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ON EXHIBITING INVENTORY SYSTEMS WITH ERLANGIAN LIFETIMES UNDER RENEWAL DEMANDS

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Abstract. A continuous review (s, S) inventory system with renewal demand in which one item is put into operation as an exhibiting piece is analyzed. The lifetime of any operating unit has Erlangian distribution, and on failure is replaced by another one from the stock and the failed item is disposed of. Replenishment of stock is instantaneous. The transient and stationary values of inventory level distribution and the mean reorder rate are obtained using the techniques of semi-regenerative processes. Decision rules for optimum s and S that minimize the long-run expected cost rate are derived. The solution for a dual model with the distribution of lifetimes and inter-demand times interchanged is also given.

Key words and phrases: Exhibiting inventory system, (s, S) policy, renewal demands, Erlangian lifetimes, inventory level distribution, mean reorder rate, optimal cost analysis.

1. Introduction

Continuous review (s, S) inventory systems have been studied by many authors in the past, and details of the work done in the last two decades can be found in the works of Aggarwal (1974), Wagner (1980), Silver (1981) and Girlich (1984). Most of these models assume that items are removed from the stock only at demand points. However, there are situations wherein items are removed at times other than demand epochs; one such case is concerned with exhibiting one of the stocked items to boost sales. This is common in the sale of electrical goods. These items have random lifetimes while in operation, and when an exhibited item fails it is removed and replaced by another one from the stock.

This aspect has been introduced in our earlier papers (refer to Kalpakam and Arivarignan (1985*a*, 1985*b*)) where the operating items were assumed to have exponential lifetimes and an item on exhibition could also be used to meet the demand. The present article deals with the case in which the items in operation have Erlangian lifetimes, and also where the exhibited item will not be used to meet the demand. This occurs in situations where the manufacturers do not wish to sell used items as these are not preferred by the customers. The demand epochs form a renewal process, and the supply of items is instantaneous. As the instants at which the items are removed from the stock do not form a renewal process, the usual analysis of inventory level process through renewal theoretic method fails. However, by identifying a suitable embedded Markov renewal process in the stochastic process of the inventory level process, expressions for the transient and limiting values of the inventory level distribution, as well as mean reorder rates, are obtained. Optimal values of s and S that minimize the steady state expected cost rate are also given. A dual model which is obtained by interchanging the distributions of inter-demand points and lifetimes of exhibiting units is considered, and it is shown that finding a solution for one of the models readily yields a solution for the other. It is interesting to note that the limiting inventory distribution for these models is also uniform, as in the case of a non-exhibiting inventory system.

2. Problem formulation and analysis

Consider an inventory stock with a maximum capacity of S units, in which one of the items is put into operation as an exhibited piece. The lifetime of an item when put into operation has an Erlangian distribution of order k and parameter μ (>0). On failure, it will be immediately replaced by another item from the stock. The items not in operation do not fail. Demands occur one at a time and the time intervals between successive demands are independently and identically distributed with common distribution function $F(\cdot)$. Let $f(t)$ denote the derivative of $F(t)$. An item being exhibited will not be used to meet any demand. When the stock level drops to s (>0), an order is placed for $Q = S - s$ (>0) units to bring the stock level back to S. Supply of items is assumed to be instantaneous. We use the following notation:

$$\bar{F}(t) = 1 - F(t) ,$$

$$g(i,t) = (\mu t)^i \exp(-\mu t)/i! \qquad i = 0, 1, 2, \ldots ,$$

$$q(i,t) = \mu(\mu t)^{i-1} \exp(-\mu t)/(i-1)! \qquad i = 1, 2, \ldots ,$$

$$G(r,t) = \sum_{n=0}^{\infty} g(r + nQk, t) \qquad r = 0, 1, 2, \ldots, kQ - 1 ,$$

$$\langle r \rangle = \begin{cases} r & \text{if} \quad r \geq 0 , \\ Q + r & \text{if} \quad r < 0 , \end{cases} \qquad r \text{ is an integer} ,$$

$$\langle\!\langle r \rangle\!\rangle = \begin{cases} r & \text{if} \quad r \geq 0 \,, \\ kQ + r & \text{if} \quad r < 0 \,, \end{cases} \qquad r \text{ is an integer} \,,$$

$[x]^+ =$ Largest integer less than or equal to x.

