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HIGHER ORDER ASYMPTOTICS IN ESTIMATION FOR TWO-SIDED WEIBULL TYPE DISTRIBUTIONS

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Abstract. We consider the estimation problem of a location parameter θ on a sample of size n from a two-sided Weibull type density $f(x-\theta) = C(\alpha)\exp(-|x-\theta|^{\alpha})$ for $-\infty < x < \infty$, $-\infty < \theta < \infty$ and $1 < \alpha < 3/2$, where $C(\alpha) = \alpha/\{2\Gamma(1/\alpha)\}$. Then the bound for the distribution of asymptotically median unbiased estimators is obtained up to the 2α-th order, i.e., the order $n^{-(2\alpha-1)/2}$. The asymptotic distribution of a maximum likelihood estimator (MLE) is also calculated up to the 2α-th order. It is shown that the MLE is not 2α-th order asymptotically efficient. The amount of the loss of asymptotic information of the MLE is given.

Key words and phrases: 2α-th order asymptotically median unbiased estimator, 2α-th order asymptotic distribution, 2α-th order asymptotic efficiency, Edgeworth expansion, maximum likelihood estimator.

1. Introduction

Higher order asymptotics have been studied by Pfanzagl and Wefelmeyer (1978, 1985), Ghosh *et al.* (1980) and Akahira and Takeuchi (1981), Akahira (1986), Akahira *et al.* (1988), among others, under suitable regularity conditions.

In non-regular cases when the regularity conditions do not necessarily hold, higher order asymptotics were discussed by Akahira and Takeuchi (1981), Akahira (1987, 1988*a*, 1988*b*), Pfanzagl and Wefelmeyer (1985), Sugiura and Naing (1987) and others.

In this paper we consider the estimation problem of a location parameter θ on a sample of size n from a two-sided Weibull type density $f(x-\theta) = C(\alpha)\exp(-|x-\theta|^{\alpha})$ for $-\infty < x < \infty$, $-\infty < \theta < \infty$ and $1 < \alpha < 3/2$, where $C(\alpha) = \alpha/\{2\Gamma(1/\alpha)\}$. It is noted that there is a Fisher information amount and a first order derivative of $f(x)$ at $x = 0$, but there is no second order one of $f(x)$ at $x = 0$. It is also seen in Akahira (1975) that the order of consistency is equal to $n^{-1/2}$ in this situation. Then we shall obtain the bound for the distribution of asymptotically median unbiased estimators of θ up to the 2α-th order, i.e., the order $n^{-(2\alpha-1)/2}$. We shall also get the asymptotic distribution of the maximum likelihood estimator (MLE) of θ up to the 2α-th order and see that the MLE is not generally 2α-th order asymptotically efficient. Further, we shall obtain the amount of the loss of asymptotic information of the MLE.

2. The 2α-th order asymptotic bound for the distribution of second order AMU estimators

Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with a two-sided Weibull type density $f(x-\theta) = C(\alpha)\exp\{-|x-\theta|^{\alpha}\}$ for $-\infty < x < \infty$ where θ is a real-valued parameter, $1 < \alpha < 3/2$ and $C(\alpha) = \alpha/\{2\Gamma(1/\alpha)\}$ with a Gamma function $\Gamma(u)$, i.e., $\Gamma(u) = \int_0^{\infty} x^{u-1}e^{-x}dx$ $(u > 0)$.

We denote by $P_{\theta,n}$ the n-fold products of probability measure P_θ with the above density $f(x-\theta)$. An estimator $\hat{\theta}_n$ of θ based on $X_1, \ldots, X_n$ is called a 2α-th order asymptotically median unbiased (AMU) estimator if for any $\eta \in R^1$, there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta: |\theta-\eta|<\delta} n^{(2\alpha-1)/2} \left| P_{\theta,n}\{\hat{\theta}_n \le \theta\} - \frac{1}{2} \right| = 0 ,$$

$$\lim_{n\to\infty} \sup_{\theta: |\theta-\eta|<\delta} n^{(2\alpha-1)/2} \left| P_{\theta,n}\{\hat{\theta}_n \ge \theta\} - \frac{1}{2} \right| = 0 .$$

We denote by $A_{2\alpha}$ the class of all best asymptotically normal and 2α-th order AMU estimators. For a $\hat{\theta}_n$ 2α-th order AMU, $G_0(t,\theta) + n^{-(2\alpha-1)/2}G_1(t,\theta)$ is defined to be the 2α-th order asymptotic distribution of $\sqrt{n}\,(\hat{\theta}_n - \theta)$ (or $\hat{\theta}_n$ for short) if for any $t \in R^1$ and each $\theta \in R^1$

$$\lim_{n\to\infty} n^{(2\alpha-1)/2} | P_{\theta,n}\{\sqrt{n}\,(\hat{\theta}_n - \theta) \le t\} - G_0(t,\theta) - n^{-(2\alpha-1)/2}G_1(t,\theta)| = 0 .$$

In order to obtain the bound for the distribution of 2α-th order AMU estimators of θ, for arbitrary but fixed θ_0, we consider the problem of testing hypothesis H: $\theta = \theta_0 + tn^{-1/2}$ $(t > 0)$ against the alternative K: $\theta = \theta_0$. Then the log-likelihood ratio test statistic Z_n is given by

$$\begin{aligned} Z_n &= \sum_{i=1}^{n} \log\{f(X_i - \theta_0)/f(X_i - \theta_0 - tn^{-1/2})\} \\ &= -\sum_{i=1}^{n} (|X_i - \theta_0|^{\alpha} - |X_i - \theta_0 - tn^{-1/2}|^{\alpha}) . \end{aligned}$$

In order to obtain the asymptotic cumulants of Z_n, we need the following lemma.

LEMMA 2.1. *If* $h_\Delta(x) = (x+\Delta)^\alpha - x^\alpha$ *for* $\Delta > 0$, *then*

$$\int_0^\infty h_\Delta^2(x)e^{-x^\alpha}dx = \alpha\Gamma\left(2-\frac{1}{\alpha}\right)\Delta^2 + \alpha(\alpha-1)\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^3 - \frac{1+\gamma}{2\alpha+1}\Delta^{2\alpha+1} + o(\Delta^{2\alpha+1}),$$

$$\int_0^\infty h_\Delta^3(x)e^{-x^\alpha}dx = \alpha^2\Gamma\left(3-\frac{2}{\alpha}\right)\Delta^3 + O(\Delta^4),$$

where

$$\gamma = \frac{\alpha(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{2(2\alpha-1)\Gamma(2-\alpha)}.$$

PROOF. First we have

$$\int_0^\infty h_\Delta^2(x)e^{-x^\alpha}dx = \int_0^\infty (x+\Delta)^{2\alpha}e^{-x^\alpha}dx - 2\int_0^\infty (x+\Delta)^\alpha x^\alpha e^{-x^\alpha}dx + \int_0^\infty x^{2\alpha}e^{-x^\alpha}dx. \tag{2.1}$$

Since for $\beta > 0$

$$\int_0^\infty x^{\beta-1}e^{-x^\alpha}dx = \frac{1}{\alpha}\Gamma\left(\frac{\beta}{\alpha}\right), \tag{2.2}$$

it follows that

$$\begin{aligned}\int_0^\infty (x+\Delta)^{2\alpha}e^{-x^\alpha}dx &= -\frac{\Delta^{2\alpha+1}}{2\alpha+1} + \frac{\alpha}{2\alpha+1}\int_0^\infty (x+\Delta)^{2\alpha+1}x^{\alpha-1}e^{-x^\alpha}dx \\ &= -\frac{\Delta^{2\alpha+1}}{2\alpha+1} + \frac{\alpha}{2\alpha+1} \\ &\quad \cdot\int_0^\infty \Big\{ x^{3\alpha} + (2\alpha+1)x^{3\alpha-1}\Delta + \alpha(2\alpha+1)x^{3\alpha-2}\Delta^2 \\ &\quad + \frac{1}{3}\alpha(2\alpha-1)(2\alpha+1)x^{3\alpha-3}\Delta^3 \Big\} e^{-x^\alpha}dx + O(\Delta^4)\end{aligned} \tag{2.3}$$

$$= \frac{1}{2\alpha+1}\Gamma\left(3+\frac{1}{\alpha}\right) + 2\Delta + \alpha\Gamma\left(3-\frac{1}{\alpha}\right)\Delta^2$$
$$+ \frac{1}{3}\alpha(2\alpha-1)\Gamma\left(3-\frac{2}{\alpha}\right)\Delta^3 - \frac{1}{2\alpha+1}\Delta^{2\alpha+1} + O(\Delta^4)\,.$$

