Ann. Inst. Statist. Math.
Vol. 41, No. 4, 753–764 (1989)

ASYMPTOTIC PROPERTIES OF SOME GOODNESS-OF-FIT TESTS BASED ON THE L_1-NORM

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(Received November 14, 1988; revised January 17, 1989)

Abstract. Some goodness-of-fit tests based on the L_1-norm are considered. The asymptotic distribution of each statistic under the null hypothesis is the distribution of the L_1-norm of the standard Wiener process on $[0, 1]$. The distribution function, the density function and a table of some percentage points of the distribution are given. A result for the asymptotic tail probability of the L_1-norm of a Gaussian process is also obtained. The result is useful for giving the approximate Bahadur efficiency of the test statistics whose asymptotic distributions are represented as the L_1-norms of Gaussian processes.

Key words and phrases: Asymptotic distribution, approximate Bahadur efficiency, L_1-norm, empirical process, goodness-of-fit tests, martingale, symmetry, Wiener process.

1. Introduction

When we want to study the asymptotic distributions of some statistics, it is often useful to investigate the asymptotic behavior of a suitable stochastic process based on observations. A typical example is to study the asymptotic behavior of the empirical process in order to derive the asymptotic distributions of some goodness-of-fit statistics such as the Kolmogorov-Smirnov statistic, Cramér-von Mises statistic, etc. In fact, the asymptotic distributions of these statistics are the distributions of the supremum norm and of the L_2-norm of the Brownian bridge, respectively. Recently, since Shepp (1982), Rice (1982) and Johnson and Killeen (1983) gave some properties of the L_1-norm of the Brownian bridge, the L_1-norm of the empirical process is available as a test statistic (cf. Shorack and Wellner (1986)).

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Besides the empirical process, the martingale term of the empirical process plays an important role in some cases. The martingale term of the uniform empirical process (see (3.1) in Section 3) converges to a standard Wiener process (cf. Khmaladze (1981) and Aki (1986)). Aki (1986) proposed some statistics based on the stochastic process, such as the supremum norm and the L_2-norm of the martingale term of the empirical process. By using Khmaladze's result, we can construct some statistics like the above for testing composite hypotheses (cf. Khmaladze (1981) and Prakasa Rao (1987)). Further, as another example, we can mention the stochastic process investigated by Aki (1987), which converges to a time-changed Wiener process as the sample size tends to infinity.

From the examples described above, we are interested in the distributions of some norms of the Wiener process. The distributions of the supremum norm and the L_2-norm of the Wiener process are well known as limit distributions of the statistics for testing symmetry (cf. Butler (1969) and Rothman and Woodroofe (1972)). In Section 2, we study the distribution of the L_1-norm of the Wiener process. In Section 3, we give some statistics whose asymptotic distributions are the same as the distribution of the L_1-norm of the Wiener process.

2. The L_1-norm of the Wiener process

Let $W(t)$ be a standard Wiener process. We set $\xi = \int_0^1 |W(t)| dt$. Kac (1946) proved that the Laplace transform $L(z)$ of ξ is given by

$$\sum_{j=1}^{\infty} \kappa_j \exp\{-(\delta_j^{3/2} z)^{2/3}\}, \tag{2.1}$$

where

$$P(y) = \frac{(2y)^{1/2}}{3}\left\{ J_{-1/3}\left(\frac{(2y)^{3/2}}{3}\right) + J_{1/3}\left(\frac{(2y)^{3/2}}{3}\right)\right\},$$

δ_j is the positive j-th root of $P'(y)$, and

$$\kappa_j = \frac{1 + 3\int_0^{\delta_j} P(y) dy}{3\delta_j P(\delta_j)}, \qquad j = 1, 2, \dots .$$

Let G_α be the positive stable distribution of order α. For fixed $0 < \alpha < 1$ the function $\exp\{-z^\alpha\}$ is the Laplace transform of G_α (cf. e.g., Feller (1966), p. 424). Zolotarev ((1957), Theorem 4) proved that the characteristic function of G_α $(0 < \alpha < 1)$ is given by

$$f(t)=\exp\left[-|t|^{\alpha}\exp\left\{-i\frac{\pi}{2}(1-|1-\alpha|)\operatorname{sgn} t\right\}\right]. \tag{2.2}$$