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

$$H(n) = \begin{cases} 1 & n \geq 0 \,, \\ 0 & n < 0 \,, \end{cases} \qquad n \text{ is an integer} \,,$$

$N = \{1, 2, \ldots\}$,

$N^0 = \{0, 1, 2, \ldots\}$,

I: Identity matrix,

$a(t) © b(t)$: $\int_0^t a(\tau) b(t - \tau) d\tau$ (convolution of any two functions $a(t)$ and $b(t)$),

$b^{(n)}(t)$: n-fold convolution of $b(t)$ with itself,

b_α^*: $\int_0^\infty b(t) e^{-\alpha t} dt$, Re $\alpha > 0$ (Laplace transform of any function $b(t)$),

A_α^*: $\int_0^\infty e^{-\alpha t} A(t) dt$, Re $\alpha > 0$ for any matrix $A(t)$.

Let $L(t)$ denote the inventory level at time t. It assumes values in the set $\{s + 1, s + 2, \ldots, s + Q\}$. As lifetimes are Erlangian of order k with parameter μ, it can be assumed that the item being exhibited is passing through k-phases where each one is distributed as a negative exponential with parameter μ and at the end of the k-th phase the exhibited item fails. Hence it is convenient to introduce a "failure-phase" process $\{X(t), t \geq 0\}$ with state space $\{0, 1, 2, \ldots, k - 1\}$, where $X(t)$ denotes the number of phases elapsed since the latest failure epoch prior to time t. In other words, if $Y(t)$ is a Poisson counting process with $Y(0) = 0$ and parameter μ, then $X(t)$ is uniquely given by

$$X(t) = Y(t) - [Y(t)/k]^+ k, \qquad t \geq 0 \,.$$

We also have the relation

$$Y(t) \equiv X(t) \bmod (k) \,,$$

in which $y \equiv x \bmod (k)$ implies that there exists some non-negative integer n such that $y = nk + x$ for real $x, y \in [0, \infty)$.

Let $0 < T_1 < T_2 < \cdots$ be the successive times of demand occurrences. In order to compute the probability distribution of $L(t)$ at any one of these points, one has to know not only the level but also the "failure-phase" at the immediately preceding point. Let $L_n = L(T_n+)$ and $X_n = X(T_n+)$. Then we have

$$\begin{aligned} &P[L_{n+1} = s+j,\ X_{n+1} = r,\ T_{n+1} - T_n \le t \mid L_k, X_k, k = 0, 1, 2, \ldots, n] \\ &\quad = P[L_{n+1} = s+j,\ X_{n+1} = r,\ T_{n+1} - T_n \le t \mid L_n, X_n], \end{aligned}$$

which implies that the process $(L, X, T) = \{L_n, X_n, T_n;\ n \in N^0\}$ is a Markov renewal process (MRP) with the state space $\{s+1, s+2, \ldots, s+Q\} \times \{0, 1, 2, \ldots, k-1\} \times [0, \infty)$. The semi-Markov kernel of this process is defined as

$$\begin{aligned} \bar{\theta}(i, u, j, v, t) &= P[L_{n+1} = s+j,\ X_{n+1} = v,\ T_{n+1} - T_n \le t \mid L_n = s+i,\ X_n = u] \\ &\qquad i, j = 1, 2, \ldots, Q; \quad u, v = 0, 1, 2, \ldots, k-1, \quad n \in N^0, \quad t \ge 0. \end{aligned}$$

The derivative of $\bar{\theta}(i, u, j, v, t)$ with respect to t, denoted by $\theta(i, u, j, v, t)$ is given in the following lemma in a compact form which is useful in the analysis.

LEMMA 2.1. *The functions $\theta(i, u, j, v, t)$ are given by*

$$\theta(i, u, j, v, t) = \begin{cases} f(t)G(\langle i-j-1\rangle k + v - u, t) & \text{if} \quad \langle i-j-1\rangle \ne 0, \\ f(t)G(\langle\langle v-u\rangle\rangle, t) & \text{if} \quad \langle i-j-1\rangle = 0. \end{cases}$$

