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CONTIGUOUS ALTERNATIVES WHICH PRESERVE CRAMÉR-TYPE LARGE DEVIATIONS FOR A GENERAL CLASS OF STATISTICS*

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Abstract. Let P_N and Q_N, $N \geq 1$, be two possible probability distributions of a random vector $X_N = (X_{N1}, \ldots, X_{NN})$, whose components are independent. Suppose P_N and Q_N have respective densities $p_N = \prod_{i=1}^{N} f(x_{Ni} - \bar{\theta}_N)$ and $q_N = \prod_{i=1}^{N} f(x_{Ni} - \theta_{Ni})$, where $\bar{\theta}_N = N^{-1} \sum_{i=1}^{N} \theta_{Ni}$, such that $\max_{1 \leq i \leq N} |\theta_{Ni} - \bar{\theta}_N| = O(N^{-1/2})$, $f(x) > 0$ for almost every real x, f is absolutely continuous, and $\sup_{-\theta_0 \leq \theta \leq \theta_0} \int_{-\infty}^{\infty} [f'(x-\theta)]^2 / f(x) dx < \infty$ for some $\theta_0 > 0$. The contiguity of $\{q_N\}$ to $\{p_N\}$ is well known. In this paper it is proven that under these conditions $\{Q_N\}$ preserves C.-T.L.D. (Cramér-type large deviation) from $\{P_N\}$ for a general class of statistics $\mathscr{F}$ which includes R-, U- and L-statistics as members. That means, for any $\{S_N = S_N(X_N)\}$ from $\mathscr{F}$, a C.-T.L.D. theorem with range $C \leq x \leq o(N^{\delta})$ (any $C \leq 0$), $0 < \delta \leq 4^{-1}$, holds for $\{S_N\}$ under $\{P_N\}$, implying that the same theorem holds for $\{S_N\}$ under $\{Q_N\}$. It also provides a quick and simple way to establish C.-T.L.D. results for statistics under $\{Q_N\}$.

Key words and phrases: Contiguous alternatives, Cramér-type large deviations, linear rank statistics, U-statistics, linear combinations of order statistics.

1. The definition and the main theorem

Let P_N and Q_N, $N \geq 1$, be two possible probability distributions of a random vector $X_N = (X_{N1}, \ldots, X_{NN})$, whose components are independent. Let $S_N = S_N(X_N)$ be a statistic. We use E_0 and E_1 to denote the expectation under P_N and Q_N, respectively (similarly, Var_0, Var_1, etc.). In what follows,

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$0<\delta\le 4^{-1}$, $A_i>0$, $i\ge 1$ are absolute constants, and $\bar{\Phi}=1-\Phi$ with Φ denoting the standard normal d.f.

DEFINITION 1.1. We say that for $\{S_N\}$ the sequence $\{Q_N\}$ preserves C.-T.L.D. (Cramér-type large deviation) with range $C\le x\le o(N^{\delta})$ $(C\le 0)$ from $\{P_N\}$ if as $N\to\infty$

$$P_N[S_N>E_0S_N+x(\mathrm{Var}_0\,S_N)^{1/2}]=\bar{\Phi}(x)(1+o(1)) \tag{1.1}$$

uniformly in $C\le x\le o(N^{\delta})$ $(C\le 0)$ implies that

$$Q_N[S_N>E_1S_N+x(\mathrm{Var}_1\,S_N)^{1/2}]=\bar{\Phi}(x)(1+o(1)) \tag{1.2}$$

uniformly in $C\le x\le o(N^{\delta})$ $(C\le 0)$.

Remark 1.1. We can extend Definition 1.1 to a family $\mathscr{F}$ of (sequences of) statistics in the following way: We say that for $\mathscr{F}$ the sequence $\{Q_N\}$ preserves C.-T.L.D. with range $C\le x\le o(N^{\delta})$ $(C\le 0)$ from $\{P_N\}$ if for every $\{S_N\}$ from $\mathscr{F}$ the sequence $\{Q_N\}$ preserves C.-T.L.D. with range $C\le x\le o(N^{\delta})$ $(C\le 0)$ from $\{P_N\}$. Furthermore, if P_N and Q_N have the respective densities p_N and q_N, we shall speak of "$\{q_N\}$ preserves ... from $\{p_N\}$". Note that if $\{q_N\}$ is contiguous to $\{p_N\}$, they by the well-known LeCam's lemmas we know that under suitable conditions $\{q_N\}$ will preserve asymptotic normality for $\{S_N\}$ from $\{p_N\}$. It is natural to ask the following question: Can $\{q_N\}$ preserve some properties which are related to but beyond the asymptotic normality for $\{S_N\}$ from $\{p_N\}$? The purpose of this paper is to give one answer to this question.

