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FAILURE RATE OF THE INVERSE GAUSSIAN-WEIBULL MIXTURE MODEL

ESSAM K. AL-HUSSAINI[1] AND NAGI S. ABD-EL-HAKIM[2]

[1]*Department of Mathematics, University of Assiut, Assiut, Egypt*
[2]*Department of Mathematics, EL-Minia University, EL-Minia, Egypt*

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Abstract. Motivated by the idea that different causes of failure of a given system could lead to different failure distributions, a mixture of two-component distributions, one of which is the two-parameter Inverse Gaussian (IG) and the other the two-parameter Weibull (W), is proposed as a failure model. The IG-W mixture model covers several types of failure rates (FR's). It is shown that depending on the parameter values, the IG-W mixture model is capable of covering six different combinations of FR's, as one of the components has an upsidedown bathtub failure rate (UBTFR) or increasing failure rate (IFR) and the other component has a decreasing failure rate (DFR), constant failure rate (CFR), or IFR. A study is made for the mixed FR based on these six combinations.

Key words and phrases: Inverse Gaussian distribution, Weibull distribution, mixture, failure rate.

1. Introduction

The density function of a mixture of any two components with density functions $f_1(t)$ and $f_2(t)$ is given by:

$$f(t) = pf_1(t) + qf_2(t)\,, \tag{1.1}$$

where p is the mixing proportion, $0 \le p \le 1$ and $q = 1 - p$. In particular, $f(t)$ is the IG-W mixture density function if $f_1(t)$ and $f_2(t)$ are the density functions of the $\mathrm{IG}(\mu, \lambda)$ and $\mathrm{W}(\alpha, c)$ distributions having the respective forms:

$$f_1(t) = \left(\frac{\lambda}{2\pi t^3}\right)^{1/2} \exp\left[-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right], \qquad t > 0, \quad (\mu > 0, \lambda > 0)\,, \tag{1.2}$$

$$(1.3)\qquad f_2(t)=\left(\frac{c}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{c-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{c}\right],\qquad t>0,\quad (\alpha>0, c>0)\,.$$

Since the component distributions are distinct from each other, the mixture distribution is identifiable. For details on the identifiability of mixtures of distributions, see for example Everitt and Hand (1980), AL-Hussaini and Ahmad (1981) and Ahmad and AL-Hussaini (1982).

The reliability function $R(t)$ corresponding to the mixed model (1.1) is given by:

$$(1.4)\qquad R(t)=pR_1(t)+qR_2(t)\,,$$

where, for $j=1, 2$, $R_j(t)$ is the reliability function corresponding to $f_j(t)$. The FR function $r(t)$ of the mixed model (1.1) is given by:

$$(1.5)\qquad r(t)=\frac{f(t)}{R(t)}=\frac{pf_1(t)+qf_2(t)}{pR_1(t)+qR_2(t)}=h(t)r_1(t)+(1-h(t))r_2(t)\,,$$

where

$$(1.6)\qquad h(t)=\frac{1}{1+g(t)}\,,\qquad g(t)=\frac{qR_2(t)}{pR_1(t)}\,,$$

and, for $j=1, 2$, $r_j(t)$ is the FR corresponding to $f_j(t)$. Also, it follows that

$$(1.7)\qquad r'(t)=h(t)r_1'(t)+(1-h(t))r_2'(t)-h(t)(1-h(t))(r_1(t)-r_2(t))^2\,.$$

Refer to Barlow and Prochan (1965) for a discussion of the failure rate properties of a mixed model.

2. Special properties of the FR of the IG-W mixture model

The $\mathrm{IG}(\mu,\lambda)$ distribution was shown by Chhikara and Folks (1977) to have an UBTFR if λ is not large relative to μ. In the latter case (i.e., if the shape parameter λ/μ gets large), the IG distribution was shown by Tweedie (1957) to be approximately truncated $N(\mu,\mu^3/\lambda)$. In this case, the distribution has IFR.

It is well known that the $\mathrm{W}(\alpha,c)$ distribution has DFR, if $c<1$; CFR, if $c=1$ and IFR, if $c>1$.

The above discussion shows that the IG-W mixture model will be adequate for situations where one cause of failure has an UBTFR or IFR and the other cause of failure has a DFR, a CFR or an IFR. The resulting six combinations of FR's that can be covered by the IG-W mixture model are therefore given by: (IFR, DFR), (IFR, CFR), (IFR, IFR), (UBTFR, DFR), (UBTFR, CFR) and (UBTFR, IFR).

To study the behaviour of the FR $r(t)$ of the IG-W mixture model, we propose the following limit values of $r(t)$ and $r'(t)$ which will be needed and can be shown to hold true.