Further, Zolotarev ((1964), Theorem 1) showed by inverting (2.2) that the distribution function of G_α is represented as

$$G(x,\alpha)=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\exp\{-V_\alpha(x,u)\}du\,, \tag{2.3}$$

where

$$V_\alpha(x,u)=x^{\alpha/(\alpha-1)}\left\{\frac{\sin\left(\alpha u+\frac{\pi}{2}K(\alpha)\right)}{\cos u}\right\}^{\alpha/(1-\alpha)}\frac{\cos\left((\alpha-1)u+\frac{\pi}{2}K(\alpha)\right)}{\cos u}\,,$$

and

$$K(\alpha)=1-|1-\alpha|\,.$$

THEOREM 2.1. *The distribution function G_ξ of ξ can be expressed in the form*

$$G_\xi(x)=\sum_{j=1}^{\infty}\kappa_j G\left(\frac{x}{\delta_j^{3/2}},\frac{2}{3}\right). \tag{2.4}$$

PROOF. As we described above, (2.4) is given by inverting $L(z)$ formally term by term using Zolotarev's result. We will justify the termwise inversion. Since G is nonnegative, it suffices to show that

$$\sum_{j=1}^{\infty}|\kappa_j|\,G\left(\frac{x}{\delta_j^{3/2}},\frac{2}{3}\right)$$

converges. Note that

$$|\kappa_j|=\left|\frac{1+3\int_0^{\delta_j}P(y)dy}{3\delta_j P(\delta_j)}\right|.$$

We denote by $A_i(z)$ the Airy function (cf. Abramowitz and Stegun (1964)). Let α_j' be the j-th negative zero of $A_i'(z)$ for each $j=1,2,\ldots$. Then the relation $\delta_j=-\alpha_j'/2^{1/3}$ holds. Since $\int_0^{\delta_j}P(y)dy=\int_0^{-\alpha_j'}A_i(-z)dz$, the formula 10.4.83 of Abramowitz and Stegun (1964) implies that $\int_0^{\delta_j}P(y)dy$ is bounded uniformly in j. Note that $\delta_j P(\delta_j) = -\alpha_j' A_i(\alpha_j')$. Johnson and Killeen (1983) showed that

$$\delta_j \geq \frac{1}{2}\left(3\pi\left(j-2+\frac{7}{12}\right)\right)^{2/3}, \quad j \geq 3\,. \tag{2.5}$$

Then the formula 10.4.60 of Abramowitz and Stegun (1964) implies that $|\kappa_j| = O(j^{-5/6})$. Further, from Theorem 1 of Feller ((1966), p. 424),

$$e^{x^{-2/3}} G\left(x, \frac{2}{3}\right) \to 0 \quad \text{as} \quad x \to 0\,.$$

Therefore, for any given $\varepsilon > 0$ and $x > 0$,

$$G\left(\frac{x}{\delta_j^{3/2}}, \frac{2}{3}\right) < \frac{\varepsilon}{1+\dfrac{\delta_j}{x^{2/3}}}$$

holds for sufficiently large j. So we have $G(x/\delta_j^{3/2}, 2/3) = O(j^{-2/3})$, and hence we obtain

$$|\kappa_j|\, G\left(\frac{x}{\delta_j^{3/2}}, \frac{2}{3}\right) = O(j^{-3/2})\,.$$

Thus, the desired result is proved.

Differentiating (2.4) with respect to x, we have the next result.

THEOREM 2.2. *The probability density function g_ξ of ξ can be written as*

$$g_\xi(x) = \sum_{j=1}^{\infty} \kappa_j \frac{2}{\pi} \frac{\delta_j^3}{x^3} \int_{-\pi/2}^{\pi/2} \exp\left\{-\frac{\delta_j^3}{x^2}\varphi(u)\right\} \varphi(u)\, du\,, \tag{2.6}$$

where

$$\varphi(u) = \frac{\sin^2\left(\dfrac{2}{3}u + \dfrac{\pi}{3}\right)\cos\left(-\dfrac{u}{3} + \dfrac{\pi}{3}\right)}{\cos^3(u)}\,.$$

Before proving Theorem 2.2, we show the next lemma.

LEMMA 2.1. *For $u \in (-\pi/2, \pi/2)$, $\varphi(u)$ is monotonically increasing* *and it holds that*

$$\lim_{u \downarrow -\pi/2} \varphi(u) = \frac{4}{27} . \tag{2.7}$$

PROOF. The formula (2.7) can be easily seen, so we prove only that $\varphi(u)$ is monotonically increasing. Differentiating $\varphi(u)$ with respect to u, we have

$$\varphi'(u) = \frac{\sin\left(\frac{2}{3}u + \frac{\pi}{3}\right)}{3\cos^4 u} g(u) ,$$

where

$$\begin{aligned} g(u) = {} & 4\cos\left(\frac{2}{3}u + \frac{\pi}{3}\right)\cos\left(\frac{u}{3} - \frac{\pi}{3}\right)\cos u \\ & + 9\cos\left(\frac{u}{3} - \frac{\pi}{3}\right)\sin\left(\frac{2}{3}u + \frac{\pi}{3}\right)\sin u \\ & + \cos u \sin\left(\frac{2}{3}u + \frac{\pi}{3}\right)\sin\left(-\frac{u}{3} + \frac{\pi}{3}\right) . \end{aligned}$$