From (2.2), we obtain

$$\int_0^\infty (x+\Delta)^\alpha x^\alpha e^{-x^\alpha}dx$$
$$= -\frac{\alpha}{\alpha+1}\int_0^\infty (x+\Delta)^{\alpha+1}(x^{\alpha-1} - x^{2\alpha-1})e^{-x^\alpha}dx$$
$$= \frac{\alpha}{\alpha+1}\int_0^\infty \left\{ x^{\alpha+1} + (\alpha+1)x^\alpha\Delta + \frac{1}{2}\alpha(\alpha+1)x^{\alpha-1}\Delta^2 + \frac{1}{6}\alpha(\alpha-1)(\alpha+1)x^{\alpha-2}\Delta^3 \right\}(x^{2\alpha-1} - x^{\alpha-1})e^{-x^\alpha}dx$$
$$- \frac{\alpha}{\alpha+1}\int_0^\infty R(\Delta)x^{\alpha-1}e^{-x^\alpha}dx + O(\Delta^4)\,,$$

where

$$R(\Delta) = (x+\Delta)^{\alpha+1} - x^{\alpha+1} - (\alpha+1)\Delta x^\alpha - \frac{1}{2}\alpha(\alpha+1)\Delta^2 x^{\alpha-1}$$
$$- \frac{1}{6}\alpha(\alpha-1)(\alpha+1)\Delta^3 x^{\alpha-2}\,.$$

Then the remainder term $R(\Delta)$ of the Taylor expansion is represented by

$$R(\Delta) = K_\alpha\int_0^\Delta (\Delta-t)^3(x+t)^{\alpha-3}dt\,,$$

where $0 \le t \le \Delta$ and $K_\alpha = \alpha(\alpha+1)(\alpha-1)(\alpha-2)/6$. Since $1 - x^\alpha < e^{-x^\alpha} < 1$, it follows that

$$\int_0^\infty (x+t)^{\alpha-3}x^{\alpha-1}(1-e^{-x^\alpha})dx \le \int_0^\infty x^{2\alpha-4}(1-e^{-x^\alpha})dx$$
$$= \int_0^1 x^{2\alpha-4}(1-e^{-x^\alpha})dx + \int_1^\infty x^{2\alpha-4}(1-e^{-x^\alpha})dx$$
$$\le \int_0^1 x^{3\alpha-4}dx + \int_1^\infty x^{2\alpha-4}dx$$

$$= \frac{\alpha}{3(\alpha-1)(3-2\alpha)}.$$

Since $\int_0^\infty (x+t)^{\alpha-3}x^{\alpha-1}dx = t^{2\alpha-3}B(\alpha, 3-2\alpha)$, we have

$$\begin{aligned}
\int_0^\infty R(\Delta)x^{\alpha-1}e^{-x^\alpha}dx &= \int_0^\infty R(\Delta)x^{\alpha-1}dx - \int_0^\infty R(\Delta)x^{\alpha-1}(1-e^{-x^\alpha})dx \\
&= K_\alpha \int_0^\Delta (\Delta-t)^3 \left\{ \int_0^\infty (x+t)^{\alpha-3}x^{\alpha-1}dx \right\} dt + o(\Delta^{2\alpha+1}) \\
&= K_\alpha B(\alpha, 3-2\alpha) \int_0^\Delta t^{2\alpha-3}(\Delta-t)^3 dt + o(\Delta^{2\alpha+1}) \\
&= K_\alpha B(\alpha, 3-2\alpha) B(2\alpha-2, 4)\Delta^{2\alpha+1} + o(\Delta^{2\alpha+1}) \\
&= -\frac{(\alpha+1)(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{4(2\alpha+1)(2\alpha-1)\Gamma(2-\alpha)}\Delta^{2\alpha+1} + o(\Delta^{2\alpha+1}).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
(2.4) \qquad & \int_0^\infty (x+\Delta)^\alpha x^\alpha e^{-x^\alpha}dx \\
&= \frac{1}{\alpha+1}\left\{ \Gamma\left(3+\frac{1}{\alpha}\right) - \Gamma\left(2+\frac{1}{\alpha}\right) \right\} \\
&\quad + \Delta + \frac{\alpha}{2}\left\{ \Gamma\left(3-\frac{1}{\alpha}\right) - \Gamma\left(2-\frac{1}{\alpha}\right) \right\}\Delta^2 \\
&\quad + \frac{1}{6}\alpha(\alpha-1)\left\{ \Gamma\left(3-\frac{2}{\alpha}\right) - \Gamma\left(2-\frac{2}{\alpha}\right) \right\}\Delta^3 \\
&\quad + \frac{\alpha(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{4(2\alpha+1)(2\alpha-1)\Gamma(2-\alpha)}\Delta^{2\alpha+1} + o(\Delta^{2\alpha+1}) \\
&= \frac{1}{\alpha}\Gamma\left(2+\frac{1}{\alpha}\right) + \Delta + \left(\frac{\alpha-1}{2}\right)\Gamma\left(2-\frac{1}{\alpha}\right)\Delta^2 \\
&\quad + \frac{1}{6}(\alpha-1)(\alpha-2)\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^3 \\
&\quad + \frac{\alpha(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{4(2\alpha+1)(2\alpha-1)\Gamma(2-\alpha)}\Delta^{2\alpha+1} + o(\Delta^{2\alpha+1}),
\end{aligned}$$

and by (2.2), $\int_0^\infty x^{2\alpha}e^{-x^\alpha}dx = \Gamma(2+1/\alpha)/\alpha$. From (2.1) to (2.4), we have

$$\int_0^\infty h_\Delta^2(x)e^{-x^\alpha}dx = \alpha\Gamma\left(2-\frac{1}{\alpha}\right)\Delta^2 + \alpha(\alpha-1)\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^3$$

$$-\frac{1+\gamma}{2\alpha+1}\Delta^{2\alpha+1} + o(\Delta^{2\alpha+1}),$$

where $\gamma = \alpha(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)/2(2\alpha-1)\Gamma(2-\alpha)$. We also obtain

$$\int_0^\infty h_\Delta^3(x)e^{-x^\alpha}dx = \int_0^\infty \{(x+\Delta)^\alpha - x^\alpha\}^3 e^{-x^\alpha}dx$$

$$= \alpha^3\Delta^3\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}dx + O(\Delta^4)$$

$$= \alpha^2\Gamma\left(3-\frac{2}{\alpha}\right)\Delta^3 + O(\Delta^4).$$

Thus we complete the proof.

In the following lemma we obtain the asymptotic mean, variance and third-order cumulant of Z_n, under H and K.

LEMMA 2.2. *The asymptotic mean, variance and third-order cumulant of Z_n are given as follows: Under K: $\theta = \theta_0$,*

$$E_{\theta_0}(Z_n) = \frac{I}{2}t^2 - \frac{k}{2}t^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}),$$

$$V_{\theta_0}(Z_n) = It^2 - kt^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}),$$

$$K_{3,\theta_0}(Z_n) = o(n^{-(2\alpha-1)/2}),$$

and under H: $\theta = \theta_0 + tn^{-1/2}$

$$E_{\theta_0+tn^{-1/2}}(Z_n) = -\frac{I}{2}t^2 + \frac{k}{2}t^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}),$$

$$V_{\theta_0+tn^{-1/2}}(Z_n) = It^2 - kt^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}),$$

$$K_{3,\theta_0+tn^{-1/2}}(Z_n) = o(n^{-(2\alpha-1)/2}),$$

where

$$I = E_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(X-\theta)\right\}^2\right]$$

$$= -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X-\theta)\right] = \alpha(\alpha-1)\Gamma(1-(1/\alpha))/\Gamma(1/\alpha)$$

and

$$k=\alpha\left\{B(\alpha+1,\alpha+1)+\frac{\gamma}{2\alpha+1}\right\}\Big/\Gamma(1/\alpha)$$

with

$$B(u,v)=\int_0^1 x^{u-1}(1-x)^{v-1}dx \qquad (u,v>0).$$

PROOF. Without loss of generality, we assume that $\theta_0=0$. Putting $\Psi_\Delta(x)=|x|^\alpha-|x-\Delta|^\alpha$ with $\Delta>0$, we have $Z_n=-\sum_{i=1}^n \Psi_\Delta(X_i)$ where $\Delta=tn^{-1/2}$. Since

$$\Psi_\Delta(x)=\begin{cases} x^\alpha-(x-\Delta)^\alpha & \text{for} \quad x\geq\Delta\,, \\ x^\alpha-(\Delta-x)^\alpha & \text{for} \quad 0\leq x<\Delta\,, \\ (-x)^\alpha-(\Delta-x)^\alpha & \text{for} \quad x<0\,, \end{cases} \tag{2.5}$$

it follows that

$$\begin{aligned} E_0[\Psi_\Delta(X)] &= C(\alpha)\left[-\int_0^\infty\{(x+\Delta)^\alpha-x^\alpha\}e^{-x^\alpha}dx\right. \\ &\qquad +\int_0^\Delta\{x^\alpha-(\Delta-x)^\alpha\}e^{-x^\alpha}dx \\ &\qquad \left.+\int_\Delta^\infty\{x^\alpha-(x-\Delta)^\alpha\}e^{-x^\alpha}dx\right] \\ &= C(\alpha)(I_1+I_2+I_3) \quad \text{(say)}\,. \end{aligned} \tag{2.6}$$

