0 \le \lambda_1 \le \cdots \le \lambda_r \le 1,
$$

where

$$
C = \frac{\pi^{r^2/2} \Gamma_r\left(\frac{1}{2}(\nu_1+\nu_2)\right)}{\Gamma_r\left(\frac{1}{2}\nu_1\right) \Gamma_r\left(\frac{1}{2}\nu_2\right) \Gamma_r\left(\frac{1}{2}r\right)}
$$

then, proceeding in the same way as before and using Lemma 3.6 we get,

$$\begin{aligned}
&f(q=r,\, S_1,\, S_2) \\
&= C_{**}(S_1,\, S_2)p(r)\sum_{j=0}^{\infty}\sum_{\tau\in P(j,p)}\left[\left(\frac{m+n}{2}\right)_{\tau} C_{\tau}\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right)\right. \\
&\cdot\int_{\mathcal{D}_{r1}}\prod_{1}^{r}\lambda_i^{(\nu_1+n_2-p-1)/2}(1-\lambda_i)^{(\nu_2-p-1)/2} \\
&\left.\cdot\prod_{i>j}(\lambda_i-\lambda_j)C_{\tau}(\Lambda_{r1})/C_{\tau}(I)d\boldsymbol{\lambda}_r\right] \\
&= C_{**}(S_1,\, S_2)p(r)\sum_{j=0}^{\infty}\sum_{\tau\in P(j,p)}\frac{1}{j!}\left(\frac{m+n}{2}\right)_{\tau} C_{\tau}\left(-\frac{1}{2}(S_2+A)^{-1}S_2\right) \\
&\cdot E_{\tau}^{r}\left(\frac{1}{2}(\nu_1+n_2-p+1),\, \frac{1}{2}(\nu_2-p+1)\right),
\end{aligned}$$

where $C_*(S_1,\, S_2)$ is some constant.

Finally, the estimate is

$$\begin{aligned}
\hat{q} &= E(q\mid S) = \sum_{r=0}^{p-1} rf(r,\, S)\Bigg/\sum_{r=0}^{p-1} f(r,\, S) \\
&= \frac{\begin{array}{c}(p-1)\phi\sum_{j=0}^{\infty}\sum_{\tau}\frac{1}{j!}\left(\frac{m+n}{2}\right)_{\tau} C_{\tau}\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right) \\ \cdot\sum_{r=0}^{p-2}F_2(r)E_{\tau}^{(r+1)}\left(\frac{1}{2}(\nu_1+n_2-p+1),\, \frac{1}{2}(\nu_2-p+1)\right)\end{array}}{\begin{array}{c}\sum_{j=0}^{\infty}\sum_{\tau}\frac{1}{j!}\left(\frac{m+n}{2}\right)_{\tau} C_{\tau}\left(-\frac{1}{2}(S_1+A)^{-1}S_2\right) \\ \cdot\sum_{r=0}^{p-1}F_1(r)E_{\tau}^{(r)}\left(\frac{1}{2}(\nu_1+n_2-p+1),\, \frac{1}{2}(\nu_2-p+1)\right)\end{array}},
\end{aligned}$$

where, for $i=1,\,2$

$$F_i(r) = \binom{p-i}{r}\phi^r(1-\phi)^{p-r-i}.$$

Here also the series in $\hat{q}$ converge if $\lambda_{\max}((1/2)(S_1+A)^{-1}S_2) < 1$.

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Appendix

The table for coefficients $D^1_\tau(n/2+\alpha,\,\beta)$ and $B^1_\tau(n/2+\alpha,\,n/2+\alpha,\,\beta)$, where $\gamma_{ni} = \Gamma(n/2+\alpha+i)\Gamma(\beta)/\Gamma(n/2+\alpha+\beta+i)$, $i = 0, 1, 2, \dots$ and $\xi_{ni} = \Gamma(n/2+\alpha+i)\beta^{n/2+\alpha+i}$, $i = 0, 1, 2, \dots$.

degree	τ	$D^1_\tau(n/2+\alpha,\,\beta)$	$B^1_\tau(n/2+\alpha,\,n/2+\alpha,\,\beta)$
0	(0)	γ_{n0}	$\xi^2_{n0}/2$
1	(1)	$(\gamma_{n0}+\gamma_{n2})/2$	$\xi_{n2}\xi_{n1}/2$
2	(2)	$(3\gamma_{n0}+2\gamma_{n1}+3\gamma_{n2})/8$	$(\xi_{n0}\xi_{n2}+\xi^2_{n1})/8$
	(1^2)	γ_{n1}	$\xi^2_{n1}/2$
3	(3)	$(5\gamma_{n0}+3\gamma_{n1}+3\gamma_{n2}+5\gamma_{n3})/16$	$(5\xi_{n0}\xi_{n3}+3\xi_{n1}\xi_{n2})/16$
	(21)	$(\gamma_{n1}+\gamma_{n2})/2$	$\xi_{n1}\xi_{n2}/2$
4	(4)	$(35\gamma_{n0}+20\gamma_{n1}+18\gamma_{n2}$ $+20\gamma_{n3}+35\gamma_{n4})/128$	$(35\xi_{n0}\xi_{n4}+20\xi_{n1}\xi_{n3}+9\xi^2_{n2})/128$
	(31)	$(3\gamma_{n1}+2\gamma_{n2}+3\gamma_{n3})/8$	$(3\xi_{n1}\xi_{n3}+\xi^2_{n2})/8$
	(2^2)	γ_{n2}	ξ^2_{n2}

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