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SOME GEOMETRIC APPLICATIONS OF THE BETA DISTRIBUTION

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Abstract. Let θ be the angle between a line and a "random" k-space in Euclidean n-space R^n. Then the random variable $\cos^2\theta$ has the beta distribution. This result is applied to show (1) in R^n there are exponentially many (in n) lines going through the origin so that any two of them are "nearly" perpendicular, (2) any N-point set of diameter d in R^n lies between two parallel hyperplanes distance $2d\{(\log N)/(n-1)\}^{1/2}$ apart and (3) an improved version of a lemma of Johnson and Lindenstrauss (1984, *Contemp. Math.*, **26**, 189–206). A simple estimate of the area of a spherical cap, and an area-formula for a neighborhood of a great circle on a sphere are also given.

Key words and phrases: Beta distribution, Spherical cap, Johnson-Lindenstrauss Lemma.

1. Introduction

The *beta distribution* Beta (p, q) has a continuous probability density inside the interval (0, 1) given by

$$B(p,q)^{-1}x^{p-1}(1-x)^{q-1}\,,$$

where $p, q > 0$ and

$$B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)\,.$$

If X and Y are independent random variables having the chi-square distributions with degrees of freedom a and b, then the random variable $X/(X+Y)$ has the beta distribution Beta $(a/2, b/2)$ (e.g., Wilks (1962), p. 187). The beta distribution is well known as an appropriate prior distribution for Bayesian inference.

By a *random point* v in Euclidean n-space R^n, we mean a point

$v = (z_1, \ldots, z_n)$ whose coordinates $z_1, \ldots, z_n$ are independent normal variables with zero mean and unit variance. In other words, v is distributed according to the n-dimensional normal distribution $N(O, I)$ with mean O (the origin) and covariance matrix the identity matrix. Let $v_1, \ldots, v_k$ be k independent random points in R^n. If $k < n$ then the vectors $\boldsymbol{Ov}_i$ $(i = 1, \ldots, k)$ span a k-dimensional linear subspace almost surely. We call this subspace a *random k-space* in R^n.

Let L be a fixed 1-space (line) in R^n, and let H be a random k-space in R^n. Let θ be the angle between L and H.

THEOREM 1.1. *The random variables* $\cos^2 \theta$ *and* $\sin^2 \theta$ *have the beta distributions* Beta $(k/2, (n-k)/2)$ *and* Beta $((n-k)/2, k/2)$, *respectively.*

This theorem seems to have appeared in many places and forms (e.g., Muirhead (1982), p. 39 and Watson (1983)). However, for the convenience of readers, we give here a short proof.

PROOF. Since the distribution $N(O, I)$ is "isotropic", we may assume that the 1-space L is a random 1-space taken in advance of the random k-space H. Then we can reverse the order: first take a random k-space H, and then take a random point v and determine the random line $L = Ov$. In this case, we may regard H as a fixed k-space, say

$$H = \{(x_1, \ldots, x_k, 0, \ldots, 0) \colon x_i \in R\}.$$

If we let $v = (z_1, \ldots, z_n)$, then since z_i's are independent normal variables with zero mean and unit variance, the sums

$$z_1^2 + \cdots + z_k^2 \quad \text{and} \quad z_{k+1}^2 + \cdots + z_n^2$$

are independent random variables having the chi-square distributions with degrees of freedom k and $n - k$, respectively. Hence

$$\cos^2 \theta = (z_1^2 + \cdots + z_k^2)/(z_1^2 + \cdots + z_n^2)$$

has the distribution Beta $(k/2, (n-k)/2)$. Similarly, $\sin^2 \theta$ has the distribution Beta $((n-k)/2, k/2)$. □

For a random point v in R^n, the point $v/|v|$ is uniformly distributed on the surface of the unit sphere S centered at the origin in R^n. (Since the probability Pr $(v = 0)$ is zero, we may assume $v \neq 0$.) Hence we have the following.