PROOF. The events that lead to the occurrence of a demand at time t (synchronizing the origin with the time of occurrence of the previous demand) can be classified into the following mutually exclusive and exhaustive set of events: (i) the stock level does not drop to s, (ii) it drops to s at least once in $(0, t)$ only due to failures of the operating units. Hence we have

$$\begin{aligned} (2.1)\qquad \theta(i, u, j, v, t) = f(t)[&H(i-j-2)g((i-j-1)k - u + v, t) \\ &+ \delta_{(i-j-1)0} H(v-u) g(v-u, t) \\ &+ q(ik-u, t) \circledcirc g((Q-j-1)k + v, t) \\ &+ q(ik-u, t) \circledcirc h_a(t) \circledcirc g((Q-j-1)k + v, t)], \\ &\qquad\qquad\qquad j \ne Q, \end{aligned}$$

$$\theta(i,u,j,v,t) = f(t)[H(i-2)g((i-1)k-u+v,t) + \delta_{(i-1)0}H(v-u)g(v-u,t) + q(ik-u,t) © g((Q-1)k+v,t) + q(ik-u,t) © h_a(t) © g((Q-1)k+v,t)], \quad j=Q, \tag{2.2}$$

where $h_a(t)$ is the renewal density of events corresponding to dropping to s, only due to consecutive failures of the exhibited item without any interruptions of demands in $(0,t)$ and satisfies the integral equation

$$h_a(t) = q(Qk,t) + q(Qk,t) © h_a(t). \tag{2.3}$$

It can be shown that

$$q(i,t) © g(j,t) = g(i+j,t), \tag{2.4}$$

$$q(i,t) © q(j,t) = q(i+j,t), \tag{2.5}$$

hence

$$q^{(n)}(Q,t) = q(nQ,t). \tag{2.6}$$

Also from (2.3), we obtain

$$h_a(t) = \sum_{n=1}^{\infty} q^{(n)}(Qk,t) = \sum_{n=1}^{\infty} q(nQk,t). \tag{2.7}$$

Making use of the results (2.4) to (2.7), we obtain from (2.1) for $j \neq Q$

$$\begin{aligned}\theta(i,u,j,v,t) &= f(t)\Big[H(i-j-2)g((i-j-1)k+v-u,t) \\ &\quad + \delta_{(i-j-1)0}H(v-u)g(v-u,t) \\ &\quad + \sum_{n=1}^{\infty} g((nQ+i-j-1)k+v-u,t)\Big] \\ &= f(t)\Big[H(i-j-2)\sum_{n=0}^{\infty} g((nQ+i-j-1)k+v-u,t) \\ &\quad + \delta_{(i-j-1)0}H(v-u)\sum_{n=0}^{\infty} g(nQk+v-u,t) \\ &\quad + \delta_{(i-j-1)0}(1-H(v-u)) \\ &\quad \cdot \sum_{n=0}^{\infty} g((nQ+i-j-1)k+Qk+v-u,t)\end{aligned}$$

$$
\begin{aligned}
&+(1-H(i-j-1)) \\
&\quad \cdot \left.\sum_{n=0}^{\infty} g(nQk+(Q+i-j-1)k+v-u,t)\right] \\
&= f(t)[H(i-j-2)G((i-j-1)k+v-u,t) \\
&\quad + \delta_{(i-j-1)0} H(v-u)G(v-u,t) \\
&\quad + \delta_{(i-j-1)0}(1-H(v-u))G(Qk+v-u,t) \\
&\quad + (1-H(i-j-1))G((Q+i-j-1)k+v-u,t)] \\
&= f(t)[(1-\delta_{(i-j-1)0})G(\langle i-j-1\rangle k+v-u,t) \\
&\quad + \delta_{(i-j-1)0}\, G(\langle\langle v-u\rangle\rangle,t)] \, .
\end{aligned}
$$

Also, it may be noted that the expression on the right-hand side of (2.2) can be obtained from that of (2.1) by substituting $j=0$. Hence arguments similar to the above for $j=Q$ yield the desired result.

3. Inventory level distribution and reorder rate

Define

$$\phi(i,u,j,v,t) = p[L(t)=s+j,\ X(t)=v \mid L_0 = s+i,\ X_0 = u] \, .$$

We note that once the level and failure phase at $T_n = \sup\{T_i < t\}$ are known, the past history of $L(t)$ and $X(t)$ prior to T_n loses its predictive value. Hence the stochastic process $\{L(t), X(t), t \geq 0\}$ is a semi-regenerative process with the embedded Markov renewal process (L, X, T). As such the functions $\phi(i,j,u,v,t)$ satisfy the Markov renewal equations

$$\phi(i,u,j,v,t) = \tilde{\phi}(i,u,j,v,t) + \sum_{r=1}^{Q}\sum_{w=0}^{k-1}\int_0^t \theta(i,u,r,w,\tau)\phi(r,w,j,v,t-\tau)\,d\tau$$
$$i,j = 1,2,\ldots,Q; \quad u,v = 0,1,\ldots,k-1 \, ,$$

where

$$\tilde{\phi}(i,u,j,v,t) = P[L(t)=s+j,\ X(t)=v,\ T_1 > t \mid L_0 = s+i,\ X_0 = u]$$
$$i,j = 1,2,\ldots,Q; \quad u,v = 0,1,\ldots,k-1 \, .$$