Putting $h_\Delta(x)=(x+\Delta)^\alpha-x^\alpha$, we have from Lemma 2.1

$$\begin{aligned} I_1+I_3 &= -\int_0^\infty\{(x+\Delta)^\alpha-x^\alpha\}e^{-x^\alpha}dx+\int_\Delta^\infty\{x^\alpha-(x-\Delta)^\alpha\}e^{-x^\alpha}dx \\ &= \int_0^\infty\{(x+\Delta)^\alpha-x^\alpha\}\{e^{-(x+\Delta)^\alpha}-e^{-x^\alpha}\}dx \\ &= \int_0^\infty h_\Delta(x)\{e^{-h_\Delta(x)}-1\}e^{-x^\alpha}dx \\ &= -\int_0^\infty h_\Delta^2(x)e^{-x^\alpha}dx+\frac{1}{2}\int_0^\infty h_\Delta^3(x)e^{-x^\alpha}dx+O(\Delta^4) \\ &= -\alpha\Gamma\left(2-\frac{1}{\alpha}\right)\Delta^2+\frac{1+\gamma}{2\alpha+1}\Delta^{2\alpha+1}+o(\Delta^{2\alpha+1})\,. \end{aligned} \tag{2.7}$$

We also obtain

$$(2.8)\qquad I_2=\int_0^{\Delta}\{x^{\alpha}-(\Delta-x)^{\alpha}\}e^{-x^{\alpha}}dx$$

$$=\int_0^{\Delta}\{x^{\alpha}-(\Delta-x)^{\alpha}\}\left\{1-x^{\alpha}+\frac{1}{2}x^{2\alpha}+O(x^{3\alpha})\right\}dx$$

$$=-\frac{\Delta^{2\alpha+1}}{2\alpha+1}+\Delta^{2\alpha+1}\int_0^{\Delta}\left(\frac{x}{\Delta}\right)^{\alpha}\left(1-\frac{x}{\Delta}\right)^{\alpha}\frac{1}{\Delta}dx+O(\Delta^{3\alpha+1})$$

$$=\left\{B(\alpha+1,\alpha+1)-\frac{1}{2\alpha+1}\right\}\Delta^{2\alpha+1}+O(\Delta^{3\alpha+1}),$$

where $B(u,v)$ denotes the Beta function. From (2.7) and (2.8), we have

$$I_1+I_2+I_3=-\alpha\Gamma\left(2-\frac{1}{\alpha}\right)\Delta^2$$

$$+\left\{B(\alpha+1,\alpha+1)+\frac{\gamma}{2\alpha+1}\right\}\Delta^{2\alpha+1}+o(\Delta^{2\alpha+1}).$$

Since $C(\alpha)=\alpha/\{2\Gamma(1/\alpha)\}$, it follows from (2.6), (2.7) and (2.8) that

$$(2.9)\qquad E_0[\Psi_{\Delta}(x)]=C(\alpha)(I_1+I_2+I_3)$$

$$=-\frac{\alpha(\alpha-1)\Gamma\left(1-\dfrac{1}{\alpha}\right)}{2\Gamma\left(\dfrac{1}{\alpha}\right)}\Delta^2$$

$$+\frac{\alpha\{B(\alpha+1,\alpha+1)+(\gamma/(2\alpha+1))\}}{2\Gamma\left(\dfrac{1}{\alpha}\right)}\Delta^{2\alpha+1}+o(\Delta^{2\alpha+1})$$

From (2.5), we obtain

$$(2.10)\qquad E_0[\Psi_{\Delta}^2(X)]=\int_{-\infty}^{0}\{(-x)^{\alpha}-(\Delta-x)^{\alpha}\}^2f(x)dx$$

$$+\int_0^{\Delta}\{x^{\alpha}-(\Delta-x)^{\alpha}\}^2f(x)dx$$

$$+\int_{\Delta}^{\infty}\{x^{\alpha}-(x-\Delta)^{\alpha}\}^2f(x)dx$$

$$=\int_0^{\infty}\{x^{\alpha}-(x+\Delta)^{\alpha}\}^2f(x)dx$$

$$+\int_0^{\Delta}\{x^{\alpha}-(\Delta-x)^{\alpha}\}^2 f(x)dx$$

$$+\int_{\Delta}^{\infty}\{x^{\alpha}-(x-\Delta)^{\alpha}\}^2 f(x)dx$$

$$=C(\alpha)(I_1'+I_2'+I_3') \qquad \text{(say)}.$$

Since $h_{\Delta}(x)=(x+\Delta)^{\alpha}-x^{\alpha}$, it follows from Lemma 2.1 that

$$(2.11)\quad I_1'+I_3'=\int_0^{\infty}\{(x+\Delta)^{\alpha}-x^{\alpha}\}^2e^{-x^{\alpha}}dx+\int_{\Delta}^{\infty}\{x^{\alpha}-(x-\Delta)^{\alpha}\}^2e^{-x^{\alpha}}dx$$

$$=\int_0^{\infty}\{(x+\Delta)^{\alpha}-x^{\alpha}\}^2\{e^{-x^{\alpha}}+e^{-(x+\Delta)^{\alpha}}\}dx$$

$$=\int_0^{\infty}h_{\Delta}^2(x)\{e^{-h_{\Delta}(x)}+1\}e^{-x^{\alpha}}dx$$

$$=2\int_0^{\infty}h_{\Delta}^2(x)e^{-x^{\alpha}}dx-\int_0^{\infty}h_{\Delta}^3(x)e^{-x^{\alpha}}dx+O(\Delta^4)$$

$$=2\alpha\Gamma\left(2-\frac{1}{\alpha}\right)\Delta^2-\frac{2(1+\gamma)}{2\alpha+1}\Delta^{2\alpha+1}+o(\Delta^{2\alpha+1}).$$

Since

$$I_2'=\int_0^{\Delta}\{x^{\alpha}-(\Delta-x)^{\alpha}\}^2e^{-x^{\alpha}}dx$$

$$=\int_0^{\Delta}\{x^{\alpha}-(\Delta-x)^{\alpha}\}^2(1-x^{\alpha})dx+O(\Delta^{3\alpha+1})$$

$$=\int_0^{\Delta}\{x^{2\alpha}-2x^{\alpha}(\Delta-x)^{\alpha}+(\Delta-x)^{2\alpha}\}dx+O(\Delta^{3\alpha+1})$$

$$=\frac{2\Delta^{2\alpha+1}}{2\alpha+1}-2\Delta^{2\alpha+1}\int_0^{\Delta}\left(\frac{x}{\Delta}\right)^{\alpha}\left(1-\frac{x}{\Delta}\right)^{\alpha}\frac{1}{\Delta}dx+O(\Delta^{3\alpha+1})$$

$$=\frac{2\Delta^{2\alpha+1}}{2\alpha+1}-2\Delta^{2\alpha+1}B(\alpha+1,\alpha+1)+O(\Delta^{3\alpha+1}),$$

we obtain from (2.10) and (2.11)

$$(2.12)\quad E_0[\Psi_{\Delta}^2(X)]=C(\alpha)(I_1'+I_2'+I_3')$$

$$=\frac{\alpha(\alpha-1)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}\Delta^2$$

$$-\frac{\alpha\{B(\alpha+1,\alpha+1)+(\gamma/(2\alpha+1))\}}{\Gamma\left(\frac{1}{\alpha}\right)}\Delta^{2\alpha+1}+o(\Delta^{2\alpha+1})$$

From (2.5), we have

$$E_0[\Psi_\Delta^3(X)]=\int_{-\infty}^{0}\{(-x)^\alpha-(\Delta-x)^\alpha\}^3 f(x)dx \tag{2.13}$$
$$+\int_0^\Delta\{x^\alpha-(\Delta-x)^\alpha\}^3 f(x)dx$$
$$+\int_\Delta^\infty\{x^\alpha-(x-\Delta)^\alpha\}^3 f(x)dx$$
$$=\int_0^\infty\{x^\alpha-(x+\Delta)^\alpha\}^3 f(x)dx$$
$$+\int_0^\Delta\{x^\alpha-(\Delta-x)^\alpha\}^3 f(x)dx$$
$$+\int_\Delta^\infty\{x^\alpha-(x-\Delta)^\alpha\}^3 f(x)dx$$
$$=C(\alpha)(I_1''+I_2''+I_3'')\quad\text{(say)}.$$

Since $h_\Delta(x)=(x+\Delta)^\alpha-x^\alpha$, it follows that

$$I_1''+I_3''=-\int_0^\infty\{(x+\Delta)^\alpha-x^\alpha\}^3 e^{-x^\alpha}dx+\int_\Delta^\infty\{x^\alpha-(x-\Delta)^\alpha\}^3 e^{-x^\alpha}dx \tag{2.14}$$
$$=\int_0^\infty\{(x+\Delta)^\alpha-x^\alpha\}^3\{e^{-(x+\Delta)^\alpha}-e^{-x^\alpha}\}dx$$
$$=\int_0^\infty h_\Delta^3(x)\{e^{-h_\Delta(x)}-1\}e^{-x^\alpha}dx\,.$$