COROLLARY 1.1. *If* $v = (z_1, \ldots, z_n)$ *is a point uniformly distributed on* *the surface of the unit sphere S centered at the origin in* R^n, *then* $z_1^2 + \cdots + z_k^2$ *is distributed according to* Beta $(k/2, (n-k)/2)$.

In this paper we present some geometric applications of Theorem 1.1 and Corollary 1.1.

2. A neighborhood of a great circle on a sphere

Let S be the unit sphere in R^n centered at the origin O, and let G be a *great circle* on S. More precisely, G is the intersection of the sphere S and a plane (2-space) through the origin O. Let G_δ denote the δ-neighborhood of G in S, that is, the set of points on S within angular distance δ from G. Then the area of G_δ can be easily calculated from Corollary 1.1.

THEOREM 2.1. area $(G_\delta) = (\sin\delta)^{n-2}$ area (S).

PROOF. Let H be the plane in R^n determined by G and let v be a random point uniformly distributed on S. Let θ be the angle between the line Ov and H. Then $\sin^2\theta$ is distributed according to Beta $((n-2)/2, 1)$. Hence $X := \sin^2\theta$ has the probability density function

$$g(x) = B((n-2)/2, 1)^{-1} x^{(n-2)/2-1} = \{(n-2)/2\} x^{(n-2)/2-1} .$$

Therefore,

$$\Pr(\theta < \delta) = \Pr(X < \sin^2\delta) = \int_0^{\sin^2\delta} g(x)\,dx = (\sin\delta)^{n-2} .$$

Thus the probability content of G_δ is $(\sin\delta)^{n-2}$, and hence

$$\text{area}(G_\delta) = (\sin\delta)^{n-2}\,\text{area}(S) . \qquad \square$$

3. The angle between two random lines

Let θ $(0 \le \theta \le \pi/2)$ be the angle between a fixed line L passing through the origin O and a random line $K = Ov$. Then, by Theorem 1.1, the variable $X := \sin^2\theta$ has the probability density function

$$f(x) = (1/B) x^{(n-1)/2-1} (1-x)^{-1/2} \qquad (0 < x < 1)$$

where $B = B((n-1)/2, 1/2)$. Since $\log\Gamma(x)$ $(x > 0)$ is a convex function (e.g., Artin (1964), p. 17), we have

$$\log\Gamma((k+1)/2) + \log\Gamma((k-1)/2) > 2\log\Gamma(k/2)$$

and hence $\log \Gamma((k+1)/2) - \log \Gamma(k/2) > \log \Gamma(k/2) - \log \Gamma((k-1)/2)$, that is,

$$\Gamma((k+1)/2)/\Gamma(k/2) > \Gamma(k/2)/\Gamma((k-1)/2) .$$

Therefore, noting that

$$\{\Gamma((k+1)/2)/\Gamma(k/2)\}\{\Gamma(k/2)/\Gamma((k-1)/2)\} = (k-1)/2$$

we have

$$\Gamma(k/2)/\Gamma((k-1)/2) < \{(k-1)/2\}^{1/2} < \Gamma((k+1)/2)/\Gamma(k/2) .$$

And since $\Gamma(1/2) = \pi^{1/2}$, we have

$$\{(n-2)/(2\pi)\}^{1/2} < B((n-1)/2, 1/2)^{-1} < \{(n-1)/(2\pi)\}^{1/2} . \tag{3.1}$$

THEOREM 3.1. *For any* $0 < \alpha < \pi/2$,

$$F_n(\alpha)(1 - o(1)) < \Pr(\theta < \alpha) < F_n(\alpha) ,$$

where $o(1) \to 0$ *as* $n \to \infty$ *and*

$$F_n(\alpha) = \frac{(\sin \alpha)^{n-1}}{(\pi(n-1)/2)^{1/2} \cos \alpha} .$$

PROOF. Letting $y = \sin^2 \alpha$, we have

$$\begin{aligned} \Pr(\theta < \alpha) &= \Pr(X < y) = \int_0^y f(x)dx \\ &< (1/B)\int_0^y (1-y)^{-1/2} x^{(n-1)/2-1} dx \\ &= 2(1/B)(1-y)^{-1/2}(n-1)^{-1} y^{(n-1)/2} = \frac{2(\sin \alpha)^{n-1}}{B(n-1)\cos \alpha} . \end{aligned}$$