In order to obtain the expressions for $\tilde{\phi}(i,u,j,v,t)$, we note that the required events, along with the condition that the next demand should occur after t units of time, can be classified into the following mutually exclusive and exhaustive cases: the stock level does not drop to s or drops to s at least once in $(0, t)$ due to consecutive failures of exhibited items. As demand occurrences and failures of exhibited items are independent, we have

$$\begin{aligned}(3.1)\quad \tilde{\phi}(i,u,j,v,t) = \bar{F}(t)[&H(i-j-1)g((i-j)k-u+v,t)\\ &+\delta_{(i-j)0}H(v-u)g((v-u),t)\\ &+q(ik-u,t) \copyright\, g((Q-j)k+v,t)\\ &+q(ik-u,t) \copyright\, h_a(t) \copyright\, g((Q-j)k+v,t)]\\ &\qquad\qquad j \neq Q\,,\end{aligned}$$

$$\begin{aligned}(3.2)\quad \tilde{\phi}(i,u,j,v,t) = \bar{F}(t)[&H(i-2)g(ik-u+v,t)\\ &+\delta_{i1}H(v-u)g(k-u+v,t)\\ &+q(ik-u,t) \copyright\, g(Qk+v,t)\\ &+q(ik-u,t) \copyright\, h_a(t) \copyright\, g((Q-j)k+v,t)]\\ &\qquad\qquad j = Q\,.\end{aligned}$$

As expressions (3.1) and (3.2) are similar to expressions (2.1) and (2.2), respectively, except that $(j-1)$ is replaced by j, simplification along similar lines to that given in Lemma 2.1 yields

$$\tilde{\phi}(i,u,j,v,t) = \begin{cases} \bar{F}(t)G(\langle i-j\rangle k+v-u,t), & \langle i-j\rangle \neq 0\,, \\ \bar{F}(t)G(\langle\langle v-u\rangle\rangle,t), & \langle i-j\rangle = 0\,. \end{cases}$$

Let the collection $D = \{1,2,\ldots,Q\} \times \{0,1,2,\ldots,k-1\}$ be arranged lexicographically as follows:

$$\{(1,0),(1,1),\ldots,(1,k-1),(2,0),(2,1),\ldots,(2,k-1),\ldots,$$
$$(Q,0),(Q,1),\ldots,(Q,k-1)\}\,.$$

Denote by $\Phi(t)$, $\Theta(t)$ and $\tilde{\Phi}(t)$ the square matrices of order kQ whose elements are $\phi(i,j,q,r,t)$, $\theta(i,j,q,r,t)$ and $\tilde{\phi}(i,j,q,r,t)$, respectively, arranged according to the above ordering of states $\{(i,q)|(i,q) \in D\}$. Then the Laplace transform of $\Phi(t)$ is given by

$$\Phi_\alpha^* = (I - \Theta_\alpha^*)^{-1}\tilde{\Phi}_\alpha^* = R_\alpha^*\tilde{\Phi}_\alpha^*\,,$$

where R_α^* is the Laplace transform of the Markov renewal kernel of the process (L, X, T) and exists for Re $\alpha > 0$ (Cinlar (1975)).

4. Steady state analysis

Consider the finite Markov chain $(L, X) = \{L_n, X_n, n \in N^0\}$ whose transition probability functions are given by

$$p(i, u, j, v) = \int_0^\infty \theta(i, u, j, v, t)\, dt\,,$$

and are positive for all $i, j = 1, 2, \ldots, Q$; $u, v = 0, 1, 2, \ldots, k-1$. Hence the chain is irreducible and this implies the existence of a unique stationary distribution π.