PROPOSITION 2.1. *For $r(t)$ and $r'(t)$, given by* (1.5) *and* (1.7), *we have*

$$\lim_{t\to 0^+}[r(t)] = \begin{cases} \infty, & c<1, \\ q/\alpha, & c=1, \\ 0, & c>1, \end{cases} \tag{2.1}$$

$$\lim_{t\to \infty}[r(t)] = \begin{cases} 0, & c<1, \\ 1/\alpha, & c=1 \quad \text{and} \quad \dfrac{\lambda}{2\mu^2} \geq \dfrac{1}{\alpha}, \\ \lambda/2\mu^2, & c>1 \quad \text{or} \quad \left(c=1 \quad \text{and} \quad \dfrac{\lambda}{2\mu^2} < \dfrac{1}{\alpha} \right), \end{cases} \tag{2.2}$$

$$\lim_{t\to 0^+}[r'(t)] = \begin{cases} -\infty, & c<1, \\ -pq/\alpha^2, & c=1, \\ +\infty, & 1<c<2, \\ 2q/\alpha^2, & c=2, \\ 0, & c>2, \end{cases} \tag{2.3}$$

$$\lim_{t\to\infty}[r'(t)] = 0\,. \tag{2.4}$$

3. Behaviour of the FR of the IG-W mixture model

The values of the five parameters involved in the mixture do affect the behaviour of the FR curve. In particular, the FR curves are studied when $c<1$, $c=1$ and $c>1$.

3.1 $c<1$

From (2.1) and (2.2) of Proposition 2.1, it was shown that if $c<1$, then $r(t)\to +\infty$ as $t\to 0^+$ and $r(t)\to 0$ as $t\to\infty$. Since $r_2'(t)<0$ for all t (the $W(\alpha, c)$ distribution has DFR if $c<1$), it follows from $r'(t)$, given by (1.7), that:

(a) If p is chosen so that $h(t)r_1'(t)$ is dominated by the other two terms in $r'(t)$, then $r'(t)<0$ on $(0,\infty)$ and the mixed IG-W model has DFR (decreases from $+\infty$ as $t\to 0^+$ to 0 as $t\to\infty$).

(b) The FR curve decreases from $+\infty$ (as $t \to 0^+$) to a point, say t_0. If p is chosen so that $h(t)r_1'(t)$ dominates the other two terms in $r'(t)$ on (t_0, t^*), then $r'(t) > 0$ on (t_0, t^*), where $t^* < t_1$, t_1 being the point at which $r_1(t)$ attains its maximum. This can happen since the IG distribution has an IFR on $(0, t_1)$. Since $r_1'(t) < 0$ on (t_1, ∞), $r'(t) < 0$ on (t_1, ∞) and the FR of the mixed model decreases to 0. On the interval (t^*, t_1), $r_1'(t) \to 0$ as $t \to t_1$ and so the term $h(t)r_1'(t)$ is dominated by the other two terms in $r'(t)$ so that $r'(t) \to 0$ on (t^*, t_1). Summarizing, if p is chosen so that $h(t)r_1'(t)$ dominates the other two terms in $r'(t)$ on (t_0, t^*), then the FR of the mixed model decreases from $+\infty$ to a minimum value t_0 on $(0, t_0)$, increases on (t_0, t^*) and decreases again on (t^*, ∞) to 0.

3.2 $c = 1$

From (2.1) and (2.2), $r(t) \to q/\alpha$ as $t \to 0^+$ and $r(t) \to \min\{\lambda/2\mu^2, 1/\alpha\}$ as $t \to \infty$. It may be observed that when $c = 1$, then $r_2'(t) = 0$. Therefore,

$$r'(t) = h(t)r_1'(t) - h(t)(1 - h(t))\left(r_1(t) - \frac{1}{\alpha}\right)^2. \tag{3.1}$$

Two possible cases arise:

(a) If $\lambda/2\mu^2 < q/\alpha$, where q is large and α is small, then we have two possibilities:

(i) $\lambda/2\mu^2 \ll q/\alpha$. In this case, the first term of (3.1) will be smaller than the second term and so $r'(t) < 0$ on $(0, \infty)$. Therefore the IG-W mixed model will have a DFR (decreases from q/α as $t \to 0^+$ to the asymptotic value of $\lambda/2\mu^2$ as $t \to \infty$).

(ii) $\lambda/2\mu^2 < q/\alpha$, but the two values are close to each other. In this case, the FR of the mixed model decreases on the interval $(0, t_0)$ from the value q/α to a point t_0 (t_0 being close to zero). With such a choice of p, $h(t)r_1'(t)$ will be greater than the second term in (3.1) on the interval (t_0, t_1), where t_1 is the point at which $r_1(t)$ attains its maximum and so $r(t)$ increases on (t_0, t_1). On (t_1, ∞) the FR $r(t)$ decreases again to the asymptotic value $\lambda/2\mu^2$.

(b) If $\lambda/2\mu^2 \geq q/\alpha$, we shall always have a FR $r(t)$ having a similar shape as that obtained in the possibility (ii) of (a) above, except that its asymptotic value will be $\min\{\lambda/2\mu^2, 1/\alpha\}$ as $t \to \infty$.