It is easy to see that $g(u)$ can be rewritten as

$$\begin{aligned} g(u) = {} & 4\cos^2\left(-\frac{u}{3} + \frac{\pi}{3}\right) + \sin^2\left(\frac{2}{3}u + \frac{\pi}{3}\right) \\ & + 4\sin^2\left(\frac{2}{3}u + \frac{\pi}{3}\right)\cos^2\left(-\frac{u}{3} + \frac{\pi}{3}\right) \\ & - \sin\left(\frac{4}{3}u + \frac{2}{3}\pi\right)\sin\left(-\frac{2}{3}u + \frac{2}{3}\pi\right) . \end{aligned}$$

Noting that

$$\sin\left(-\frac{2}{3}u + \frac{2}{3}\pi\right) = \sin\left(\frac{2}{3}u + \frac{\pi}{3}\right)$$

and

$$\cos\left(-\frac{u}{3} + \frac{\pi}{3}\right) = \sin\left(\frac{u}{3} + \frac{\pi}{6}\right) ,$$

we set $t = \sin(u/3 + \pi/6)$. Then we can obtain that

$$g(u) = -4t^4(8t^2 - 9).$$

Thus we see that $g(u)$ is positive since t is between 0 and 1 when $u \in (-\pi/2, \pi/2)$. This completes the proof.

PROOF OF THEOREM 2.2. Let $0 < a < b < \infty$ be given constants. Then, it suffices to show that the right-hand side of (2.6) converges uniformly in $x \in [a, b]$. Let δ be a positive constant. It is easy to see that, if $v > 0$, $(1/\delta)\exp(\delta v) > v$ holds. Then, since $\varphi(u)$ is positive for $-\pi/2 < u < \pi/2$, we have

$$\begin{aligned}(2.8) \qquad & \int_{-\pi/2}^{\pi/2} \exp\left\{-\frac{\delta_j^3}{x^2}\varphi(u)\right\}\varphi(u)du \\ & \leq \frac{1}{\delta}\int_{-\pi/2}^{\pi/2} \exp\left\{-\left(\frac{\delta_j^3}{x^2} - \delta\right)\varphi(u)\right\}du .\end{aligned}$$

Take $\delta = \delta_1^3/2b^2$. Then, for each $x \in [a, b]$ and for every $j = 1, 2, \ldots$, it holds that $\delta_j^3/x^2 - \delta > 0$. Therefore, from Lemma 2.1, we see that (2.8) is less than

$$\frac{1}{\delta}\pi \exp\left\{-\frac{4}{27}\left(\frac{\delta_j^3}{b^2} - \delta\right)\right\}.$$

Now it is easy to see the result by considering the orders of δ_j and κ_j.

Table 1 was obtained by calculating (2.4) numerically. The series (2.4) converges very rapidly. It is indeed seen that

$$\sum_{j=11}^{\infty} |\kappa_j|\, G\left(\frac{x}{\delta_j^{3/2}}, \frac{2}{3}\right) < 2.0 \times 10^{-12} \quad \text{for} \quad 0 < x < 2 .$$

So, we truncated the summation of (2.4) at $j = 10$. For the calculation of δ's and κ's we used some formulas listed in Abramowitz and Stegun (1964).

We made Fig. 1 by using the formula (2.6).

Besides the theoretical work, we checked the feasibility of Monte Carlo calculations for problems like this. Since the theoretical values of the percentage points are given in Table 1, we can compare the values obtained by Monte Carlo calculations with the theoretical values. We tried to evaluate the percentage points of $\xi = \int_0^1 |W(t)|dt$ by simulations of size $(10{,}000{,}000, N)$, where 10,000,000 is the number of repetitions of experi-

Table 1.

x	$P(\xi \leq x)$	x	$P(\xi \leq x)$	x	$P(\xi \leq x)$
.1914	.05	.4511	.50	.9891	.91
.2239	.10	.4882	.55	1.0204	.92
.2514	.15	.5293	.60	1.0550	.93
.2774	.20	.5753	.65	1.0941	.94
.3032	.25	.6275	.70	1.1390	.95
.3295	.30	.6876	.75	1.1924	.96
.3570	.35	.7586	.80	1.2588	.97
.3861	.40	.8458	.85	1.3481	.98
.4173	.45	.9607	.90	1.4911	.99