For the rest of this paper, we assume that P_N and Q_N, $N\ge 1$, have respective densities

$$p_N=\prod_{i=1}^{N}f(x_{Ni}-\bar{\theta}_N)\quad\text{and}\quad q_N=\prod_{i=1}^{N}f(x_{Ni}-\theta_{Ni}) \tag{1.3}$$

$\left(\bar{\theta}_N=N^{-1}\sum_{i=1}^{N}\theta_{Ni}\right)$ such that

$$\max_{1\le i\le N}|\theta_{Ni}-\bar{\theta}_N|=O(N^{-1/2})\,, \tag{1.4}$$

(1.5) $f(x)>0$ for almost every real x, f is absolutely continuous, and $\sup_{-\theta_0\le\theta\le\theta_0}\int_{-\infty}^{\infty}[f'(x-\theta)]^2/f(x)\,dx<\infty$ for some $\theta_0>0$.

Note that (1.5) is fulfilled by, for example, normal, double exponential, logistic and extreme value (minimum) densities, and $\{q_N\}$ is contiguous to $\{p_N\}$. We also note that (1.5) is a location version of Assumption (V) of Hušková (1977). For simplicity we confine ourselves here to contiguous location alternatives, but extensions of the results to more general contiguous alternatives are possible.

In Theorem 1.1, we shall give general conditions on $\{S_N\}$ such that $\{q_N\}$ will preserve C.-T.L.D. for $\{S_N\}$ from $\{p_N\}$. In Section 2, four applications of Theorem 1.1 are considered. It is proven that for (simple or signed) linear rank-, U- and L-statistics: (i) $\{q_N\}$ preserves C.-T.L.D. from $\{p_N\}$ (see Theorems 2.1–2.4) and, consequently, (ii) C.-T.L.D. results hold under $\{q_N\}$ (see Corollaries 2.1–2.4). The validity of C.-T.L.D. results for (simple or signed) linear rank- and L-statistics are already known under general alternatives by the work of Seoh *et al.* (1985). The results in Corollaries 2.1, 2.2 and 2.4 of the present paper indicate that under contiguous alternatives $\{q_N\}$ (here each X_{Ni} is continuous with a density function $f(X_{Ni}-\theta_{Ni})$ satisfying (1.4) and (1.5)), some conditions of Seoh *et al.* (1985) can be relaxed (see Remarks 2.1 and 2.3 for details). C.-T.L.D. results for U-statistics under the null hypothesis were studied by Malevich and Abdalimov (1979). The results in Corollary 2.3 extend their results to contiguous alternatives $\{q_N\}$.

The results of this paper depend heavily on Lemma 3.6 of Hušková (1977). An adapted version of that lemma is given by the following

LEMMA 1.1. *Let $\{Y_N = Y_N(X_N)\}$ be any sequence of statistics. Assume* (1.3)–(1.5). *Then for all $N \geq N_0$ and real $k > 0$ (N_0 is some integer not depending on k),*

$$E_1 |Y_N|^{2k} \leq A_1\{E_0 |Y_N|^{4k}\}^{1/2} . \tag{1.6}$$

PROOF. Put $B_N = \left[\prod_{i=1}^{N} f(X_{Ni} - \bar{\theta}_N) > 0\right]$. Then (1.5) implies that $E_1 I(B_N') = 0$, $N \geq 1$, where $I(\cdot)$ denotes the indicator function. Thus $E_1 |Y_N|^{2k} = E_1\{|Y_N|^{2k} I(B_N)\}$. The rest follows from (3.7)–(3.11) of Hušková (1977).

Let $\Delta_N = (\theta_{N1} - \bar{\theta}_N, \ldots, \theta_{NN} - \bar{\theta}_N)$. For any statistic W_N ($= W_N(X_N)$), we denote

$$W_{N+\Delta}(= W_{N+\Delta}(X_N)) = W_N(X_N + \Delta_N) ,$$

$$W_{N-\Delta} = W_N(X_N - \Delta_N), \qquad W_N^{(i)} = W_N - E_i W_N ,$$

$$W_{N+\Delta}^{(i)} = W_{N+\Delta} - E_i W_{N+\Delta} \quad \text{and} \quad W_{N-\Delta}^{(i)} = W_{N-\Delta} - E_i W_{N-\Delta}, \qquad i = 0, 1 .$$

THEOREM 1.1. *Assume* (1.3)–(1.5). *Furthermore, assume*:

(i) $\liminf_{N\to\infty} \mathrm{Var}_0\, S_N > \tau^2$ *for some* $\tau > 0$.

