Ann. Inst. Statist. Math.
Vol. 41, No. 1, 139-148 (1989)

ESTIMATION OF THE SCALE PARAMETER OF A POWER LAW PROCESS USING POWER LAW COUNTS

A. D. DHARMADHIKARI, U. V. NAIK-NIMBALKAR AND S. BHYRI*

Department of Statistics, University of Poona, Pune 411 007, India

(Received August 13, 1987; revised April 28, 1988)

Abstract. Analogous to Kingman's Poisson Counts, power law counts are defined. Further, these are used to obtain the maximum likelihood estimator of the scale parameter of a power law process. Comparison of this estimator is done with those obtained by using other sampling schemes. Also, cost comparisons are done under the assumption of equal asymptotic variances under different sampling schemes.

Key words and phrases: Poisson sampling, power law process, power law counts, power law process sampling.

1. Introduction

The study of various point processes has been dealt with by many authors. Cox and Lewis (1966) have discussed in detail the classical inference problems in point processes along with applications. Cox and Isham (1980) study random collection of point occurrences from the theoretical point of view though they do not consider the development of techniques for the statistical analysis of the data from such processes. Cox (1970) contains a systematic study of the theory of renewal processes. Billingsley (1961) deals with the problems related to the statistical inference for Markov processes. Basawa and Prakasa Rao (1980) contains a vast study of various stochastic processes and their inference problems. Kingman (1963) has introduced the concept of Poisson Sampling (PS) and has proved that the stochastic structure of a process is completely determined by the distribution of the Poisson Counts associated with it. Basawa (1974) has suggested three sampling schemes for the maximum likelihood estimation of the parameters of a renewal and a Markov renewal process using PS. Schuh and Tweedie (1979) have given numerical evidence to show that in certain instances sampling at random time points is more advantageous

*Work done while the author was with the Department of Statistics, University of Poona. Current address: Dept. of Mathematics, CME, Pune.

than sampling at fixed time points. McDunnough and Wolfson (1980) compare fixed versus random sampling for estimating the parameters of a Poisson process and a pure birth process. Using the variance-covariance matrix and the cost factor, Baba (1982) has made a detailed comparison of the estimators of the parameters of $M/M/1$ queue obtained from PS with those obtained from complete observations.

It has been established that a non-homogeneous Poisson process with power law rate function, as described by Ascher (1981) serves as a good model for reliability growth. Even though this process is also known as the Weibull process, this can lead to confusion as indicated in Ascher (1981). Therefore in this article we refer to such a process as a power law process. Several applications of this process are given in Bain (1978). If instead of the time points of the events, the recorded data is the number of events in the specified intervals, one could use the PS. However, as it is shown in Section 3, the solution of the likelihood equation requires iterative procedures in this case. Hence the technique of power law (PL) counts is developed. The corresponding likelihood equation has an explicit analytic solution. In Section 2, power law process (PLP) sampling scheme is defined and the probability distributions of 'PL Counts' are derived. In Section 3, under various sampling schemes, solutions to the likelihood equations and in turn the maximum likelihood estimators are obtained. The estimators are compared in Section 4. The criterion used for comparison is the 'asymptotic variance' and the 'cost analysis'. In the last section, we give a numerical example to illustrate the usage of PLP sampling scheme of Type II.

2. PL counts

DEFINITION 2.1. Power law process (PLP): A PLP denoted by $\mathrm{PL}(\theta,\beta)$ with scale parameter θ and shape parameter β is a non-homogeneous Poisson process with intensity

$$\lambda(t)=\frac{\beta}{\theta}(t/\theta)^{\beta-1}, \quad (\beta,\theta>0), \quad t\geq 0 .$$

DEFINITION 2.2. Power law count process: Let $\tau=\{T_0, T_1, T_2,\ldots\}$ be the sequence of random variables where T_n indicates the time point of the n-th transition in a discrete state point process $\{X(t),\ t\geq 0\}$. Assume that $\{X(t),\ t\geq 0\}$ makes only finitely many transitions in any finite time interval. Let $\tau'=(T_0', T_1', T_2',\ldots)$ be a random sequence corresponding to a $\mathrm{PL}(\gamma,\beta)$, where τ' is independent of τ. Let N_n = number of transitions in a stochastic process $\{X(t),\ t\geq 0\}$ during $(T_{n-1}', T_n']$, and N_n^* = number of transitions in $\mathrm{PL}(\gamma,\beta)$ during the n-th interval $(T_{n-1}, T_n]$, $n\geq 1$, $T_0=T_0'=0$. N_n, $\mathbb{N}=\{N_n,\ n=1, 2,\ldots\}$ and $\mathbb{N}^*=\{N_n^*,\ n=1, 2,\ldots\}$ are a PL-count, the PL-count process and the dual count process, respectively. Further, it is assumed that for each m, n, $T_m \neq T'_n$, so that $\mathbb{N}$ determines $\mathbb{N}^*$ uniquely and vice-versa.

DEFINITION 2.3. PLP sampling: The data obtained by recording the observations on a PL-count process is referred to as PLP sampling.