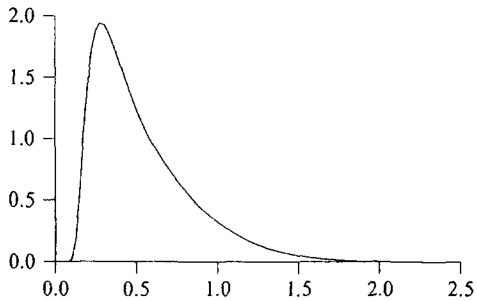

Fig. 1. The probability density function g_ξ of ξ.

ments and N is the number of uniform random numbers used for approximating a sample path of a Wiener process. The results of the simulations are given in the following table. The series of random numbers we used was generated by a physical process available in the computer system of the Institute of Statistical Mathematics, which gives a quite satisfactory random character even in the case of a very long sequence. Considering the feasibility by the computer, we took $N = 64$, 128, 256, 512 and 1024.

From Table 2, we see that, for the cases $N = 512$ and $N = 1024$, the values given by the simulations coincide with the corresponding theoretical

Table 2.

Prob.	Theoretical	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
.05	.1914	.1895	.1903	.1907	.1911	.1912
.10	.2239	.2224	.2230	.2234	.2237	.2237
.30	.3295	.3287	.3290	.3293	.3294	.3295
.50	.4511	.4507	.4509	.4508	.4511	.4512
.70	.6275	.6278	.6276	.6274	.6277	.6274
.90	.9607	.9602	.9605	.9606	.9608	.9605
.95	1.1390	1.1369	1.1380	1.1378	1.1391	1.1388
.99	1.4911	1.4828	1.4870	1.4888	1.4888	1.4900

values up to about the third decimal digit. We think that Monte Carlo calculations can be effective for similar problems for which theoretical results have not been obtained yet. For example, we can mention a weighted L_1-norm of the Wiener process.

Next we will state a theorem on the asymptotic tail probability of the L_1-norm of a Gaussian process. The theorem will be useful for giving the approximate Bahadur efficiency of test statistics whose asymptotic distributions are represented as L_1-norms of Gaussian processes (cf. Bahadur (1960)).

THEOREM 2.3. *Let X be a Gaussian process with values in $C[0,1]$. Suppose that the covariance function $R(s,t)\geq 0$ for each s and $t\in[0,1]$. Then it holds that*

$$\lim_{t\to\infty}\frac{1}{t^2}\log P\left\{\int_0^1 |X(s)|ds>t\right\}=-\frac{1}{2\int_0^1\int_0^1 R(s,t)dsdt}\,.$$

PROOF. The idea of the proof is due to Marcus and Shepp (1971). Consider a step function φ on $[0,1]$ which is represented as

$$\varphi(x)=\sum_{i=1}^{n}\varepsilon_i I_{[(i-1)/n,\,i/n)}(x)\,, \tag{2.9}$$

where $I_A(\cdot)$ is the indicator function of the set A and $\varepsilon_i=1$ or -1 for each $i=1,2,\ldots,n$. Let Φ be the totality of the step functions which are written in the form (2.9) for an integer n. Then it holds that

$$\int_0^1 |X(s)|ds=\sup_{\varphi\in\Phi}\int_0^1 X(s)\varphi(s)ds\,.$$

Let n be a fixed integer. Suppose that a step function φ is written as (2.9) for the integer n. Setting $Y_i=\int_{(i-1)/n}^{i/n}X(t)dt$, we have

$$X_\varphi=\int_0^1 X(t)\varphi(t)dt=\sum_{i=1}^{n}\varepsilon_i Y_i\,.$$

From the assumption of the theorem, we can note that

$$\operatorname{cov}(Y_i,Y_j)=\int_{(i-1)/n}^{i/n}\left(\int_{(j-1)/n}^{j/n}R(s,t)dt\right)ds\geq 0\,,$$

for each $1\leq i\leq j\leq n$. Then we can easily see that $\varepsilon_i=1$ for all i (or $\varepsilon_i=-1$ for all i) maximize

$$\begin{aligned}\sigma_\varphi^2 &= \operatorname{var} X_\varphi \\ &= \sum_{i=1}^{n} \varepsilon_i^2 \operatorname{var}(Y_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \varepsilon_i\varepsilon_j \operatorname{cov}(Y_i, Y_j)\,,\end{aligned}$$

under the condition that n is fixed. But, since the maximum value

$$\max \sigma_\varphi^2 = \int_0^1\int_0^1 R(s,t)\,ds\,dt \qquad \text{(when } n \text{ is fixed)}$$

does not depend on n, we conclude that

$$\sigma^2 = \sup_{\varphi\in\Phi} \operatorname{var} X_\varphi = \int_0^1\int_0^1 R(s,t)\,ds\,dt\,.$$

Hence, the desired result follows from Theorem 2.5 of Marcus and Shepp (1971).