Since

$$\int_0^\infty h_\Delta^4(x)e^{-x^\alpha}dx=\int_0^\infty\{(x+\Delta)^\alpha-x^\alpha\}^4 e^{-x^\alpha}dx$$
$$=\alpha^4\Delta^4\int_0^\infty x^{4\alpha-4}e^{-x^\alpha}dx+o(\Delta^4)=O(\Delta^4)\,,$$

it follows from (2.14) that

$$I_1''+I_3''=O(\Delta^4)\,. \tag{2.15}$$

We also have

(2.16) $$I_2'' = \int_0^{\Delta} \{x^{\alpha} - (\Delta - x)^{\alpha}\}^3 e^{-x^{\alpha}} dx = O(\Delta^{3\alpha+1}) .$$

From (2.13), (2.15) and (2.16), we obtain

(2.17) $$E_0[\Psi_{\Delta}^3(X)] = O(\Delta^4) .$$

Putting $\Delta = tn^{-1/2}$ with $t > 0$, we obtain from (2.9), (2.12) and (2.17)

$$\begin{aligned}
E_0[Z_n] &= -nE_0[\Psi_{tn^{-1/2}}(X)] \\
&= \frac{\alpha(\alpha-1)\Gamma\left(1-\dfrac{1}{\alpha}\right)}{2\Gamma\left(\dfrac{1}{\alpha}\right)} t^2 \\
&\quad - \frac{\alpha\{B(\alpha+1,\alpha+1) + (\gamma/(2\alpha+1))\}}{2\Gamma\left(\dfrac{1}{\alpha}\right)} t^{2\alpha+1} n^{-(2\alpha-1)/2} \\
&\quad + o(n^{-(2\alpha-1)/2}) \\
&= \frac{I}{2} t^2 - \frac{k}{2} t^{2\alpha+1} n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}) , \\
V_0(Z_n) &= nV_0(\Psi_{tn^{-1/2}}(X)) \\
&= \frac{\alpha(\alpha-1)\Gamma\left(1-\dfrac{1}{\alpha}\right)}{\Gamma\left(\dfrac{1}{\alpha}\right)} t^2 \\
&\quad - \frac{\alpha\{B(\alpha+1,\alpha+1) + (\gamma/(2\alpha+1))\}}{\Gamma\left(\dfrac{1}{\alpha}\right)} t^{2\alpha+1} n^{-(2\alpha-1)/2} \\
&\quad + o(n^{-(2\alpha-1)/2}) \\
&= It^2 - kt^{2\alpha+1} n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}) , \\
K_{3,0}(Z_n) &= -nK_{3,0}(\Psi_{tn^{-1/2}}(X)) \\
&= -nE_0[\{\Psi_{tn^{-1/2}}(X) - E_0[\Psi_{tn^{-1/2}}(X)]\}^3] \\
&= o(n^{-(2\alpha-1)/2}) ,
\end{aligned}$$

where

$$I = E_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(X-\theta)\right\}^2\right] = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X-\theta)\right]$$
$$= \alpha(\alpha-1)\Gamma(1-(1/\alpha))/\Gamma(1/\alpha)$$

and

$$k = \alpha\left\{B(\alpha+1,\alpha+1) + \frac{\gamma}{2\alpha+1}\right\}\Big/\Gamma(1/\alpha).$$

In a similar way to the case under K: $\theta = 0$, we can obtain the asymptotic mean, variance and third-order cumulants under H: $\theta = tn^{-1/2}$. Thus we complete the proof.

In order to get the bound for the 2α-th order asymptotic distribution of 2α-th order AMU estimators, we need the following.

LEMMA 2.3. *Assume that the asymptotic mean, variance and third order cumulant of Z_n, under the distributions $P_{\theta,n}(\theta = \theta_0, \theta_0 + tn^{-1/2})$, are given by the following form.*

$$E_\theta(Z_n) = \mu(t,\theta) + n^{-(2\alpha-1)/2}C_1(t,\theta) + o(n^{-(2\alpha-1)/2}),$$
$$V_\theta(Z_n) = \upsilon^2(t,\theta) + n^{-(2\alpha-1)/2}C_2(t,\theta) + o(n^{-(2\alpha-1)/2}),$$
$$K_{3,\theta}(Z_n) = o(n^{-(2\alpha-1)/2}).$$

Then

$$P_{\theta,n}\{Z_n \leq a_0\} = \frac{1}{2} + o(n^{-(2\alpha-1)/2}),$$

if and only if

$$a_0 = \mu(t,\theta) + C_1(t,\theta)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}).$$

The proof is essentially given in Akahira and Takeuchi ((1981), pp. 132, 133).

In the following theorem we obtain the 2α-th asymptotic bound for the distribution of 2α-th order AMU estimators of θ.

THEOREM 2.1. *The bound for the 2α-th order asymptotic distribution of 2α-th order AMU estimators of θ is given by*

$$\Phi(t) - C_0|t|^{2\alpha}\phi(t)n^{-(2\alpha-1)/2}\operatorname{sgn} t + o(n^{-(2\alpha-1)/2}),$$

that is, for any $\hat{\theta}_n \in A_{2\alpha}$

$$P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n - \theta) \le t\} \le \Phi(t) - C_0 t^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}) \quad \textit{for all} \quad t > 0\,,$$

$$P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n - \theta) \le t\} \ge \Phi(t) + C_0 |t|^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}) \quad \textit{for all} \quad t < 0\,,$$

where

$$C_0 = \frac{\alpha\{B(\alpha+1, \alpha+1) + (\gamma/(2\alpha+1))\}}{2I^{\alpha+(1/2)}\Gamma(1/\alpha)}$$

and $\Phi(t)$ *and* $\phi(t)$ *denote the standard normal distribution function and its density function, respectively.*

PROOF. Without loss of generality, we assume that $\theta_0 = 0$. We consider the case when $t > 0$. In order to choose a_0 such that

$$P_{tn^{-1/2},n}\{Z_n \le a_0\} = \frac{1}{2} + o(n^{-(2\alpha-1)/2})\,, \tag{2.18}$$

we have by Lemmas 2.2 and 2.3

$$a_0 = -\frac{I}{2}t^2 + \frac{k}{2}t^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2})\,.$$

Since

$$P_{0,n}\{Z_n \ge a_0\} = P_{0,n}\{-(Z_n - It^2 - a_0) \le It^2\}\,, \tag{2.19}$$

putting $W_n = -(Z_n - It^2 - a_0)$, we have from Lemma 2.2

$$E_0(W_n) = kt^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2})\,, \tag{2.20}$$

$$V_0(W_n) = It^2 - kt^{2\alpha+1}n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2})\,, \tag{2.21}$$

$$K_{3,0}(W_n) = o(n^{-(2\alpha-1)/2})\,. \tag{2.22}$$

We obtain by (2.19) to (2.22) and the Edgeworth expansion

$$\begin{aligned} P_{0,n}\{Z_n \ge a_0\} &= P_{0,n}\{W_n \le It^2\} \\ &= \Phi(\sqrt{I}\,t) - \frac{k}{2\sqrt{I}}t^{2\alpha}\phi(\sqrt{I}\,t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2})\,. \end{aligned} \tag{2.23}$$

Here, from (2.18), the definition of Z_n and the fundamental lemma of Neyman-Pearson it is noted that a test with the rejection region $\{Z_n \geq a_0\}$ is the most powerful test of level $1/2 + o(n^{-(2\alpha-1)/2})$.

Let $\hat{\theta}_n$ be any 2α-th order AMU estimator. Putting $A_{\hat{\theta}_n} = \{\sqrt{n}\,\hat{\theta}_n \leq t\}$, we have

$$P_{tn^{-1/2},n}(A_{\hat{\theta}_n}) = P_{tn^{-1/2},n}\{\hat{\theta}_n \leq tn^{-1/2}\} = \frac{1}{2} + o(n^{-(2\alpha-1)/2}) .$$

Then it is seen that $\chi_{A_{\hat{\theta}_n}}$ of the indicator of $A_{\hat{\theta}_n}$ is a test of level $1/2 + o(n^{-(2\alpha-1)/2})$. From (2.23), we obtain for any $\hat{\theta}_n \in A_{2\alpha}$

$$\begin{aligned} P_{0,n}\{\sqrt{n}\,\hat{\theta}_n \leq t\} &\leq P_{0,n}\{W_n \leq It^2\} \\ &= \Phi(\sqrt{I}\,t) - \frac{k}{2\sqrt{I}}\,t^{2\alpha}\phi(\sqrt{I}\,t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}) , \end{aligned}$$

that is,

$$P_{0,n}\{\sqrt{In}\,\hat{\theta}_n \leq t\} \leq \Phi(t) - C_0 t^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}) \tag{2.24}$$

for all $t > 0$, where

$$C_0 = \frac{k}{2I^{\alpha+(1/2)}} = \frac{\alpha\{B(\alpha+1, \alpha+1) + (\gamma/(2\alpha+1))\}}{2I^{\alpha+(1/2)}\Gamma(1/\alpha)} .$$

Hence we see that the bound for the 2α-th order distribution of 2α-th order AMU estimators for all $t > 0$ is given by (2.24). In a similar way to the case $t > 0$, we can obtain the 2α-th order bound for all $t < 0$. Thus we complete the proof.