(ii) *There exist statistics* $\{T_N(=T_N(X_N))\}$ *such that*

$$(1.7) \qquad P_N[|S_N^{(0)} - T_N^{(0)}| > N^{-\delta}] = O(\varepsilon_N), \qquad E_0|S_N - T_N|^4 \le A_2^4 N^{-8\delta},$$

eventually, and

$$(1.8) \qquad P_N[|T_{N+\Delta}^{(0)} - T_N^{(0)}| > N^{-\delta}] = O(\varepsilon_N), \qquad \mathrm{Var}_0\,(T_{N+\Delta} - T_N) = O(N^{-4\delta}),$$

where $\varepsilon_N = \exp(-aN^{2\delta})$ *and* $a > 0$ *is a constant not depending on* N. *Then for* $\{S_N\}$ *the sequence* $\{q_N\}$ *preserves C.-T.L.D. with range* $C \le x \le o(N^\delta)$ $(C \le 0)$ *from* $\{p_N\}$.

PROOF. Suppose (1.1) is true. It follows from Lemma 1.1, (1.7), and from Hölder's inequality that for all sufficiently large N,

$$E_1|S_N - T_N| \le A_1\{E_0|S_N - T_N|^2\}^{1/2} \le A_1 A_2 N^{-\delta}$$

and

$$(1.9) \qquad \begin{aligned} D_{N1} &\le Q_N[|S_N - T_N| > (A_2+1)N^{-\delta}] \\ &\le A_1\{P_N[|S_N - T_N| > (A_2+1)N^{-\delta}]\}^{1/2} \\ &\le A_1\{P_N[|S_N^{(0)} - T_N^{(0)}| > N^{-\delta}]\}^{1/2} = O(\varepsilon_N^{1/2}), \end{aligned}$$

where $D_{N1} = Q_N[|S_N^{(1)} - T_N^{(1)}| > (A_2 + A_1A_2 + 1)N^{-\delta}]$. Now, put

$$\sigma_{N0}^2 = \mathrm{Var}_0\, S_N, \qquad D_{N2} = Q_N[|T_N^{(1)} - T_{N-\Delta}^{(1)}| > N^{-\delta}],$$
$$D_{N3} = Q_N[|T_{N-\Delta}^{(1)} - S_{N-\Delta}^{(1)}| > N^{-\delta}]$$

and

$$H_N^{\pm}(x) = Q_N[S_{N-\Delta}^{(1)} > (x \pm A_3\tau^{-1}N^{-\delta})\sigma_{N0}],$$

where $A_3 = A_2 + A_1A_2 + 3$. Note that

$$D_{N2} = P_N[|T_{N+\Delta}^{(0)} - T_N^{(0)}| > N^{-\delta}],$$
$$D_{N3} = P_N[|T_N^{(0)} - S_N^{(0)}| > N^{-\delta}]$$

and

$$H_N^{\pm}(x) = P_N[S_N^{(0)} > (x \pm A_3\tau^{-1}N^{-\delta})\sigma_{N0}] .$$

By Slutsky's argument and (i) of Theorem 1.1,

$$-\sum_{i=1}^{3} D_{Ni} + H_N^{+}(x) \leq Q_N[S_N^{(1)} > x\sigma_{N0}] \leq H_N^{-}(x) + \sum_{i=1}^{3} D_{Ni} \tag{1.10}$$

for all $N \geq N_1$ (N_1 is an integer not depending on x). By (1.1) and by Lemma A1 of Vandemaele and Veraverbeke (1982), we get $H_N^{\pm}(x) = (\bar{\Phi}(x))(1 + o(1))$ uniformly in $C \leq x \leq o(N^{\delta})$ ($C \leq 0$), which, together with (1.7)–(1.10) and the fact that $\varepsilon_N < \varepsilon_N^{1/2} = (\bar{\Phi}(x))o(1)$ uniformly in $C \leq x \leq o(N^{\delta})$ ($C \leq 0$), leads to

$$Q_N[S_N > E_1S_N + x(\mathrm{Var}_0\, S_N)^{1/2}] = \bar{\Phi}(x)(1 + o(1)) \tag{1.11}$$

uniformly in $C \leq x \leq o(N^{\delta})$ ($C \leq 0$). By (1.11) and by Lemma A1 of Vandemaele and Veraverbeke (1982), the proof will be concluded if we show that

$$[(\mathrm{Var}_0\, S_N)^{-1}\, \mathrm{Var}_1\, S_N]^{1/2} = 1 + O(N^{-2\delta}) .$$