This is less than $F_n(\alpha)$ by (3.1). Letting $t = 1 - (1/n)^{1/2}$,

$$\begin{aligned} \int_0^y f(x)dx &> \int_{ty}^y f(x)dx > (1/B)\int_{ty}^y (1-ty)^{-1/2} x^{(n-1)/2-1} dx \\ &= \frac{2(1/B)}{(n-1)(1-ty)^{1/2}} y^{(n-1)/2}(1 - t^{(n-1)/2}) \end{aligned}$$

$$= \frac{2(1/B)}{(n-1)(1-y)^{1/2}} y^{(n-1)/2}(1 - t^{(n-1)/2})\{(1-y)/(1-ty)\}^{1/2} .$$

Here, $\{(1-y)/(1-ty)\}^{1/2} = 1 - o(1)$, and since

$$\log t^{(n-1)/2} = ((n-1)/2) \log (1 - n^{-1/2})$$
$$< ((n-1)/2)(-n^{-1/2}) < -(n^{1/2} - 1)/2 \to -\infty ,$$

we have $t^{(n-1)/2} = o(1)$. Therefore,

$$\Pr(\theta < \alpha) > \frac{2(\sin \alpha)^{n-1}}{B(n-1)\cos \alpha}(1 - o(1)) > \frac{(n-2)^{1/2}}{(n-1)^{1/2}} F_n(\alpha)(1 - o(1))$$
$$= F_n(\alpha)(1 - o(1)) . \qquad \square$$

Note that in the n-dimensional normal distribution $N(O, I)$, $\Pr(\theta < \alpha)$ corresponds to the probability content of the "double" cone with axis L and angular radius α. Thus we have the following.

COROLLARY 3.1. *Let $C(\alpha)$ $(0 < \alpha < \pi/2)$ be a spherical cap of angular radius α on the surface S of a unit sphere in R^n. Then*

$$\text{area}(C(\alpha))/\text{area}(S) < F_n(\alpha)/2 = (\sin \alpha)^{n-1}/\{(2\pi(n-1))^{1/2} \cos \alpha\}$$

and both sides are asymptotically $(n \to \infty)$ equal.

THEOREM 3.2. *For any $0 < \alpha < \pi/2$, there exist more than*

$$F_n(\alpha)^{-1} = (\pi(n-1)/2)^{1/2} \cos \alpha (\sin \alpha)^{-(n-1)}$$

lines in R^n going through the origin O such that any two of them determine an angle greater than α.

PROOF. Consider a collection of lines passing through the origin in R^n such that

(*) any two lines in the collection determine an angle $> \alpha$.

Let $\Lambda = \{L_1, \ldots, L_m\}$ be such a collection of lines which is *maximal* in the sense that no line can be added to Λ without violating the condition (*). Consider a random line $K = Ov$, and let θ_i be the angle between L_i and K. Then,

$$\Pr(\theta_i < \alpha \text{ for some } i) < m \cdot \Pr(\theta_1 < \alpha) < m \cdot F_n(\alpha) .$$

Hence, if $m \le F_n(\alpha)^{-1}$ then $\Pr(\theta_i < \alpha$ for some $i) < 1$, that is,

$$\Pr(\theta_i > \alpha \text{ for all } i) > 0 .$$

Thus, if $m \le F_n(\alpha)^{-1}$, then there exists a line K such that all angles between K and L_i $(i = 1, \ldots, m)$ are greater than α, contradicting the maximality of Λ. □

Thus, in R^n, there are exponentially many lines going through the origin so that all angles of them are greater than, say 89°. For other related results, see, for example, Erdös and Füredi (1983).

A line going through the center of a sphere in R^n meets the surface of the sphere at two points. Hence, letting $\alpha = 2\delta$ in the above theorem, we have the following.