Remark 2.1. The result of Theorem 2.1 holds for $2/3 < \alpha < 1$, where the information amount I must be expressed as $\alpha^2\Gamma(2 - (1/\alpha))/\Gamma(1/\alpha)$. The proof is omitted since it is essentially similar to the above.

3. The 2α-th order asymptotic distribution of the maximum likelihood estimator

In this section we obtain the 2α-th order asymptotic distribution of the maximum likelihood estimator (MLE) and compare it with the 2α-th order asymptotic bound obtained in the previous section.

We denote by θ_0 and $\hat{\theta}_{ML}$ the true parameter and the MLE, respectively. It is seen that for real t, $\hat{\theta}_{ML} < \theta_0 + tn^{-1/2}$ if and only if

$$(\partial/\partial\theta)\sum_{i=1}^{n}\log f(X_i-\theta_0-tn^{-1/2})<0 .$$

Without loss of generality, we assume that $\theta_0=0$. Hence we see that for each real t

$$\hat{\theta}_{\mathrm{ML}}<tn^{-1/2} \quad \text{if and only if} \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(d/dx)\log f(X_i-tn^{-1/2})>0 . \tag{3.1}$$

Since

$$\frac{d}{dx}\log f(x)= -\alpha|x|^{\alpha-1}\operatorname{sgn} x ,$$

we put

$$\begin{aligned} U_n &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(d/dx)\log f(X_i-tn^{-1/2}) \\ &= \frac{\alpha}{\sqrt{n}}\sum_{i=1}^{n}|X_i-tn^{-1/2}|^{\alpha-1}\operatorname{sgn}(X_i-tn^{-1/2}) . \end{aligned} \tag{3.2}$$

In order to obtain the asymptotic cumulants of U_n, we need the following lemma.

LEMMA 3.1. *If* $h_\Delta(x)=(x+\Delta)^\alpha-x^\alpha$ *for* $\Delta>0$, *then*

$$\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta(x)dx=\Gamma\left(2-\frac{1}{\alpha}\right)\Delta+\frac{\alpha-1}{2}\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^2 - \frac{\gamma}{2\alpha}\Delta^{2\alpha}+o(\Delta^{2\alpha}) ,$$

$$\int_0^\infty x^{2\alpha-2}e^{-x^\alpha}h_\Delta(x)dx=\Gamma\left(3-\frac{2}{\alpha}\right)\Delta+\frac{1}{2}(\alpha-1)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta^2 + o(\Delta^2) , \tag{3.3}$$

$$\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}h_\Delta(x)dx=\left(3-\frac{3}{\alpha}\right)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta+O(\Delta^2) , \tag{3.4}$$

$$\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta^2(x)dx=\alpha\Gamma\left(3-\frac{2}{\alpha}\right)\Delta^2+O(\Delta^3) ,$$

$$\int_0^\infty x^{2\alpha-2}e^{-x^\alpha}h_\Delta^2(x)dx=\alpha\Gamma\left(4-\frac{3}{\alpha}\right)\Delta^2+O(\Delta^3) , \tag{3.5}$$

(3.6) $$\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}h_\Delta^2(x)dx = O(\Delta^2),$$

$$\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta^3(x)dx = O(\Delta^3),$$

(3.7) $$\int_0^\infty x^{2\alpha-2}e^{-x^\alpha}h_\Delta^3(x)dx = O(\Delta^3).$$

PROOF. From (2.2), we have

$$\begin{aligned}
&\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta(x)dx \\
&=\int_0^\infty (x+\Delta)^\alpha x^{\alpha-1}e^{-x^\alpha}dx-\int_0^\infty x^{2\alpha-1}e^{-x^\alpha}dx \\
&=-\frac{1}{\alpha+1}\int_0^\infty (x+\Delta)^{\alpha+1}\{(\alpha-1)x^{\alpha-2}-\alpha x^{2\alpha-2}\}e^{-x^\alpha}dx-\frac{1}{\alpha} \\
&=-\frac{1}{\alpha+1}\int_0^\infty \left\{x^{\alpha+1}+(\alpha+1)\Delta x^\alpha+\frac{\alpha(\alpha+1)}{2}\Delta^2x^{\alpha-1}\right\} \\
&\qquad\cdot\{(\alpha-1)x^{\alpha-2}-\alpha x^{2\alpha-2}\}e^{-x^\alpha}dx \\
&\quad-\frac{\alpha-1}{\alpha+1}\int_0^\infty R^*(\Delta)x^{\alpha-2}e^{-x^\alpha}dx-\frac{1}{\alpha}+O(\Delta^3),
\end{aligned}$$

where

$$R^*(\Delta)=(x+\Delta)^{\alpha+1}-x^{\alpha+1}-(\alpha+1)\Delta x^\alpha-\frac{1}{2}\alpha(\alpha+1)\Delta^2x^{\alpha-1}.$$

In a similar way to (2.4), we obtain

$$\begin{aligned}
&\int_0^\infty R^*(\Delta)x^{\alpha-2}e^{-x^\alpha}dx \\
&=\frac{1}{2}\alpha(\alpha+1)(\alpha-1)\int_0^\Delta(\Delta-t)^2\left(\int_0^\infty(x+t)^{\alpha-2}x^{\alpha-2}e^{-x^\alpha}dx\right)dt \\
&=\frac{1}{2}\alpha(\alpha+1)(\alpha-1)B(2\alpha-2,3)B(\alpha-1,3-2\alpha)\Delta^{2\alpha}+o(\Delta^{2\alpha}) \\
&=\frac{(\alpha+1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{4(2\alpha-1)\Gamma(2-\alpha)}\Delta^{2\alpha}+o(\Delta^{2\alpha}).
\end{aligned}$$

Hence, we have

$$\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta(x)$$

$$= -\frac{1}{\alpha+1}\left\{(\alpha-1)\int_0^\infty x^{2\alpha-1}e^{-x^\alpha}dx - \alpha\int_0^\infty x^{3\alpha-1}e^{-x^\alpha}dx\right.$$

$$+ (\alpha-1)(\alpha+1)\Delta\int_0^\infty x^{2\alpha-2}e^{-x^\alpha}dx$$

$$- \alpha(\alpha+1)\Delta\int_0^\infty x^{3\alpha-2}e^{-x^\alpha}dx$$

$$+ \frac{1}{2}\alpha(\alpha-1)(\alpha+1)\Delta^2\int_0^\infty x^{2\alpha-3}e^{-x^\alpha}dx$$

$$\left. - \frac{1}{2}\alpha^2(\alpha+1)\Delta^2\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}dx\right\}$$

$$- \frac{1}{\alpha} - \frac{(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{4(2\alpha-1)\Gamma(2-\alpha)}\Delta^{2\alpha} + o(\Delta^{2\alpha})$$

$$= -\frac{1}{\alpha+1}\left\{\frac{\alpha-1}{\alpha} - 2 + \frac{1}{\alpha}(\alpha-1)(\alpha+1)\Gamma\left(2-\frac{1}{\alpha}\right)\Delta\right.$$

$$- (\alpha+1)\Gamma\left(3-\frac{1}{\alpha}\right)\Delta$$

$$+ \frac{1}{2}(\alpha-1)(\alpha+1)\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^2$$

$$\left. - \frac{1}{2}\alpha(\alpha+1)\Gamma\left(3-\frac{2}{\alpha}\right)\Delta^2\right\}$$

$$- \frac{1}{\alpha} - \frac{(\alpha-1)\Gamma(\alpha-1)\Gamma(3-2\alpha)}{4(2\alpha-1)\Gamma(2-\alpha)}\Delta^{2\alpha} + O(\Delta^3)$$

$$= \Gamma\left(2-\frac{1}{\alpha}\right)\Delta + \frac{\alpha-1}{2}\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^2 - \frac{\gamma}{2\alpha}\Delta^{2\alpha} + o(\Delta^{2\alpha})\,.$$

In a similar way to the above, we can obtain (3.3) and (3.4). We also have

$$\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta^2(x)dx$$

$$= \int_0^\infty (x+\Delta)^{2\alpha}x^{\alpha-1}e^{-x^\alpha}dx - 2\int_0^\infty (x+\Delta)^\alpha x^{2\alpha-1}e^{-x^\alpha}dx$$

$$+ \int_0^\infty x^{3\alpha-1}e^{-x^\alpha}dx$$

$$=\int_0^\infty \{x^{2\alpha}+2\alpha x^{2\alpha-1}\Delta+\alpha(2\alpha-1)x^{2\alpha-2}\Delta^2\}x^{\alpha-1}e^{-x^\alpha}dx$$

$$-2\int_0^\infty \left\{x^\alpha+\alpha x^{\alpha-1}\Delta+\frac{1}{2}\alpha(\alpha-1)x^{\alpha-2}\Delta^2\right\}x^{2\alpha-1}e^{-x^\alpha}dx$$

$$+\frac{2}{\alpha}+O(\Delta^3)$$

$$=\alpha\Gamma\left(3-\frac{2}{\alpha}\right)\Delta^2+O(\Delta^3),$$

and similarly get (3.5) and (3.6). From (2.2), we obtain

$$\int_0^\infty x^{\alpha-1}e^{-x^\alpha}h_\Delta^3(x)dx=\int_0^\infty x^{\alpha-1}e^{-x^\alpha}(\alpha x^{\alpha-1}\Delta)^3dx+o(\Delta^3)$$

$$=\alpha^2\Gamma\left(4-\frac{3}{\alpha}\right)\Delta^3+o(\Delta^3)$$

$$=O(\Delta^3),$$

and similarly have (3.7). Thus we complete the proof.