3.3 $c > 1$

From (2.1) and (2.2), $r(t) \to 0$ as $t \to 0^+$ and $r(t) \to \lambda/2\mu^2$ as $t \to \infty$. Suppose that $t^* = \min\{t_1^*, t_2^*\}$, where t_1^* and t_2^* are the modes of the $\mathrm{IG}(\mu, \lambda)$ and $\mathrm{W}(\alpha, c)$ density functions, respectively. From (1.5), both $f_1(t)$ and $f_2(t)$ in the numerator of $r(t)$ increase on $(0, t^*)$, whereas the denominator decreases on the same interval. Therefore, $r(t)$ increases on $(0, t^*)$. Rewrite $r'(t)$, given by (1.7), as follows:

$$(3.2) \qquad r'(t) = h(t)r_1'(t) + (1 - h(t))\left[\left(\frac{c-1}{t}\right)r_2(t) - h(t)(r_2(t) - r_1(t))^2\right].$$

Since $r_1(t) \to \lambda/2\mu^2$ as $t \to \infty$ and when $c > 1$, $r_2(t) \to \infty$ and $h(t) \to 1$, there exists a $t_3 > t^*$ such that $r'(t) < 0$ on (t_3, ∞). That is, $r(t)$ decreases on (t_3, ∞) reaching the asymptotic value of $\lambda/2\mu^2$ as $t \to \infty$. On the interval (t^*, t_3) two cases arise:

(a) If either λ is close to one and c is large or λ is large and c is close to one, then for large values of p, there exists a t_1, $t^* < t_1 < t_3$ such that $r(t)$ continues to increase on (t^*, t_1), decreases on (t_1, t_2), $t_1 < t_2 < t_3$ and then increases again on (t_2, t_3). The decrease of $r(t)$ on (t_1, t_2) is due to the large difference between $r_1(t)$ and $r_2(t)$ on this interval, which causes the third term in (3.2) to dominate the other two terms and hence $r'(t) < 0$ on (t_1, t_2).

Summarizing, we find in these cases that $r(t)$ first increases on $(0, t_1)$, decreases on (t_1, t_2), increases again on (t_2, t_3) and finally decreases to the asymptotic value of $\lambda/2\mu^2$ on (t_3, ∞).

(b) For moderate values of both λ and c, $r(t)$ continues to increase on (t_1, t_3). Therefore, $r(t)$ increases on $(0, t_3)$, and decreases on (t_3, ∞) to reach the asymptotic value of $\lambda/2\mu^2$.

If either λ is close to one and c is large or λ is large and c is close to one, then for small values of p, $r(t)$ takes a shape similar to that described for moderate values of both λ and c.

Figure 1 illustrates the mixed FR with components having (UBTFR,

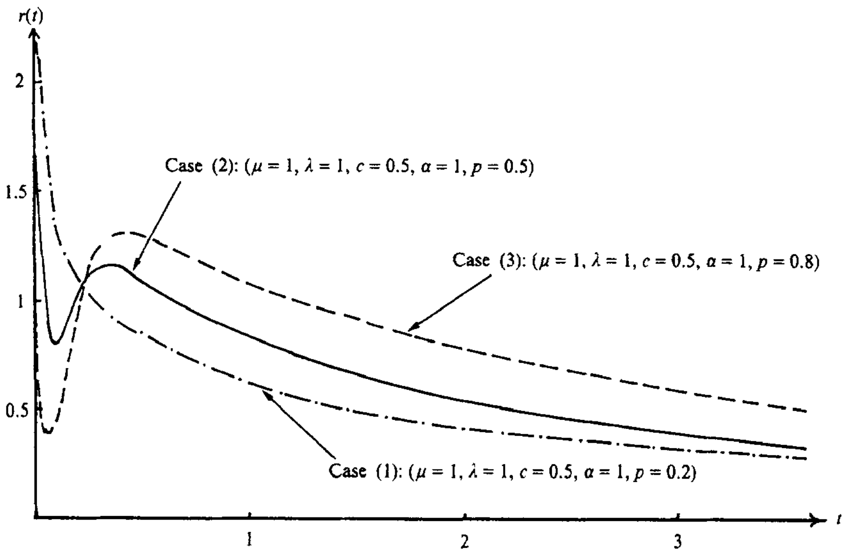

Fig. 1. Mixed FR with components having (UBTFR, DFR).

DFR): IG(1, 1), W(0.5, 1) for different values of p. It may be observed that Case (1) of Fig. 1 is an example of a DFR for the mixture, yet one of the two components does not have a DFR.

REFERENCES

Ahmad, K. E. and AL-Hussaini, E. K. (1982). Remarks on the non-identifiability of finite mixtures, *Ann. Inst. Statist. Math.*, **34**, 543–544.

AL-Hussaini, E. K. and Ahmad, K. E. (1981). On the identifiability of finite mixtures of distributions, *IEEE Trans. Inform. Theory*, **27**, 664–668.

Barlow, R. E. and Prochan, R. (1965). *Mathematical Theory of Reliability*, p. 37, Wiley, New York.

Chhikara, R. S. and Folks, J. L. (1977). The Inverse Gaussian distribution as a lifetime model, *Technometrics*, **19**, 461–468.

Everitt, B. S. and Hand, D. J. (1980). *Finite Mixture Distributions*, p. 57, Chapman & Hall, London–New York.

Tweedie, M. C. K. (1957). Statistical properties of Inverse Gaussian distributions, I, *Ann. Math. Statist.*, **28**, 362–377.