LEMMA 4.1. *The stationary distribution*

$$\pi = (\pi^{iu}), \quad \textit{is given by} \quad \pi^{iu} = \frac{1}{Qk}\, \forall\, (i, u) \in D\,.$$

PROOF. We first show that the transition probability matrix (tpm) P (with entries $p(i, u, j, v)$ arranged according to the lexicographic ordering) is doubly Markov. Consider

$$\begin{aligned}
(4.1) \quad & \sum_{i=1}^{Q} \sum_{u=0}^{k-1} p(i, u, j, v) \\
&= \int_0^\infty f(t) \sum_{i=1}^{Q} \sum_{u=0}^{k-1} [\delta_{(i-j-1)0}\, G(\langle\!\langle v - u \rangle\!\rangle, t) \\
&\qquad + (1 - \delta_{(i-j-1)0}) G(\langle i - j - 1 \rangle k + v - u, t)]\, dt \\
&= \int_0^\infty f(t) \left[\sum_{d=v-k+1}^{v} G(\langle\!\langle d \rangle\!\rangle, t) \right. \\
&\qquad \left. + \sum_{r=1}^{Q-1} \sum_{d=v-k+1}^{v} G(rk + d, t) \right] dt\,,
\end{aligned}$$

where $d = v - u$ and for fixed j, as i varies from 1 to Q, $\langle i - j - 1 \rangle$ takes values in $\{0, 1, 2, \ldots, Q-1\}$. Hence

$$\begin{aligned}
& \sum_{i=1}^{Q} \sum_{u=0}^{k-1} p(i, u, j, v) \\
&= \int_0^\infty f(t) \left[\sum_{d=v-k+1}^{-1} G(Qk + d, t) + \sum_{d=0}^{v} G(d, t) \right. \\
&\qquad \left. + \sum_{r=1}^{Q-1} \left\{ \sum_{d=v-k+1}^{-1} G(rk + d, t) + \sum_{d=0}^{v} G(rk + d, t) \right\} \right] dt \\
&= \int_0^\infty f(t) \left[\sum_{r=1}^{Q} \sum_{d=v-k+1}^{-1} G(rk + d, t) + \sum_{r=0}^{Q-1} \sum_{d=0}^{v} G(rk + d, t) \right] dt
\end{aligned}$$

(by combining the first term with the third and the second with the fourth term)

$$
\begin{aligned}
&= \int_0^\infty f(t) \Bigg[\sum_{r=1}^{Q} \sum_{d=v-k+1}^{-1} G((r-1)k + k + d, t) \\
&\qquad + \sum_{r=0}^{Q-1} \sum_{d=0}^{v} G(rk + d, t) \Bigg] dt \\
&= \int_0^\infty f(t) \Bigg[\sum_{r=0}^{Q-1} \sum_{d=v-k+1}^{-1} G(rk + k + d, t) + \sum_{r=0}^{Q-1} \sum_{d=0}^{v} G(rk + d, t) \Bigg] dt \\
&= \int_0^\infty f(t) \Bigg[\sum_{r=0}^{Q-1} \sum_{d=v+1}^{k-1} G(rk + d, t) + \sum_{r=0}^{Q-1} \sum_{d=0}^{v} G(rk + d, t) \Bigg] dt \\
&= \int_0^\infty f(t) \sum_{r=0}^{Q-1} \sum_{d=0}^{k-1} G(rk + d, t) dt \\
&= \int_0^\infty f(t) \sum_{r=0}^{kQ-1} G(r, t) dt \\
&= \int_0^\infty f(t) \sum_{r=0}^{kQ-1} \sum_{n=0}^{\infty} g(r + nQk, t) dt \\
&= \int_0^\infty f(t) \sum_{n=0}^{\infty} g(n, t) dt \\
&= \int_0^\infty f(t) dt, \quad \text{as} \quad \sum_{n=0}^{\infty} g(n, t) = 1 \\
&= 1 .
\end{aligned}
$$

Hence the tpm P is doubly Markov. This implies that the stationary distribution is uniform over the Qk states. Hence

$$\pi^{iu} = 1/Qk, \quad i = 1, 2, \ldots, Q; \quad u = 0, 1, \ldots, k-1 . \tag{4.2}$$

We now prove the following theorem.