Remark 2.1. In particular, the case where X is a standard Wiener process is known as Marlow's theorem (cf. Marcus and Shepp (1971)). Similarly, as the proof of the above theorem, it is easy to prove the corresponding theorem of the multiparameter Gaussian processes.

3. Goodness-of-fit tests

3.1 *A generalized test for symmetry*

Let $0 < \alpha < 1$ be a given constant and let $X_1, X_2, \ldots, X_n$ be independent random variables having a common continuous distribution function F on $(0, 1)$. Our problem is to test the hypothesis H_1 based on the observations $X_1, X_2, \ldots, X_n$,

H_1: There exists a continuous distribution function G^* defined on $(0, 1)$ such that

$$F(t) = \begin{cases} \alpha G^*(2t) & \text{if} \quad 0 < t \le \dfrac{1}{2}\,, \\[2ex] \alpha + (1-\alpha)(1 - G^*(2-2t)) & \text{if} \quad \dfrac{1}{2} < t < 1\,, \end{cases}$$

holds.

If $\alpha = 1/2$, then the corresponding hypothesis means that F is symmetric about $1/2$. We define, for $i = 1, \ldots, n$,

$$Y_i = \begin{cases} 2X_i & \text{if} \quad X_i \le \dfrac{1}{2}, \\ 2(1 - X_i) & \text{if} \quad X_i > \dfrac{1}{2}, \end{cases}$$

$$\xi_i = \begin{cases} \sqrt{\dfrac{1-\alpha}{\alpha}} & \text{if} \quad X_i \le \dfrac{1}{2}, \\ -\sqrt{\dfrac{\alpha}{1-\alpha}} & \text{if} \quad X_i > \dfrac{1}{2}, \end{cases}$$

and

$$u_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i I_{[0,t]}(Y_i), \qquad 0 \le t \le 1 .$$

From Theorem 2.1 of Aki (1987) and the proof of Theorem 3.1 of Aki (1987), we see that $u_n(t)$ converges weakly to the process $W(G^*(t))$ in $D[0, 1]$, if F satisfies the hypothesis H_1. Therefore, if we define

$$T_n = \int_0^1 |u_n(t)| dH_n(t) ,$$

where $H_n(t)$ is the empirical distribution function of the variables $Y_1, Y_2, \ldots, Y_n$, then T_n converges in distribution to the L_1-norm of the Wiener process under the hypothesis H_1.

For more information about such tests, see Aki (1987), where the meaning of the hypothesis and some properties of statistics by other norms of $u_n(t)$ are discussed.

3.2 *The L_1-norm of the martingale term of the empirical process*

Let $\{F_\theta; \theta \in \Theta\}$ be a set of continuous distribution functions on $[0, 1]$. We assume that there exists $\theta_0 \in \Theta$ such that F_{θ_0} is the uniform distribution over $[0, 1]$. Let $X_1, X_2, \ldots, X_n$ be independent random variables having a common distribution function F_θ. F_n denotes the empirical distribution function for $X_1, X_2, \ldots, X_n$. Let

$$W_n(t) = \sqrt{n} \left(F_n(t) - \int_0^t \frac{1 - F_n(s)}{1 - s} ds \right). \tag{3.1}$$

We consider testing the hypothesis that $\theta = \theta_0$ by the test statistic $T_n = \int_0^1 | W_n(t)| dt$. Then, from Theorem 2.1 of Aki (1986), under the null hypothesis, T_n converges in distribution to $\int_0^1 |W(t)|dt$ as $n \to \infty$. From Marlow's theorem (cf. Theorem 2.3 in this paper), it holds that

$$\lim_{t\to\infty} \frac{1}{t^2} \log P\left\{\int_0^1 |W(s)|ds > t\right\} = -\frac{3}{2}.$$

Similarly, as the proof of Theorem 4.2 of Aki (1986), it is easy to see that $\{T_n\}$ is a standard sequence in the Bahadur sense if F_θ satisfies Conditions A and B of Aki (1986) for every $\theta \in \Theta$. Let

$$b(\theta) = \int_0^1 \left| F_\theta(t) - \int_0^t \frac{1 - F_\theta(s)}{1 - s} ds \right| dt.$$

Then, the approximate Bahadur slope of T_n is given by $3b^2(\theta)$.

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