By the fact that $\mathrm{Var}_1\, S_{N-\Delta}^{(1)} = \mathrm{Var}_0\, S_N$, we have

$$\begin{aligned} &|\mathrm{Var}_1\, S_N - \mathrm{Var}_0\, S_N| \\ &\quad = |\mathrm{Var}_1\, (S_N^{(1)} - S_{N-\Delta}^{(1)}) + 2\, \mathrm{cov}_1\, (S_N^{(1)} - S_{N-\Delta}^{(1)}, S_{N-\Delta}^{(1)})| \\ &\quad \leq D_{N4} + 2(D_{N4}\, \mathrm{Var}_0\, S_N)^{1/2} , \end{aligned}$$

where $D_{N4} = E_1(S_N^{(1)} - S_{N-\Delta}^{(1)})^2$. Now, (1.6) implies that for all $N \geq N_0$,

$$\begin{aligned} D_{N4} &\leq 3\{E_1(S_N^{(1)} - T_N^{(1)})^2 + E_1(T_N^{(1)} - T_{N-\Delta}^{(1)})^2 + E_1(T_{N-\Delta}^{(1)} - S_{N-\Delta}^{(1)})^2\} \\ &\leq 3\{4A_1[E_0(S_N - T_N)^4]^{1/2} + E_0(T_{N+\Delta}^{(0)} - T_N^{(0)})^2 + 4E_0(S_N - T_N)^2\} . \end{aligned}$$

The proof follows.

Remark 1.2. Note that in Theorem 1.1 we may pick $T_N = S_N$ for all $N \geq 1$. However, if S_N is a sum of dependent r.v.'s, then generally we will pick T_N such that: (i) It is a sum of independent r.v.'s. (ii) (1.7) is either already known to be true or can be established. Moreover, in such situations, (1.8) involves the tail probabilities of sums of independent r.v.'s with vanishing expectations and, therefore, some standard theorems can be applied.

It is easy to see that (1.9)–(1.11) remain true if condition (ii) of Theorem 1.1 is replaced by the following weaker condition:

(ii)′ There exist statistics $\{T_N(=T_N(X_N))\}$ such that

$$P_N[|S_N^{(0)}-T_N^{(0)}|>N^{-\delta}]=O(\varepsilon_N),\qquad E_0|S_N-T_N|^2=O(N^{-2\delta}),$$

and

$$P_N[|T_{N+\Delta}^{(0)}-T_N^{(0)}|>N^{-\delta}]=O(\varepsilon_N),$$

where ε_N is defined in Theorem 1.1.

We therefore immediately have the following

COROLLARY 1.1. *Assume* (1.3)–(1.5) *and condition* (ii)′. *Furthermore, assume that there exists some* $\{\sigma_N\}$ *such that* $\liminf_{N\to\infty}\sigma_N>0$ *and* $P_N[S_N>E_0S_N+x\sigma_N]=\bar{\Phi}(x)(1+o(1))$ *uniformly in* $C\le x\le o(N^\delta)$ $(C\le 0)$. *Then* (1.11) *remains true if in it* $(\mathrm{Var}_0\, S_N)^{1/2}$ *is replaced by* σ_N.

COROLLARY 1.2. *Assume* (1.1), (1.3)–(1.5), *condition* (i) *of Theorem* 1.1, *and condition* (ii)′. *Then* (1.11) *holds true.*

2. Applications: Contiguous alternatives which preserve Cramér-type large deviations for R-, U- and L-statistics

In this section, we give four applications of Theorem 1.1. Let F denote the c.d.f. associated with f. In the sequel we suppress the index N whenever it causes no confusion.

Application 1. (*Simple linear rank statistics*) Let R_{Ni} be the rank of X_{Ni} among $X_{N1},\ldots,X_{NN}$. Let $c_{N1},\ldots,c_{NN}$ be constants satisfying

$$\begin{aligned}&\sum_{i=1}^{N}(c_{Ni}-\bar{c}_N)^2=1,\qquad \max_{1\le i\le N}|c_{Ni}-\bar{c}_N|\le A_4N^{-1/3},\\ &\left|\sum_{i=1}^{N}(c_{Ni}-\bar{c}_N)^3\right|\le A_4N^{-1/2},\qquad N\ge 1,\end{aligned}\tag{2.1}$$

where $\bar{c}_N=N^{-1}\sum_{i=1}^{N}c_{Ni}$. Let $a_N(1),\ldots,a_N(N)$ be scores generated by a non-constant function $\varphi(u)$, $0<u<1$, according to either

$$a_N(i)=\varphi(i/(N+1))\quad\text{or}\quad a_N(i)=E\varphi(U_N^{(i)}),\qquad 1\le i\le N,\tag{2.2}$$

where $U_N^{(i)}$ denotes the i-th order statistic in a sample of size N from the uniform (0, 1) distribution, and φ satisfies

$$|\varphi(u)-\varphi(v)| \le A_5|u-v| \quad \text{for all} \quad u, v \in (0,1)\,. \tag{2.3}$$