In the following lemma, we obtain the asymptotic mean, variance and third order cumulant of U_n.

LEMMA 3.2. *The asymptotic mean, variance and third order cumulant of U_n are given for $t>0$ as follows*:

$$E_0(U_n)=-It+\frac{k'}{2}t^{2\alpha}n^{-(2\alpha-1)/2}+o(n^{-(2\alpha-1)/2}),$$

$$V_0(U_n)=I+o(n^{-(2\alpha-1)/2}),$$

$$K_{3,0}(U_n)=o(n^{-(2\alpha-1)/2}),$$

where I is given in Lemma 2.2 and

$$k'=\frac{\alpha^2\{B(\alpha,\alpha+1)+(\gamma/\alpha)\}}{\Gamma(1/\alpha)}.$$

PROOF. Putting $\Delta=tn^{-1/2}$, we obtain

(3.8) $E_0[|X-\Delta|^{\alpha-1}\operatorname{sgn}(X-\Delta)]$

$$=C(\alpha)\left\{\int_\Delta^\infty(x-\Delta)^{\alpha-1}e^{-x^\alpha}dx-\int_0^\Delta(\Delta-x)^{\alpha-1}e^{-x^\alpha}dx\right.$$

$$
\begin{aligned}
& \left. - \int_{-\infty}^{0} (\Delta - x)^{\alpha-1} e^{-|x|^\alpha} dx \right\} \\
&= C(\alpha) \left\{ \int_0^\infty x^{\alpha-1} e^{-(x+\Delta)^\alpha} dx - \int_0^\infty (x+\Delta)^{\alpha-1} e^{-x^\alpha} dx \right. \\
& \qquad \left. - \int_0^\Delta (\Delta - x)^{\alpha-1} e^{-x^\alpha} dx \right\} \\
&= C(\alpha) \left[\int_0^\infty x^{\alpha-1} \{ e^{-(x+\Delta)^\alpha} - e^{-x^\alpha} \} dx - \int_0^\infty \{ (x+\Delta)^{\alpha-1} - x^{\alpha-1} \} e^{-x^\alpha} dx \right. \\
& \qquad \left. - \int_0^\Delta (\Delta - x)^{\alpha-1} e^{-x^\alpha} dx \right] \\
&= C(\alpha)(J_1 + J_2 + J_3) \qquad \text{(say)} .
\end{aligned}
$$

Putting $h_\Delta(x) = (x+\Delta)^\alpha - x^\alpha$, we have from Lemma 3.1

$$
\begin{aligned}
(3.9) \qquad J_1 &= \int_0^\infty x^{\alpha-1} \{ e^{-(x+\Delta)^\alpha} - e^{-x^\alpha} \} dx \\
&= \int_0^\infty x^{\alpha-1} e^{-x^\alpha} \{ e^{-h_\Delta(x)} - 1 \} dx \\
&= - \int_0^\infty x^{\alpha-1} e^{-x^\alpha} h_\Delta(x) dx + \frac{1}{2} \int_0^\infty x^{\alpha-1} e^{-x^\alpha} h_\Delta^2(x) dx + O(\Delta^3) \\
&= - \Gamma\left(2 - \frac{1}{\alpha}\right) \Delta - \frac{\alpha-1}{2} \Gamma\left(2 - \frac{2}{\alpha}\right) \Delta^2 + \frac{\alpha}{2} \Gamma\left(3 - \frac{2}{\alpha}\right) \Delta^2 \\
& \quad + \frac{\gamma}{2\alpha} \Delta^{2\alpha} + o(\Delta^{2\alpha}) \\
&= - \left(1 - \frac{1}{\alpha}\right) \Gamma\left(1 - \frac{1}{\alpha}\right) \Delta + \frac{\alpha-1}{2} \Gamma\left(2 - \frac{2}{\alpha}\right) \Delta^2 \\
& \quad + \frac{\gamma}{2\alpha} \Delta^{2\alpha} + o(\Delta^{2\alpha}) .
\end{aligned}
$$

From (2.2), we obtain

$$
\begin{aligned}
(3.10) \qquad - J_2 &= \int_0^\infty (x+\Delta)^{\alpha-1} e^{-x^\alpha} dx - \int_0^\infty x^{\alpha-1} e^{-x^\alpha} dx \\
&= - \frac{\Delta^\alpha}{\alpha} + \int_0^\infty (x+\Delta)^\alpha x^{\alpha-1} e^{-x^\alpha} dx - \frac{1}{\alpha} .
\end{aligned}
$$

Since by a similar way to (2.4)

$$\int_0^{\infty}(x+\Delta)^{\alpha}x^{\alpha-1}e^{-x^{\alpha}}dx$$

$$=\frac{1}{\alpha}+\Gamma\left(2-\frac{1}{\alpha}\right)\Delta+\frac{\alpha-1}{2}\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^2-\frac{\gamma}{2\alpha}\Delta^{2\alpha}+o(\Delta^{2\alpha}),$$

it follows from (3.10) that

$$J_2=\frac{\Delta^{\alpha}}{\alpha}-\Gamma\left(2-\frac{1}{\alpha}\right)\Delta-\frac{\alpha}{2}\left(1-\frac{1}{\alpha}\right)\Gamma\left(2-\frac{2}{\alpha}\right)\Delta^2+\frac{\gamma}{2\alpha}\Delta^{2\alpha}+o(\Delta^{2\alpha}). \tag{3.11}$$

We also have

$$\begin{aligned}-J_3&=\int_0^{\Delta}(\Delta-x)^{\alpha-1}e^{-x^{\alpha}}dx\\&=\int_0^{\Delta}(\Delta-x)^{\alpha-1}(1-x^{\alpha})dx+O(\Delta^{3\alpha})\\&=\frac{\Delta^{\alpha}}{\alpha}-\Delta^{2\alpha}\int_0^1\left(1-\frac{x}{\Delta}\right)^{\alpha-1}\left(\frac{x}{\Delta}\right)^{\alpha}\frac{1}{\Delta}dx+O(\Delta^{3\alpha})\\&=\frac{\Delta^{\alpha}}{\alpha}-B(\alpha,\alpha+1)\Delta^{2\alpha}+O(\Delta^{3\alpha}).\end{aligned} \tag{3.12}$$

From (3.8), (3.9), (3.11) and (3.12), we obtain

$$\begin{aligned}&E_0[|X-\Delta|^{\alpha-1}\operatorname{sgn}(X-\Delta)]\\&=C(\alpha)\left\{-\frac{2(\alpha-1)}{\alpha}\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Delta+\left(B(\alpha,\alpha+1)+\frac{\gamma}{\alpha}\right)\Delta^{2\alpha}\right\}+o(\Delta^{2\alpha})\\&=-(\alpha-1)\frac{\Gamma(1-(1/\alpha))}{\Gamma(1/\alpha)}\Delta+\frac{\alpha\{B(\alpha,\alpha+1)+(\gamma/\alpha)\}}{2\Gamma(1/\alpha)}\Delta^{2\alpha}+o(\Delta^{2\alpha}).\end{aligned} \tag{3.13}$$

Next, we have

$$E_0[|X-\Delta|^{2\alpha-2}] \tag{3.14}$$

$$
\begin{aligned}
&= C(\alpha)\left\{\int_{\Delta}^{\infty}(x-\Delta)^{2\alpha-2}e^{-x^\alpha}dx+\int_{-\infty}^{\Delta}(\Delta-x)^{2\alpha-2}e^{-|x|^\alpha}dx\right\} \\
&= C(\alpha)\left\{\int_0^{\infty}x^{2\alpha-2}e^{-(x+\Delta)^\alpha}dx+\int_0^{\infty}(x+\Delta)^{2\alpha-2}e^{-x^\alpha}dx\right. \\
&\qquad\qquad\qquad\qquad \left.+\int_0^{\Delta}(\Delta-x)^{2\alpha-2}e^{-x^\alpha}dx\right\} \\
&= C(\alpha)\left[2\int_0^{\infty}x^{2\alpha-2}e^{-x^\alpha}dx+\int_0^{\infty}\{e^{-(x+\Delta)^\alpha}-e^{-x^\alpha}\}x^{2\alpha-2}dx\right. \\
&\qquad +\int_0^{\infty}\{(x+\Delta)^{2\alpha-2}-x^{2\alpha-2}\}e^{-x^\alpha}dx \\
&\qquad \left.+\int_0^{\Delta}(\Delta-x)^{2\alpha-2}e^{-x^\alpha}dx\right] . \\
&= C(\alpha)[J_1'+J_2'+J_3'+J_4'] \quad \text{(say)} .
\end{aligned}
$$