THEOREM 4.1. *The limiting inventory level distribution is independent of the initial state and is given by*

$$\phi_j = \frac{1}{Q}, \quad j = 1, 2, \ldots, Q .$$

PROOF. As the chain (L, X) is irreducible and recurrent, the MRP (L, X, T) is also irreducible and recurrent. Moreover the derivative of the SMK of the MRP (L, X, T) exists. This implies that the MRP is aperiodic. As $\tilde{\phi}(i, j, u, v, t)$ is non-negative and Riemann integrable, we obtain (Cinlar (1975), p. 347)

(4.3) $$\phi(j,v)=\lim_{t\to\infty}\phi(i,u,j,v,t)$$
$$=\sum_{i=1}^{\infty}\sum_{u=0}^{k-1}\pi^{iu}\int_0^{\infty}\tilde{\phi}(i,u,j,v,t)dt\Big/\sum_{i=1}^{Q}\sum_{u=0}^{Q-1}\pi^{iu}m(i,u)\,,$$

where $m(i,u)$ is the mean sojourn time in state (i,u) which in the present case is equal to m, the mean interval time between demands. Substituting for π^{iu} from (4.2), we obtain

(4.4) $$\phi(j,v)=(1/Qk)\int_0^{\infty}\sum_{i=1}^{Q}\sum_{u=0}^{k-1}\tilde{\phi}(i,j,u,v,t)dt/m$$
$$=(1/Qk)\int_0^{\infty}\bar{F}(t)\sum_{i=1}^{Q}\sum_{u=0}^{k-1}[\delta_{\langle i-j\rangle 0}G(\langle\!\langle v-u\rangle\!\rangle,t)$$
$$+(1-\delta_{\langle i-j\rangle 0})G(\langle i-j\rangle k+v-u,t)]dt/m\,.$$

The integrand in (4.4) is similar to the one given in (4.1) except that $j-1$ is replaced by j. Hence following the derivations given before, we obtain

$$\phi(j,v)=(1/Qk)\int_0^{\infty}\bar{F}(t)dt/m$$
$$=1/Qk\qquad\left(\text{as}\int_0^{\infty}\bar{F}(t)dt=m\right)$$
$$j=1,2,\ldots,Q\quad\text{and}\quad v=0,1,\ldots,k-1\,.$$

The limiting distribution of inventory level ϕ_j is given by

$$\phi_j=\sum_{v=0}^{k-1}\phi(j,v)=1/Q\,.$$

Hence the theorem.

Thus the result regarding the limiting inventory level distribution being uniform for an (s,S) system with renewal demands and instantaneous supply (Sivazlin (1974)) would remain true even if we include failure of exhibited items with Erlangian lifetimes in the model.

The mean inventory level $E(L)$ in the stationary case is given by

(4.5) $$E(L)=s+\sum_{n=1}^{Q}n\phi_n=s+\frac{Q+1}{2}\,.$$

5. Reorders

In order to obtain the mean reorder rate, we consider the counting process associated with the point event b: stock level reaching s from above. Let $N_b(t)$ denote the number of b-events in $(0, t]$. Then the conditional first order product density of b-events (Srinivasan (1974)) denoted by $h(i, u, v, t)$ is defined as

$$h(i,u,v,t) = \lim_{\Delta \to 0} P[N_b(t+\Delta) - N_b(t) = 1, X(t+\Delta) = v \mid L_0 = s+i, X_0 = u]/\Delta$$
$$i = 1, 2, \ldots, Q; \quad u, v = 0, 1, 2, \ldots, k-1 .$$

To obtain $h(i, u, v, t)$, we observe that the stock level drops to s either due to a demand or a failure. Further, we have the following mutually exclusive and exhaustive cases:

(i) no demand in $(0, t]$ and a reorder in $(t, t+\Delta)$ due to a failure,

(ii) a reorder occurs in $(t, t+\Delta)$ due to the first demand,

(iii) a demand occurs at $w\ (<t)$ with $L(w+) = s + y_1$ and $X(u) = y_2$, and a reorder in $(t, t+\Delta)$.