COROLLARY 3.2. *On the surface of a sphere in R^n, more than*

$$2F_n(2\delta)^{-1} = (2\pi(n-1))^{1/2} \cos 2\delta (\sin 2\delta)^{-(n-1)}$$

caps of angular radius δ $(0 < \delta < \pi/4)$ can be packed.

4. A random hyperplane

Let L be a fixed 1-space, and K be a random 1-space, H a random $(n-1)$-space in R^n. Let θ_K be the angle between the two lines L and K, and θ_H be the angle between L and the hyperplane H. Then by Theorem 1.1, $\sin^2\theta_K$ $(= 1 - \cos^2\theta_K)$ has the same distribution as $\cos^2\theta_H$. Therefore,

$$\begin{aligned} (4.1) \qquad \Pr(\theta_H > \delta) &= \Pr(\cos^2\theta_H < \cos^2\delta) \\ &= \Pr(\sin^2\theta_K < \sin^2(\pi/2 - \delta)) \\ &< F_n(\pi/2 - \delta) = \frac{(\cos\delta)^{n-1}}{(\pi(n-1)/2)^{1/2}\sin\delta} . \end{aligned}$$

THEOREM 4.1. *Any N-point set V in a ball of radius r in R^n lies between some two parallel hyperplanes at distance*

$$2r\{(2 \cdot \log N)/(n-1)\}^{1/2}$$

apart from each other.

For example, if $N = O(n)$ and n tends to infinity, then

$$\{(2 \cdot \log N)/(n-1)\}^{1/2} \to 0 .$$

Thus, for any $N = O(n)$ points in a ball of radius r in R^n, there exists a hyperplane which is close to all these N points, provided that n is sufficiently large.

PROOF. Let $V = \{v_1, \ldots, v_N\}$. By translating V, if necessary, we may suppose the ball of radius r containing V is centered at the origin. If $2 \log N \geq n - 1$, then the theorem is trivial. So we consider the case $2 \log N < n - 1$. Let L_i be the line Ov_i, $i = 1, \ldots, N$. Let H be a random $(n-1)$-space and let θ_i be the angle between L_i and H. Suppose, for a moment, $N \cdot F_n(\pi/2 - \delta) < 1$, where δ is defined by

$$\sin \delta = \{(2 \cdot \log N)/(n-1)\}^{1/2}, \qquad 0 < \delta < \pi/2 .$$

Then applying (4.1),

$$\Pr(\theta_i > \delta \text{ for some } i) < N \cdot F_n(\pi/2 - \delta) < 1 ,$$

and hence $\Pr(\theta_i < \delta \text{ for all } i = 1, \ldots, N) > 0$. Therefore, there exists a hyperplane H such that

$$\theta_i < \delta \qquad \text{for all} \quad i = 1, \ldots, N .$$

In this case the distance between v_i and H is less than

$$r \sin \delta < r\{(2 \cdot \log N)/(n-1)\}^{1/2} .$$

Hence V is sandwiched between the two hyperplanes, each parallel to H, at distance $r\{(2 \cdot \log N)/(n-1)\}^{1/2}$ apart from H.

Now we show that $N \cdot F_n(\pi/2 - \delta) < 1$.

$$\begin{aligned} F_n(\pi/2 - \delta)^{-1} &= \{\pi(n-1)/2\}^{1/2} \sin \delta \, (1 - \sin^2 \delta)^{-(n-1)/2} \\ &= (\pi \log N)^{1/2} (1 - \sin^2 \delta)^{-(n-1)/2} \\ &= (\pi \log N)^{1/2} \exp\{-((n-1)/2) \log(1 - \sin^2 \delta)\} . \end{aligned}$$

Since $\log(1-t) < -t$ for $0 < t < 1$,

$$\begin{aligned} F_n(\pi/2 - \delta)^{-1} &> (\pi \log N)^{1/2} \exp\{(n-1)(\sin \delta)^2/2\} \\ &= (\pi \log N)^{1/2} \exp(\log N) \end{aligned}$$

$$= N(\pi \log N)^{1/2} > N .$$

Hence $N \cdot F_n(\pi/2 - \delta) < 1$. □

For a finite point set in Euclidean space, the distance between the farthest pair in the set is called the *diameter* of the set. Jung's theorem (e.g., Danzer *et al.* (1963)) asserts that if a point set V in R^n has diameter d, then V is contained in a ball of radius $d\{n/(2n+2)\}^{1/2}$. Hence we have the following.