From (2.2), we obtain

$$
J_1' = 2\int_0^{\infty}x^{2\alpha-2}e^{-x^\alpha}dx=\frac{2}{\alpha}\Gamma\left(2-\frac{1}{\alpha}\right)=\frac{2}{\alpha}\left(1-\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right). \tag{3.15}
$$

Since

$$
\begin{aligned}
J_2' &= \int_0^{\infty}\{e^{-(x+\Delta)^\alpha}-e^{-x^\alpha}\}x^{2\alpha-2}dx \\
&= \int_0^{\infty}x^{2\alpha-2}e^{-x^\alpha}\{e^{-h_\Delta(x)}-1\}dx \\
&= -\int_0^{\infty}x^{2\alpha-2}e^{-x^\alpha}h_\Delta(x)dx+\frac{1}{2}\int_0^{\infty}x^{2\alpha-2}e^{-x^\alpha}h_\Delta^2(x)dx \\
&\quad -\frac{1}{6}\int_0^{\infty}x^{2\alpha-2}e^{-x^\alpha}h_\Delta^3(x)dx+O(\Delta^3) ,
\end{aligned}
$$

it follows from Lemma 3.1 that

$$
J_2' = -\Gamma\left(3-\frac{2}{\alpha}\right)\Delta+(\alpha-1)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta^2+O(\Delta^3) . \tag{3.16}
$$

From (2.2), we have

$$
J_3' = \int_0^{\infty}\{(x+\Delta)^{2\alpha-2}-x^{2\alpha-2}\}e^{-x^\alpha}dx \tag{3.17}
$$

$$= -\frac{1}{2\alpha-1}\Delta^{2\alpha-1} + \frac{\alpha}{2\alpha-1}\int_0^\infty (x+\Delta)^{2\alpha-1}x^{\alpha-1}e^{-x^\alpha}dx$$

$$-\frac{1}{\alpha}\Gamma\left(\frac{2\alpha-1}{\alpha}\right)$$

$$= -\frac{1}{2\alpha-1}\Delta^{2\alpha-1} - \frac{1}{\alpha}\Gamma\left(\frac{2\alpha-1}{\alpha}\right)$$

$$+\frac{\alpha}{2\alpha-1}\left\{\int_0^\infty x^{3\alpha-2}e^{-x^\alpha}dx + (2\alpha-1)\Delta\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}dx\right.$$

$$\left. + (\alpha-1)(2\alpha-1)\Delta^2\int_0^\infty x^{3\alpha-4}e^{-x^\alpha}dx\right\} + o(\Delta^2)$$

$$= -\frac{1}{2\alpha-1}\Delta^{2\alpha-1} - \frac{1}{\alpha}\Gamma\left(2-\frac{1}{\alpha}\right) + \frac{1}{2\alpha-1}\Gamma\left(3-\frac{1}{\alpha}\right)$$

$$+\Gamma\left(3-\frac{2}{\alpha}\right)\Delta + (\alpha-1)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta^2 + O(\Delta^3)$$

$$= -\frac{1}{2\alpha-1}\Delta^{2\alpha-1} + \Gamma\left(3-\frac{2}{\alpha}\right)\Delta + (\alpha-1)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta^2$$

$$+ o(\Delta^2)\,.$$

We also obtain

$$(3.18)\quad J_4' = \int_0^\Delta (\Delta - x)^{2\alpha-2}e^{-x^\alpha}dx$$

$$= \int_0^\Delta (\Delta-x)^{2\alpha-2}dx - \int_0^\Delta x^\alpha(\Delta-x)^{2\alpha-2}dx + O(\Delta^{4\alpha-1})$$

$$= \frac{\Delta^{2\alpha-1}}{2\alpha-1} - \Delta^{3\alpha-1}\int_0^\Delta\left(\frac{x}{\Delta}\right)^\alpha\left(1-\frac{x}{\Delta}\right)^{2\alpha-2}\frac{1}{\Delta}dx + O(\Delta^{4\alpha-1})$$

$$= \frac{\Delta^{2\alpha-1}}{2\alpha-1} - B(\alpha+1, 2\alpha-1)\Delta^{3\alpha-1} + O(\Delta^{4\alpha-1})\,.$$

From (3.14) to (3.18), we have

$$E_0[|X-\Delta|^{2\alpha-2}] = \frac{\alpha-1}{\alpha\Gamma(1/\alpha)}\Gamma\left(1-\frac{1}{\alpha}\right)$$

$$+\frac{\alpha(\alpha-1)}{\Gamma(1/\alpha)}\Gamma\left(3-\frac{3}{\alpha}\right)\Delta^2 + o(\Delta^2)\,,$$

hence, by (3.13)

$$(3.19)\qquad V_0(|X-\Delta|^{\alpha-1}\operatorname{sgn}(X-\Delta))$$
$$=\frac{\alpha-1}{\alpha\Gamma(1/\alpha)}\Gamma\left(1-\frac{1}{\alpha}\right)$$
$$+(\alpha-1)\left[\frac{\alpha}{\Gamma(1/\alpha)}\Gamma\left(3-\frac{3}{\alpha}\right)\right.$$
$$\left.-(\alpha-1)\left\{\frac{\Gamma(1-(1/\alpha))}{\Gamma(1/\alpha)}\right\}^2\right]\Delta^2+o(\Delta^2)\,.$$

Third, we have

$$(3.20)\qquad E_0[|X-\Delta|^{3\alpha-3}\operatorname{sgn}(X-\Delta)]$$
$$=C(\alpha)\left\{\int_\Delta^\infty(x-\Delta)^{3\alpha-3}e^{-x^\alpha}dx-\int_0^\Delta(\Delta-x)^{3\alpha-3}e^{-x^\alpha}dx\right.$$
$$\left.-\int_{-\infty}^0(\Delta-x)^{3\alpha-3}e^{-|x|^\alpha}dx\right\}$$
$$=C(\alpha)\left[\int_0^\infty x^{3\alpha-3}\{e^{-(x+\Delta)^\alpha}-e^{-x^\alpha}\}dx\right.$$
$$-\int_0^\infty\{(x+\Delta)^{3\alpha-3}-x^{3\alpha-3}\}e^{-x^\alpha}dx$$
$$\left.-\int_0^\Delta(\Delta-x)^{3\alpha-3}e^{-x^\alpha}dx\right]$$
$$=C(\alpha)(J_1''+J_2''+J_3'')\qquad\text{(say)}\,.$$

Since

$$J_1''=\int_0^\infty x^{3\alpha-3}\{e^{-(x+\Delta)^\alpha}-e^{-x^\alpha}\}dx=\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}\{e^{-h_\Delta(x)}-1\}dx$$
$$=-\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}h_\Delta(x)dx+\frac{1}{2}\int_0^\infty x^{3\alpha-3}e^{-x^\alpha}h_\Delta^2(x)dx+o(\Delta^2)\,,$$

it follows by Lemma 3.1 that

$$(3.21)\qquad J_1''=-3\left(1-\frac{1}{\alpha}\right)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta+O(\Delta^2)\,.$$

From (2.2), we have

(3.22)
$$
\begin{aligned}
-J_2'' &= \int_0^\infty \{(x+\Delta)^{3\alpha-3} - x^{3\alpha-3}\} e^{-x^\alpha} dx \\
&= 3(\alpha-1)\Delta \int_0^\infty x^{3\alpha-4} e^{-x^\alpha} dx + o(\Delta) \\
&= 3\left(1-\frac{1}{\alpha}\right)\Gamma\left(3-\frac{3}{\alpha}\right)\Delta + o(\Delta),
\end{aligned}
$$

and also

(3.23)
$$
\begin{aligned}
-J_3'' &= \int_0^\Delta (\Delta - x)^{3\alpha-3} e^{-x^\alpha} dx \\
&= \frac{\Delta^{3\alpha-2}}{3\alpha-2} - \Delta^{4\alpha-2}\int_0^\Delta \left(\frac{x}{\Delta}\right)^\alpha \left(1-\frac{x}{\Delta}\right)^{3\alpha-3} \frac{1}{\Delta} dx + O(\Delta^{5\alpha-2}) \\
&= \frac{\Delta^{3\alpha-2}}{3\alpha-2} + O(\Delta^{4\alpha-2}).
\end{aligned}
$$

From (3.20) to (3.23), we obtain

$$
E_0[|X-\Delta|^{3\alpha-3}\operatorname{sgn}(X-\Delta)] = -\frac{3(\alpha-1)}{\Gamma(1/\alpha)}\Gamma\left(3-\frac{3}{\alpha}\right)\Delta + o(\Delta),
$$

hence

(3.24)
$$
\begin{aligned}
&K_{3,0}(|X-\Delta|^{\alpha-1}\operatorname{sgn}(X-\Delta)) \\
&= E_0[\{|X-\Delta|^{\alpha-1}\operatorname{sgn}(X-\Delta) - E_0[|X-\Delta|^{\alpha-1}\operatorname{sgn}(X-\Delta)]\}^3] \\
&= O(\Delta).
\end{aligned}
$$