THEOREM 2.1. *Assume* (1.3)–(1.5) *and* (2.1)–(2.3). *Then for* $\left\{S_N = \sum_{i=1}^N (c_{Ni}-\bar{c}_N)a_N(R_{Ni})\right\}$ *the sequence* $\{q_N\}$ *preserves C.-T.L.D. with range* $C \le x \le o(N^{1/6})$ $(C \le 0)$ *from* $\{p_N\}$.

PROOF. Put $T_N = \sum_{i=1}^N (c_i - \bar{c})\varphi(F(X_i - \bar{\theta}))$, $\tau_N = \|\varphi - \bar{\varphi}\|_2$ (>0) with $\bar{\varphi} = \int_0^1 \varphi(u)du$. From the proof of the theorem in Kallenberg (1982), we know (1.7) with $\delta = 6^{-1}$ and (i) of Theorem 1.1 are satisfied. Now, $T_{N+\Delta} - T_N = \sum_{i=1}^N (c_i - \bar{c})\, Y_i$ with $Y_i = \varphi(F(X_i + \theta_i - 2\bar{\theta})) - \varphi(F(X_i - \bar{\theta}))$. Note that (1.5) implies that f is bounded on $(-\infty, \infty)$ (cf. Lemma I.2.4a of Hájek and Sidák (1967)). It follows that $|Y_i| \le \|\varphi'\|_\infty \|f\|_\infty |\theta_i - \bar{\theta}|$, all $N \ge 1$, $i = 1, \ldots, N$. Hence, (1.8) is satisfied with $\delta = 6^{-1}$, as a consequence of (1.4), (2.1) and an application of Theorem 2 of Hoeffding (1963). The proof follows.

By (the proof of) Theorem 2.1, Corollaries 1.1 and 1.2, and by the theorem in Kallenberg (1982), we immediately have

COROLLARY 2.1. *Under the conditions and notations of Theorem* 2.1, *the relation*

$$Q_N[S_N > E_1 S_N + x b_N] = \bar{\Phi}(x)(1 + o(1)) \tag{2.4}$$

is true uniformly in $C \le x \le o(N^{1/6})$ $(C \le 0)$ *for each of the following three cases*: $b_N = (\mathrm{Var}_1\, S_N)^{1/2}$, $b_N = \|\varphi - \bar{\varphi}\|_2$ *or* $b_N = (\mathrm{Var}_0\, S_N)^{1/2}$.

Application 2. (*Signed linear rank statistics with regression constants*) Let R^+_{Ni} denote the rank of $|X_{Ni}|$ among $|X_{N1}|, \ldots, |X_{NN}|$. Define sgn $(x) = 1$ or -1 according as $x \ge 0$ or < 0. Let $d_{N1}, \ldots, d_{NN}$ be constants satisfying

$$\sum_{i=1}^N d_{Ni}^2 = 1, \qquad \max_{1 \le i \le N} |d_{Ni}| \le A_6 N^{-1/3}\,. \tag{2.5}$$

THEOREM 2.2. *Assume* (1.3) and (1.4) *with* $\bar{\theta}_N = 0$ *for all* $N \ge 1$, (1.5), (2.2), (2.3), (2.5), $\varphi(0+) = 0$, *and F is symmetric about zero. Then for* $\left\{S_N = \sum_{i=1}^N d_{Ni} a_N(R^+_{Ni})\, \mathrm{sgn}\,(X_{Ni})\right\}$ *the conclusion of Theorem* 2.1 *holds.*