COROLLARY 4.1. *Let V be an N-point set of diameter d in R^n. Then V lies between some two parallel hyperplanes at distance $2d\{(\log N)/(n-1)\}^{1/2}$ apart from each other.*

5. Tails of Beta (p, q)

Let X be distributed according to the beta distribution Beta (p, q). The mean and variance of X are

$$\mu = p/(p+q), \qquad \sigma^2 = pq/\{(p+q)^2(p+q+1)\} ,$$

respectively. Then $(X - \mu)/\sigma$ is asymptotically normally distributed with zero mean and unit variance, when p and q both tend to infinity (e.g., Moran (1968), p. 329). Apart from the asymptotic normality, it seems to be useful to evaluate the probabilities

$$\Pr(X/\mu < 1 - \varepsilon) \qquad \text{and} \qquad \Pr(X/\mu > 1 + \varepsilon)$$

for a given constant $\varepsilon > 0$. In this section, we consider this problem for large p, q. The result (Theorem 5.1) will be used in the next section.

We start with evaluating the probability density function $f(x)$ of the beta distribution Beta (p, q):

$$f(x) = B(p, q)^{-1} x^{p-1}(1-x)^{q-1} \quad (0 < x < 1) .$$

By Stirling's formula,

$$\Gamma(s) = (2\pi/s)^{1/2}(s/e)^s e^{\xi(s)} \qquad \text{for} \qquad s > 0 ,$$

where

$$\xi(s) = \sum_{n=0}^{\infty} \{(s + n + 1/2) \log (1 + (s+n)^{-1}) - 1\} < 1/(12s) ,$$

see, for example, Artin ((1964), p. 24). Since the function $\xi(s)$ is monotone decreasing,

$$\begin{aligned}B(p,q)^{-1} &= \Gamma(p+q)/\{\Gamma(p)\Gamma(q)\} \\ &= \{pq/(2\pi(p+q))\}^{1/2}\{(p+q)/p\}^p\{(p+q)/q\}^q e^{\xi(p+q)-\xi(p)-\xi(q)} \\ &< \{pq/(2\pi(p+q))\}^{1/2}\{(p+q)/p\}^p\{(p+q)/q\}^q \\ &= (p+q)^{3/2}(2\pi pq)^{-1/2}\{(p+q)/p\}^{p-1}\{(p+q)/q\}^{q-1} \\ &= (p+q)^{3/2}(2\pi pq)^{-1/2}(1/\mu)^{p-1}((p/q)/\mu)^{q-1},\end{aligned}$$

where $\mu = p/(p+q)$. Hence

$$f(x) < C\{x/\mu\}^{p-1}\{(p/q)(1-x)/\mu\}^{q-1}, \qquad C := (p+q)^{3/2}(2\pi pq)^{-1/2}.$$

Letting $x = (1+t)\mu$ $(-1 < t < q/p)$, we have

$$f((1+t)\mu) < C(1+t)^{p-1}(1-pt/q)^{q-1}.$$

In the following we suppose $p, q \geq 1$.

LEMMA 5.1. *Suppose* $0 < \varepsilon < 1, p\varepsilon^2 > 6$. *Then*
(1) $f((1-t)\mu) < C\exp(-pt^2/2) < C\exp(-p\varepsilon t/2)$ *for* $\varepsilon < t < 1$, *and*
(2) $f((1+t)\mu) < C\exp(-pa_\varepsilon t)$ *for* $\varepsilon < t < q/p$,
where $\mu = p/(p+q)$ *and* $a_\varepsilon = \varepsilon/2 - \varepsilon^2/3$.