Putting $\Delta = tn^{-1/2}$ with $t>0$, we have from (3.2), (3.13), (3.19) and (3.24)

$$
\begin{aligned}
E_0(U_n) &= \alpha\sqrt{n}\, E_0[|X-tn^{-1/2}|^{\alpha-1}\operatorname{sgn}(X-tn^{-1/2})] \\
&= -\alpha(\alpha-1)\frac{\Gamma(1-(1/\alpha))}{\Gamma(1/\alpha)} t \\
&\quad + \frac{\alpha^2\{B(\alpha,\alpha+1)+(\gamma/\alpha)\}}{2\Gamma(1/\alpha)} t^{2\alpha} n^{-(2\alpha-1)/2} \\
&\quad + o(n^{-(2\alpha-1)/2}) \\
&= -It + \frac{k'}{2} t^{2\alpha} n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}), \\
V_0(U_n) &= \alpha^2 V_0(|X-tn^{-1/2}|^{\alpha-1}\operatorname{sgn}(X-tn^{-1/2}))
\end{aligned}
$$

$$= \alpha(\alpha-1)\frac{\Gamma(1-(1/\alpha))}{\Gamma(1/\alpha)} + o(n^{-(2\alpha-1)/2})$$

$$= I + o(n^{-(2\alpha-1)/2}),$$

$$K_{3,0}(U_n) = \frac{\alpha^3}{\sqrt{n}} K_{3,0}(|X - tn^{-1/2}|^{\alpha-1} \operatorname{sgn}(X - tn^{-1/2}))$$

$$= o(n^{-(2\alpha-1)/2}),$$

where $I = \alpha(\alpha-1)\Gamma(1-(1/\alpha))/\Gamma(1/\alpha)$ and $k' = \alpha^2\{B(\alpha, \alpha+1) + (\gamma/\alpha)\}/\Gamma(1/\alpha)$. This completes the proof.

In the following theorem, we obtain the 2α-th order asymptotic distribution of the MLE of θ.

THEOREM 3.1. *The 2α-th order asymptotic distribution of the MLE $\hat{\theta}_{\mathrm{ML}}$ of θ is given by*

$$\text{(3.25)} \qquad P_{\theta,n}\{\sqrt{In}(\hat{\theta}_{\mathrm{ML}} - \theta) \le t\} = \Phi(t) - C_1|t|^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} \operatorname{sgn} t + (n^{-(2\alpha-1)/2}),$$

where $C_1 = (2\alpha+1)C_0$ with $C_0 = \alpha\{B(\alpha+1, \alpha+1) + (\gamma/(2\alpha+1))\}/\{2I^{\alpha+(1/2)}\Gamma(1/\alpha)\}$, and also the MLE is not 2α-th order asymptotically efficient in the sense that its 2α-th order asymptotic distribution does not uniformly attain the bound given in Theorem 2.1.

PROOF. Since the density $f(x) = C(\alpha)e^{-|x|^\alpha}$ is symmetric about the origin, we see that the MLE of θ is a 2α-th order AMU. We consider the case when $t > 0$. Using the Edgeworth expansion, we have by (3.1), (3.2) and Lemma 3.2

$$P_{\theta,n}\{\sqrt{n}(\hat{\theta}_{\mathrm{ML}} - \theta) \le t\}$$

$$= P_{\theta,n}\{U_n \le 0\}$$

$$= \Phi(\sqrt{I}\,t) - \frac{k'}{2\sqrt{I}} t^{2\alpha}\phi(\sqrt{I}\,t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}),$$

that is,

$$\text{(3.26)} \qquad P_{\theta,n}\{\sqrt{In}(\hat{\theta}_{\mathrm{ML}} - \theta) \le t\} = \Phi(t) - \frac{k'}{2I^{\alpha+(1/2)}} t^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2})$$

$$= \Phi(t) - C_1 t^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}),$$

where $C_1 = k'/\{2I^{\alpha+(1/2)}\}$.

In a similar way to the case $t > 0$, we obtain for $t < 0$

$$(3.27)\qquad P_{\theta,n}\{\sqrt{In}(\hat{\theta}_{ML} - \theta) \le t\} = \Phi(t) + C_1|t|^{2\alpha}\phi(t)n^{-(2\alpha-1)/2} + o(n^{-(2\alpha-1)/2}).$$

Hence (3.26) and (3.27) imply (3.25). Since $k' = \alpha^2\{B(\alpha, \alpha+1) + (\gamma/\alpha)\}/\Gamma(1/\alpha)$ and $C_0 = \alpha\{B(\alpha+1, \alpha+1) + (\gamma/(2\alpha+1))\}/\{2I^{\alpha+(1/2)}\Gamma(1/\alpha)\}$, it is seen that $C_1 = (2\alpha+1)C_0$. Since $C_1 > C_0$ for $1 < \alpha < 3/2$, it follows from Theorem 2.1 and (3.25) that the MLE is not 2α-th order asymptotically efficient in the sense that its 2α-th order asymptotic distribution does not uniformly attain the bound given in Theorem 2.1. This completes the proof.

Remark 3.1. In the double exponential distribution case, that is, the case when $\alpha = 1$, it is shown in Akahira and Takeuchi (1981) that the bound for the second order asymptotic distribution of the second order AMU estimators of θ is given by

$$\Phi(t) - \frac{t^2}{6}\phi(t)n^{-1/2}\operatorname{sgn} t + o(n^{-1/2}),$$

and the second order distribution of the MLE of θ, i.e., the median of $X_1, \ldots, X_n$, is given by

$$(3.28)\qquad \Phi(t) - \frac{t^2}{2}\phi(t)n^{-1/2}\operatorname{sgn} t + o(n^{-1/2}).$$

The results coincide with the case when $\alpha = 1$ is substituted in the formulae of Theorems 2.1 and 3.1, but note that the proofs of these theorems do not include the case for $\alpha = 1$ since it does not automatically hold that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ for $\alpha = 1$.

4. The amount of the loss of asymptotic information of the maximum likelihood estimator

In the section we obtain the amount of the loss of asymptotic information of the MLE $\hat{\theta}_{ML}$ using its second order asymptotic distribution (3.25). Differentiating the right-hand side of (3.25) w.r.t. t, we have the second order asymptotic density $g(t)$ of $\sqrt{In}(\hat{\theta}_{ML} - \theta)$ as follows:

$$(4.1)\qquad g(t) = \phi(t)\{1 - C_1(2\alpha|t|^{2\alpha-1} - |t|^{2\alpha+1})n^{-(2\alpha-1)/2}\} + o(n^{-(2\alpha-1)/2})$$

for $-\infty < t < \infty$.

In general, we obtain for $\alpha > 0$

(4.2) $$\int_{-\infty}^{\infty} |t|^{\alpha}\phi(t)dt = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} t^{\alpha}e^{-t^2/2}dt$$

$$= \frac{2^{\alpha/2}}{\sqrt{\pi}}\int_0^{\infty} u^{\{(\alpha+1)/2\}-1}e^{-u}du$$

(after transformation $u = t^2/2$)

$$= \frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right).$$

Since, for sufficiently large n,

$$\frac{d}{dt}\log g(t) = -t - C_1\{2\alpha(2\alpha-1)|t|^{2\alpha-2} - (2\alpha+1)|t|^{2\alpha}\}$$
$$\cdot n^{-(2\alpha-1)/2}\operatorname{sgn} t + o(n^{-(2\alpha-1)/2}),$$

it follows from (4.1) and (4.2) that the asymptotic information amount I_{ML} of the MLE is given by

$$I_{\mathrm{ML}} = nI\int_{-\infty}^{\infty}\left\{\frac{d}{dt}\log g(t)\right\}^2 g(t)dt$$

$$= nI\int_{-\infty}^{\infty}\phi(t)[t^2 + C_1\{4\alpha(2\alpha-1)|t|^{2\alpha-1}$$
$$-2(3\alpha+1)|t|^{2\alpha+1} + |t|^{2\alpha+3}\}n^{-(2\alpha-1)/2}]dt$$
$$+ o(n^{(3/2)-\alpha})$$

$$= nI\left\{1 - \frac{2^{\alpha+(3/2)}}{\sqrt{\pi}}C_1\Gamma(\alpha+1)n^{-(2\alpha-1)/2}\right\} + o(n^{(3/2)-\alpha}).$$

Hence, the amount of the loss L of asymptotic information of the MLE is given by

$$L = nI - I_{\mathrm{ML}} = \frac{2^{\alpha+(3/2)}}{\sqrt{\pi}}C_1 I\Gamma(\alpha+1)n^{(3/2)-\alpha} + o(n^{(3/2)-\alpha})$$

$$= \frac{2^{\alpha+(1/2)}\alpha(2\alpha+1)\Gamma(\alpha+1)\{B(\alpha+1,\alpha+1) + (\gamma/(2\alpha+1))\}}{\sqrt{\pi}\, I^{\alpha-(1/2)}\Gamma(1/\alpha)}n^{(3/2)-\alpha}$$
$$+ o(n^{(3/2)-\alpha}).$$

In a similar way to the above, it follows from (3.28) that, in the double-exponential distribution case, namely, when $\alpha = 1$, the amount of the loss of asymptotic information of the MLE is given by $2\sqrt{2n/\pi} + o(\sqrt{n})$, since $I = 1$.

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