The above cases yield

$$\begin{aligned} h(i,u,v,t) &= \delta_{v0}\bar{F}(t)(q(ik-u,t) + q(ik-u,t) \copyright h_a(t)) \\ &\quad + \theta(i,Q,u,v,t) \\ &\quad + \sum_{y_1=1}^{Q}\sum_{y_2=0}^{k-1}\int_0^t \theta(i,y_1,u,y_2,w)h(y_1,y_2,v,t-w)\,dw \end{aligned} \tag{5.1}$$
$$i = 1, 2, \ldots, Q; \quad u, v = 0, 1, 2, \ldots, k-1 .$$

Let

$$\begin{aligned} \tilde{h}(i,u,v,t) &= \delta_{v0}\bar{F}(t)\{q(ik-u,t) + q(ik-u,t) \copyright h_a(t)\} \\ &\quad + \theta(i,u,Q,v,t) . \end{aligned} \tag{5.2}$$

Define matrices $H(t)$ and $\tilde{H}(t)$ of order $kQ \times k$ as follows:

$$H(t) = ((h(i,u,v,t)))_{(i,u) \in D,\, v=0,1,2,\ldots,k-1} ,$$
$$\tilde{H}(t) = ((\tilde{h}(i,u,v,t)))_{(i,u) \in D,\, v=0,1,2,\ldots,k-1} .$$

Then from the system of equations (5.1) the Laplace transform H_a^* of $H(t)$ is obtained as

$$H_\alpha^* = R_\alpha^* \tilde{H}_\alpha^* .$$

THEOREM 5.1. *The mean reorder rate $E(R)$ in the stationary case is given by*

$$E(R) = \frac{1}{Q}\left(\frac{\mu}{k} + \frac{1}{m}\right).$$

PROOF. The steady state mean reorder rate $\gamma(\upsilon)$ when the exhibited item is in phase υ is given by

$$\gamma(\upsilon) = \lim_{t\to\infty} h(i, u, \upsilon, t) . \tag{5.3}$$

Applying arguments similar to those given in deriving (4.3) to equation (5.3), we obtain

$$\begin{aligned} \gamma(\upsilon) &= \sum_{i=1}^{Q} \sum_{u=0}^{k-1} \pi^{iu} \int_0^\infty \tilde{h}(i, u, \upsilon, t)dt/m \\ &= \frac{1}{mQk} \sum_{i=1}^{Q} \sum_{u=0}^{k-1} \int_0^\infty \tilde{h}(i, u, \upsilon, t)dt . \end{aligned} \tag{5.4}$$

Substituting from equations (2.5) to (2.6), in (5.2), we have

$$\begin{aligned} \tilde{h}(i, u, \upsilon, t) = \delta_{\upsilon 0} \bar{F}(t) \sum_{n=0}^{\infty} q(ik - u + nkQ, t) \\ + \theta(i, Q, u, \upsilon, t) . \end{aligned}$$

Also, we have

$$\begin{aligned} &\sum_{i=1}^{Q} \sum_{u=0}^{k-1} \delta_{\upsilon 0} \bar{F}(t) \sum_{n=0}^{\infty} q(ik - u + nkQ, t) \\ &= \delta_{\upsilon 0} \bar{F}(t) \mu \sum_{i=1}^{Q} \sum_{u=0}^{k-1} \sum_{n=0}^{\infty} g(ik - u + nkQ - 1, t) , \\ &\qquad\qquad (\text{since } q(i, t) = \mu g(i - 1, t)) \\ &= \delta_{\upsilon 0} \bar{F}(t) \sum_{n=0}^{\infty} \sum_{i=1}^{Q} \sum_{d=1}^{k} g((i - 1)k + d + nkQ - 1, t) \\ &\qquad\qquad (\text{where } d = k - u) \\ &= \delta_{\upsilon 0} \bar{F}(t) \mu \sum_{n=0}^{\infty} \sum_{u=1}^{Qk} g(u + nkQ - 1, t) \\ &= \delta_{\upsilon 0} \bar{F}(t) \mu \sum_{i=0}^{\infty} g(i, t) \end{aligned}$$

$$= \delta_{v0}\bar{F}(t)\mu$$

and

$$\sum_{i=1}^{Q}\sum_{u=0}^{k-1}\theta(i, Q, u, v, t)$$
$$= f(t)\sum_{i=1}^{\infty}\sum_{u=0}^{k-1}[(1-\delta_{(i-1)0})G((i-1)k+v-u, t) + \delta_{(i-1)0}G(\langle\langle v-u\rangle\rangle, t)]$$
$$= f(t)\,,$$

as shown in the simplification of equation (4.1). Hence from (5.3), we have

$$\gamma(v) = (1/Qk)\int_0^{\infty}(\bar{F}(t)\mu\delta_{v0} + f(t))dt/m$$
$$= (\mu\delta_{v0}/Qk + 1/Qkm)\,.$$

Thus the expected number of reorders per unit time in the steady state, $E(R)$, is given by

$$E(R) = \sum_{v=0}^{k-1}\gamma(v) = (\mu/k + 1/m)/Q\,. \tag{5.5}$$

Hence the result.