PROOF. (1) Using the inequality $\log(1-t) \leq -t - t^2/2 - t^3/3$ $(t < 1)$,

$$\begin{aligned}\log f((1-t)\mu) &< \log C + (p-1)\log(1-t) + (q-1)\log(1+pt/q) \\ &< \log C + (p-1)(-t-t^2/2-t^3/3) + (q-1)pt/q \\ &< \log C + p(-t-t^2/2-t^3/3) + 2t + pt \\ &< \log C - pt^2/2 - pt^3/3 + 2t.\end{aligned}$$

Since $pt^2 > 6$ for $t > \varepsilon$, this is less than

$$\log C - pt^2/2 = \log\{C\exp(-pt^2/2)\}.$$

(2) First, let $G(t) = (1+t)^{p-1}(1-pt/q)^{q-1}\exp\{p(t^2/2 - t^3/3)\}$. Then $\log G(0) = 0$, and

$$\{\log G(t)\}' = (p-1)/(1+t) - p(q-1)/(q-pt) + pt - pt^2$$

$$< p/(1+t) - p + pt - pt^2 = pt^2/(1+t) - pt^2 < 0 .$$

Hence $G(t) < 1$. Next, let

$$H(t) = (1+t)^{p-1}(1 - pt/q)^{q-1} \exp(pa_\varepsilon t) .$$

Then $H(\varepsilon) = G(\varepsilon) < 1$. And

$$\begin{aligned}\{\log H(t)\}' &= (p-1)/(1+t) - p(q-1)/(q-pt) + p(\varepsilon/2 - \varepsilon^2/3) \\ &< p/(1+t) - p + p(\varepsilon/2 - \varepsilon^2/3) \\ &= p\{3\varepsilon(1+t) - 6t - 2\varepsilon^2(1+t)\}/\{6(1+t)\} .\end{aligned}$$

Since $3\varepsilon(1+t) - 6t < 0$ for $t > \varepsilon$, we have $\log H(t) < 0$. Hence (2) follows. □

THEOREM 5.1. *Let X be a random variable having the beta distribution* Beta (p, q). *For an ε, $0 < \varepsilon < 1$, suppose $p\varepsilon^2 > 6$. Then*

$$\Pr\{X < (1-\varepsilon)\mu\} < (2/\varepsilon)\{(p+q)/(2\pi pq)\}^{1/2} \exp(-p\varepsilon^2/2) ,$$
$$\Pr\{X > (1+\varepsilon)\mu\} < (1/a_\varepsilon)\{(p+q)/(2\pi pq)\}^{1/2} \exp\{-p(\varepsilon^2/2 - \varepsilon^3/3)\} ,$$

and hence

$$\Pr\{|X - \mu| > \varepsilon\mu\} < (2/a_\varepsilon)\{(p+q)/(2\pi pq)\}^{1/2} \exp\{-p(\varepsilon^2/2 - \varepsilon^3/3)\} ,$$

where $\mu = p/(p+q)$ and $a_\varepsilon = \varepsilon/2 - \varepsilon^2/3$.

PROOF. By Lemma 5.1(1), $\Pr\{X < (1-\varepsilon)\mu\}$ is

$$\begin{aligned}\int_0^{(1-\varepsilon)\mu} f(x)dx &< \mu\int_\varepsilon^1 f((1-t)\mu)dt < C\mu\int_\varepsilon^\infty \exp(-p\varepsilon t/2)dt \\ &< (2/\varepsilon)\{(p+q)/(2\pi pq)\}^{1/2} \exp(-p\varepsilon^2/2) .\end{aligned}$$

Similarly, using Lemma 5.1(2), $\Pr\{X > (1+\varepsilon)\mu\}$ is less than

$$(1/a_\varepsilon)\{(p+q)/(2\pi pq)\}^{1/2} \exp\{-p(\varepsilon^2/2 - \varepsilon^3/3)\} .$$

Thus we have the theorem. □

6. Johnson-Lindenstrauss lemma

Johnson and Lindenstrauss (1984) proved an interesting lemma (see

Theorem 6.1) concerning a nearly isometric embedding of n point set in R^n into surprisingly lower dimensions. A slightly improved version of this lemma was given in Frankl and Maehara (1988). Here we present a short proof of this lemma in more improved form.