PROOF. Put $T_N = \sum_{i=1}^{N} d_i \varphi(F^*(|X_i|)) \operatorname{sgn}(X_i)$, where $F^*(x) = 2F(x) - 1$ or 0 according as $x \geq 0$ or < 0. Put $\tau_N = \|\varphi\|_2$ (> 0). Using Lemma 4.2 of Seoh *et al.* (1985) and arguing as in the proof of Theorem 2.1, we know (1.7) with $\delta = 6^{-1}$ and (i) of Theorem 1.1 are satisfied. Now, $T_{N+\Delta} - T_N = \sum_{i=1}^{N} d_i Y_i$ with $Y_i = \varphi(F^*(|X_i + \theta_i|)) \operatorname{sgn}(X_i + \theta_i) - \varphi(F^*(|X_i|)) \operatorname{sgn}(X_i)$. It is easy to see that $|Y_i| \leq 2|\theta_i| \, \|\varphi'\|_\infty \|f\|_\infty$ for $X_i \geq \max(0, -\theta_i)$ or $< \min(0, -\theta_i)$. From the fact that $\lim_{\varepsilon \to 0+} \varphi(F^*(\varepsilon)) = 0$, it can be seen that $|Y_i| \leq 4|\theta_i| \, \|\varphi'\|_\infty \|f\|_\infty$ for $\min(0, -\theta_i) \leq X_i < \max(0, -\theta_i)$. The proof follows by arguments analogous to those in Theorem 2.1.

By (the proof of) Theorem 2.2, Corollaries 1.1 and 1.2, and by Theorem 4.1 of Seoh *et al.* (1985), we have

COROLLARY 2.2. *Under the conditions and notations of Theorem* 2.2, (2.4) *is true uniformly in* $C \leq x \leq o(N^{1/6})$ ($C \leq 0$) *for each of the following three cases*: $b_N = (\operatorname{Var}_1 S_N)^{1/2}$, $b_N = \|\varphi\|_2$ *or* $b_N = (\operatorname{Var}_0 S_N)^{1/2}$.

Remark 2.1. Under contiguous alternatives specified by (1.3)–(1.5), the results in Corollaries 2.1 and 2.2 indicate that for (simple or signed) linear rank statistics the assumptions on score functions in Seoh *et al.* (1985) can be relaxed (they assumed $\|\varphi''\|_\infty < \infty$, whereas we only assume (2.3)).

Application 3. (*U-statistics*) Throughout this application, we assume that $\bar{\theta}_N = 0$ for all $N \geq 1$ in (1.3) and (1.4). Consider the statistics

$$U_N = \binom{N}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq N} h(X_{Ni_1}, \ldots, X_{Ni_m}), \qquad N \geq m, \tag{2.6}$$

where $m \geq 1$ is a fixed integer and the kernel function $h(x_1, \ldots, x_m)$ is symmetric in its m arguments. We denote $g(x) = E_0[h(X_{N1}, \ldots, X_{Nm}) | X_{N1} = x]$ for all real x. We assume that

$$E_0 h(X_{N1}, \ldots, X_{Nm}) = 0, \qquad E_0 g^2(X_{N1}) = \sigma_g^2 > 0, \tag{2.7}$$

$$E_0 |h(X_{N1}, \ldots, X_{Nm})|^p \leq A_7^p p^{rp}, \tag{2.8}$$

and there exists some constant $t_0 > 0$ such that

$$\begin{aligned} E_0 |h(X_{N1} + t, X_{N2}, \ldots, X_{Nm}) - h(X_{N1}, X_{N2}, \ldots, X_{Nm})|^p \\ \leq (|t|^l A_8 p^r)^p \end{aligned} \tag{2.9}$$

for all $p = 1, 2, \ldots$ and $|t| \leq t_0$, where $r \geq 0$ is a constant not depending on p (A_8 may depend on t_0), and $l = (2 + 2r)(3 + 2r)^{-1}$.

Remark 2.2. Many commonly used kernel functions (examples: $h(x) = x^q$, $q = 1, 2, \ldots$; $h(x_1, x_2) = x_1 x_2$; $h(x_1, x_2) = |x_1 - x_2|$; $h(x_1, x_2) = 2^{-1}(x_1 - x_2)^2$) satisfy (2.8) and (2.9) if $E_0|X_{N1}|^p$, $p = 1, 2, \ldots$, satisfies a condition similar to (2.8).

THEOREM 2.3. *Assume* (1.3) and (1.4) *with* $\bar{\theta}_N = 0$ *for all* $N \geq 1$, (1.5), *and* (2.7)–(2.9). *Then for* $\{S_N = m^{-1} N^{1/2} U_N\}$ *the sequence* $\{q_N\}$ *preserves C.-T.L.D. with range* $C \leq x \leq o(N^\alpha)$ $(C \leq 0)$ *from* $\{p_N\}$, *where* $\alpha = [2(3 + 2r)]^{-1}$.