6. Optimal cost analysis

In this section we determine the optimal decision variables s and Q that minimize the long-run expected cost rate. The relevant costs considered are the set up cost K per order, the unit cost c per item, the holding cost h per item per unit time, the disposal cost v per item and the return r per item sold. The total expected cost rate $C(s, Q)$ in the steady state as given by

$$\begin{aligned} C(s, Q) &= (K + cQ)E(R) + hE(L) + v\mu/k - rE(R)[Q - \mu/E(R)k] \\ &= (K + \overline{c-r}Q)E(R) + hE(L) + (v + r)\mu/k\,. \end{aligned} \tag{6.1}$$

Substituting for $E(L)$ and $E(R)$ from (4.5) and (5.5), respectively, we obtain

$$C(s, Q) = \frac{K}{Q}[\mu/k + 1/m] + hQ/2 + hs + h/2 \tag{6.2}$$
$$+ (v + c)\mu/k + (c - r)/m .$$

Though $C(s, Q)$ given in (6.1) generally need not be a separable function in s and Q, in this particular situation $C(s, Q)$ seen from (6.2) turns out to be a separable function of s and Q. Hence each one of the optimal values s^* and Q^* can be obtained independent of the other. As s takes positive integral values, it can be seen that $s^* = 1$. The optimum Q^* can be obtained by minimizing the function

$$\bar{C}(Q) = \frac{K}{Q}\left(\frac{\mu}{k} + \frac{1}{m}\right) + hQ/2 ,$$

over the set of positive integral values.

Consider

$$D(Q) = \bar{C}(Q + 1) - \bar{C}(Q) \tag{6.3}$$
$$= -K\left(\frac{\mu}{k} + \frac{1}{m}\right)\Big/ Q(Q + 1) + \frac{h}{2} .$$

As $(K/Q(Q + 1))(\mu/k + 1/m)$ decreases as Q increases, $D(Q)$ changes sign at most once. This implies that $\bar{C}(Q)$ possesses a unique minimum at $Q = Q^*$. For this value Q^*, the following conditions hold good.

$$D(Q^*) > 0 \quad \text{and} \quad D(Q^* - 1) \leq 0 .$$

Using (6.3), the above conditions can be written as

$$2K\left(\frac{\mu}{k} + \frac{1}{m}\right)\Big/h < Q^*(Q^* + 1)$$

and

$$2K\left(\frac{\mu}{k} + \frac{1}{m}\right)\Big/h \geq Q^*(Q^* - 1) .$$

Combining the above two inequalities, we obtain the necessary and sufficient condition for optimum Q^* as

$$Q^*(Q^* - 1) \leq 2K\left(\frac{\mu}{k} + \frac{1}{m}\right)\Big/h < Q^*(Q^* + 1) . \tag{6.4}$$

7. A dual model

Here we consider a model in which the interval time between two successive demands is distributed as Erlang with order k and parameter μ (>0) and the lifetime of an exhibited item has an arbitrary pdf $f(\cdot)$ with mean equal to m. The operating doctrine is (s, S) policy with instantaneous supply of orders and an item being exhibited will not be used to meet the demand. As this model can be obtained from the earlier one by interchanging the distribution of interval time between two successive demands and the distribution of lifetime of the exhibited item, the formulation and analysis is similar to the one given earlier, except that the regeneration points $\{T_n\}$ will now correspond to the epochs at which items are put into operation. As such the expression for $E(L)$ and $E(R)$ remain the same. The long-run total expected cost rate $\hat{C}(s, Q)$ for the dual model is given by

$$\begin{aligned}\hat{C}(s, Q) &= (K + cQ)E(R) + hE(L) + \frac{v}{m} \\ &\quad - r\left(QE(R) - \frac{v}{m}\right) \\ &= C(s, Q) + (v + r)\left(\frac{1}{m} - \frac{\mu}{R}\right),\end{aligned}$$

where $C(s, Q)$ is the cost rate given in (6.1). Hence in this case as well the optimum s and Q denoted by s^* and Q^* will be same as s^* and Q^*, the optimum values of the earlier model. Thus finding the solution to one of the models readily yields the solution to its dual problem.

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