LEMMA 6.1. *For an* ε, $0<\varepsilon<1$, *let* p, q *be positive numbers such that*

$$q \geq p \geq 2(\varepsilon^2/2-\varepsilon^3/3)^{-1}\log(2q).$$

If X *is a random variable having the distribution* Beta (p,q) *then*

$$\Pr\{|X-\mu|>\varepsilon\mu\}<2(\log 2q)^{-1/2}(2q)^{-2},$$

where $\mu=p/(p+q)$, *the mean of* Beta (p,q).

PROOF. From the condition imposed on p, q, it follows easily that $p\varepsilon^2>6$. Hence applying Theorem 5.1,

$$\begin{aligned}\Pr(|X-\mu|>\varepsilon\mu) &< (2/a_\varepsilon)\{(p+q)/(2\pi pq)\}^{1/2}\exp\{-p(\varepsilon^2/2-\varepsilon^3/3)\}\\ &< (2/a_\varepsilon)\{2q/(2\pi pq)\}^{1/2}\exp\{\log(2q)^{-2}\}\\ &< (2/a_\varepsilon)\{\varepsilon^2/2-\varepsilon^3/3\}^{1/2}\{2\pi\log(2q)\}^{-1/2}(2q)^{-2}\\ &= 2\{2\pi(1/2-\varepsilon/3)\}^{-1/2}(\log 2q)^{-1/2}(2q)^{-2}\\ &< 2(\log 2q)^{-1/2}(2q)^{-2}.\end{aligned}$$

□

THEOREM 6.1. (Johnson-Lindenstrauss lemma) *For an* ε $(0<\varepsilon<1)$ *and an integer* n, *let* k *be positive integers such that*

$$k \geq 4(\varepsilon^2/2-\varepsilon^3/3)^{-1}\log n.$$

Then for any n*-point set* V *in Euclidean space, there exists a map* f *from* V *to a* k*-space such that*

$$1-\varepsilon<|f(u)-f(v)|^2/|u-v|^2<1+\varepsilon \quad \textit{for all } u, v \textit{ of } V, u\neq v.$$

Remark. In Johnson and Lindenstrauss (1984), the function $4(\varepsilon^2/2-\varepsilon^3/3)$ of ε is not specified, and in Frankl and Maehara (1988), an extra condition $(n>k^2)$ is imposed.

PROOF. If $n\leq k$, then the theorem is trivial. Hence we may suppose that $n>k$. Let $V=\{v_1,\ldots,v_n\}$ be any n point set in R^{n+k}. Let H be a random k-space in R^{n+k}, and w_i be the projection of v_i on H. Then for any $i, j\ (i \neq j)$,

$$X_{ij} := |w_i - w_j|^2 / |v_i - v_j|^2$$

is distributed according to Beta $(k/2, n/2)$. Hence by Lemma 6.1,

$$\Pr\{|X_{ij} - k/(n+k)| > \varepsilon k/(n+k)\} < 2(\log n)^{-1/2} n^{-2} .$$

Therefore, the probability that

$$|X_{ij} - k/(n+k)| > \varepsilon k/(n+k) \qquad \text{for some} \quad i, j \quad (i \neq j)$$

is less than $\binom{n}{2} 2(\log n)^{-1/2} n^{-2} < 1$. Thus there exists a k-space H for which

$$(1-\varepsilon)k/(n+k) < |w_i - w_j|^2 / |v_i - v_j|^2 < (1+\varepsilon)k/(n+k) \qquad (i \neq j) .$$

Then letting $f(v_i) = \{(n+k)/k\}^{1/2} w_i$, we have a desired map f. □

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