PROOF. Put $T_N = N^{-1/2} \sum_{i=1}^{N} g(X_i)$, $N \geq m$. From the proof of the theorem in Vandemaele (1983) (see also, Malevich and Abdalimov (1979)), we know (1.7) with $\delta = \alpha$ and (i) of Theorem 1.1 is fulfilled (in fact, it was established that $E_0|S_N - T_N|^{2p} = O(N^{-p})$ for any $p \geq 1$ and, consequently, $\sigma_g^{-1}(\mathrm{Var}_0\, S_N)^{1/2} = 1 + O(N^{-1/2})$). Now, by (1.4), (2.9) and by Marcinkiewicz-Zygmund inequality (cf. Chow and Teicher (1978), p. 356), we get for all real $k \geq 1$ and $N \geq N_2$ (N_2 is an integer not depending on k),

$$\begin{aligned} (2.10) \qquad E_0|T_{N+\Delta}^{(0)} - T_N^{(0)}|^{2k} &\leq N^{-1} A_9^k k^k \sum_{i=1}^{N} E_0|g(X_i + \theta_i) - g(X_i)|^{2k} \\ &\leq (A_{10} k^{(1+2r)} N^{-l})^k \,. \end{aligned}$$

Put $k(N) = N^{2\alpha}(A_{10}^{-1} e^{-1})^{1/(1+2r)}$. Applying Markov's inequality, we get

$$(2.11) \qquad P_N[|T_{N+\Delta}^{(0)} - T_N^{(0)}| > N^{-\alpha}] \leq e^{-k(N)} \,,$$

eventually. Hence (1.8) is satisfied with $\delta = \alpha$. The proof follows.

From Lemmas 1.(b) and A1 of Vandemaele and Veraverbeke (1982), and from (the proof of) Theorem 2.3 and Corollaries 1.1 and 1.2, we immediately obtain the following

COROLLARY 2.3. *Under the conditions and notations of Theorem* 2.3, (2.4) *is true uniformly in* $C \leq x \leq o(N^\alpha)$ $(C \leq 0)$ *for each of the following three cases*: $b_N = (\mathrm{Var}_1\, S_N)^{1/2}$, $b_N = \sigma_g$ *or* $b_N = (\mathrm{Var}_0\, S_N)^{1/2}$.

Application 4. (*L-statistics*) Let $X_N^{(i)}$ denote the i-th order statistic among $\{X_{Nj}\colon 1 \leq j \leq N\}$. Consider the statistic

$$L_N = N^{-1} \sum_{i=1}^{N} z_{Ni} X_N^{(i)} ,$$

where z_{Ni}'s are real numbers satisfying

$$\sum_{i=1}^{N} \left[z_{Ni} - N \int_{(i-1)/N}^{i/N} J(u)\, du \right]^2 = O(N^{-1}) \tag{2.12}$$

with $J(u)$, $0 < u < 1$, being a non-constant function satisfying $\|J''\|_\infty < \infty$. Assume that

$$\sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x)) J(F(y)) [F(\min(x, y)) - F(x) F(y)]\, dx\, dy \tag{2.13}$$
$$> 0 ,$$

$$\sup_{N > 1} |\bar{\theta}_N| < \infty \quad \text{and} \quad E_0 |X_{N1} - \bar{\theta}_N|^p \le A_{11}^p p^{rp} , \tag{2.14}$$

all $p = 1, 2, \dots$, where $r \ge 0$ is a constant not depending on p.

THEOREM 2.4. *Assume* (1.3)–(1.5) *and* (2.12)–(2.14). *Then for* $\{S_N = \sqrt{N}\, L_N\}$ *the conclusion of Theorem* 2.3 *holds.*

PROOF. Put (R_{Ni} was defined in Application 1)

$$T_N = \sqrt{N} \sum_{i=1}^{N} \int_{(i-1)/N}^{i/N} J(u)\, du X_N^{(i)} , \tag{2.15}$$

$$W_N = N^{-1/2} \sum_{i=1}^{N} J(i/(N+1)) X_N^{(i)} = N^{-1/2} \sum_{i=1}^{N} J(R_{Ni}/(N+1)) X_{Ni} . \tag{2.16}$$

Under (2.12)–(2.14), by Lemma 2 and the result $\lim_{N \to \infty} \mathrm{Var}_0\, S_N = \sigma^2$ of Vandemaele and Veraverbeke ((1982), p. 424) and by arguments similar to those in deriving (2.11), we know (1.7) with $\delta = \alpha$ and (i) of Theorem 1.1 are fulfilled, and moreover, for all $N \ge N_3$ (an integer not depending on k) and real $k \ge 1$,

$$\begin{aligned} E_0 | T_{N+\Delta}^{(0)} - W_{N+\Delta}^{(0)} |^{2k} &= E_1 | T_N^{(1)} - W_N^{(1)} |^{2k} \\ &\le 2^{2k} A_1 \{ E_0 | T_N - W_N |^{4k} \}^{1/2} \le (A_{12} k^{2r} N^{-1})^k \end{aligned} \tag{2.17}$$

which can be seen by simple computations and by (1.6). Introduce

$$\rho_{Ni} = R_{Ni} (N+1)^{-1} ,$$
$$\rho_{Nii} = E_0 [\rho_{Ni} | X_{Ni}]$$

$$= (N+1)^{-1}(1 + (N-1)F(X_{Ni} - \bar{\theta}_N)),$$

$$V_N = N^{-1/2} \sum_{i=1}^{N} [J(\rho_{Nii}) + (\rho_{Ni} - \rho_{Nii})J'(\rho_{Nii})]X_{Ni},$$

$$\hat{V}_N = \sum_{i=1}^{N} E_0[V_N | X_{Ni}] - (N-1)E_0 V_N.$$

By (1.6), (2.14), and by arguments analogous to those in deriving (2.17) and in deriving Lemmas 3.3–3.5 of Seoh *et al.* (1985) (with slight modifications), we get for all $N \geq N_0$ and real $k \geq 1$,

$$\text{(2.18)} \qquad \begin{aligned} &E_0 | W_{N+\Delta}^{(0)} - \hat{V}_{N+\Delta}^{(0)} |^{2k} \\ &\quad \leq 2^{2k} A_1 [E_0 | W_N - \hat{V}_N |^{4k}]^{1/2} \leq (A_{13} k^{2(1+r)} N^{-1})^k, \end{aligned}$$

where $\hat{V}_N = N^{-1/2} \sum_{j=1}^{N} \hat{V}_{N,j}$ and

$$\text{(2.19)} \qquad \begin{aligned} \hat{V}_{N,j} &= J(\rho_{jj})X_j + (N+1)^{-1} \sum_{k \neq j} E_0[(u(X_k - X_j) \\ &\qquad - F(X_k - \bar{\theta}))J'(\rho_{kk})X_k | X_j] \\ &= J(\rho_{jj})X_j + (N+1)^{-1}(N-1)\int_{-\infty}^{\infty} [u(x - X_j) - F(x - \bar{\theta})] \\ &\qquad \cdot J'(\upsilon_N(x))x\,dF(x - \bar{\theta}) \end{aligned}$$

with $u(x) = (1 + \operatorname{sgn}(x))/2$ and $\upsilon_N(x) = (N+1)^{-1}(1 + (N-1)F(x - \bar{\theta}))$. Using the fact that $\|f\|_\infty < \infty$, it is not hard to see that $|\hat{V}_{N+\Delta,j} - \hat{V}_{N,j}| \leq A_{14}|\theta_j - \bar{\theta}| \cdot (|X_j| + |\theta_j - \bar{\theta}| + 1)$. Hence, by (1.4), (2.14) and by Marcinkiewicz-Zygmund inequality, we get $E_0|\hat{V}_{N+\Delta}^{(0)} - \hat{V}_N^{(0)}|^{2k} \leq (A_{15}k^{1+2r}N^{-1})^k$ for all $N \geq N_4$ (an integer not depending on k) and real $k \geq 1$, which, together with (2.17), (2.18) and arguments similar to those in deriving (2.11), implies (1.8) is satisfied with $\delta = \alpha$. The proof follows.

By Theorem 2 of Vandemaele and Veraverbeke (1982), and by (the proof of) Theorem 2.4 and Corollaries 1.1 and 1.2, we quickly get

COROLLARY 2.4. *Under the conditions and notations of Theorem* 2.4, (2.4) *is true uniformly in* $C \leq x \leq o(N^\alpha)$ $(C \leq 0)$ *for each of the following three cases*: $b_N = (\mathrm{Var}_1\, S_N)^{1/2}$, $b_N = \sigma$ or $b_N = (\mathrm{Var}_0\, S_N)^{1/2}$.

Remark 2.3. Corollary 2.4 can be viewed as an extension of Theorem 2.1 (for L-statistics under contiguous alternatives) of Seoh *et al.* (1985) because they only dealt with the case $r = 0$ (due to their condition (2.1)). In fact, it is not possible to deal with the case $r > 0$ by their method (i.e., applying Theorem 1 of Feller (1943) directly to $\hat{V}_N$).

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