Next we obtain the distribution of the PL-count process. Let τ and τ' be as mentioned earlier where τ' is independent of τ. Using the properties of a PL process it may be seen that

$$\Pr[N_j^* = k_j; j = 1, 2, \ldots, n] = E_\tau\left[\frac{e^{-(T_n/\gamma)^\beta} T_1^{\beta k_1}(T_2^\beta - T_1^\beta)^{k_2}\cdots(T_n^\beta - T_{n-1}^\beta)^{k_n}}{\gamma^{\beta k(n)} k_1! k_2! \cdots k_n!}\right], \tag{2.1}$$

where $k(n) = k_1 + k_2 + \cdots + k_n$, $\Pr[\cdot]$ is the probability of '$\cdot$' and $E_\tau[\cdot]$ denotes the expectation with respect to the joint distribution of T_1, $T_2, \ldots, T_n$. Since for every m, T'_m has absolutely continuous distribution, $T'_m = T_n$ with probability zero for any m and n. Thus the sequence $\mathbb{N}^*$ determines the sequence $\mathbb{N}$ with probability one, so that (2.1) enables us to compute the distribution of the PL-count process $\mathbb{N}$.

In the following discussion $\{X(t), t \geq 0\}$ is considered as $\mathrm{PL}(\theta, \beta)$ with θ unknown and β known. We note that using time-transformation $y = t^\beta$, the process $\mathrm{PL}(\theta, \beta)$ may be studied as a homogeneous Poisson process (HPP). However, usage of PS for this HPP gives rise to likelihood equation (3.6) derived later.

THEOREM 2.1. *Let $\tau = (T_1, T_2, \ldots)$ be a sequence corresponding to $\mathrm{PL}(\theta, \beta)$, θ unknown, β known and $\tau' = (T'_1, T'_2, \ldots)$ a sequence corresponding to $\mathrm{PL}(\gamma, \beta)$, which is independent of τ. Then*

[A] *$\{N_n^*, n \geq 1\}$ are independent and identically distributed (iid) as*

$$\Pr[N_n^* = k] = pq^k, \quad (k = 0, 1, \ldots; n = 1, 2, \ldots), \tag{2.2}$$

[B] *$\{N_n, n \geq 1\}$ are iid as*

$$\Pr[N_n = k] = qp^k, \quad (k = 0, 1, \ldots; n = 1, 2, \ldots), \tag{2.3}$$

[C]

$$\Pr[N_1^* + N_2^* + \cdots + N_k^* = m] = \binom{k+m-1}{m} p^k q^m, \quad (m = 0, 1, \ldots; k = 1, 2, \ldots), \tag{2.4}$$

and

[D]

$$\Pr[N_1 + N_2 + \cdots + N_k = m] = \binom{k+m-1}{m} q^k p^m, \tag{2.5}$$

$$(m = 0, 1, \ldots;\ k = 1, 2, \ldots),$$

where $p = \gamma^\beta/(\theta^\beta + \gamma^\beta)$ *and* $q = 1 - p$.

PROOF. $\tau = (T_1, T_2, \ldots)$ being a sequence corresponding to $\mathrm{PL}(\theta, \beta)$, the joint probability density function of $(T_1, T_2, \ldots)$ as given by Bain (1978) is,

$$f_{T_1, T_2, \ldots, T_n}(t_1, t_2, \ldots, t_n) = (\beta/\theta)^n e^{-(t_n/\theta)^\beta} \left[\frac{t_1 t_2 \cdots t_n}{\theta^n} \right]^{\beta - 1}, \tag{2.6}$$

$$0 = t_0 < t_1 < \cdots < t_n < \infty .$$

From (2.1) and (2.6),

$$\begin{aligned}
&\Pr[N_j^* = k_j; j = 1, 2, \ldots, n] \\
&= E_\tau \left[\frac{e^{-(T_n/\gamma)^\beta} T_1^{\beta k_1} (T_2^\beta - T_1^\beta)^{k_2} \cdots (T_n^\beta - T_{n-1}^\beta)^{k_n}}{\gamma^{\beta k(n)} k_1! \cdots k_n!} \right] \\
&= \frac{\beta^n}{\left(\prod_{j=1}^n k_j! \right) \gamma^{\beta k(n)} \theta^{n\beta}} \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} t_1^{\beta k_1} (t_2^\beta - t_1^\beta)^{k_2} \cdots (t_n^\beta - t_{n-1}^\beta)^{k_n} \\
&\quad \cdot \left\{ \exp\left[- t_n^\beta \left(\frac{1}{\theta^\beta} + \frac{1}{\gamma^\beta} \right) \right] \right\} \cdot (t_1 t_2 \cdots t_n)^{\beta - 1} dt_1 dt_2 \cdots dt_n .
\end{aligned}$$

Substituting $t_1^\beta = x_i$, $i = 1, 2, \ldots, n$ and simplifying the above expression reduces to

$$\frac{\theta^{\beta k(n)} \gamma^{n\beta} \Gamma(k(n) + n) \prod_{j=2}^n \sum_{i_j=0}^{k_j} \left[(-1)^{k_j - i_j} \binom{k_j}{i_j} \Big/ (k(j) - i_j + j - 1) \right]}{(\theta^\beta + \gamma^\beta)^{k(n)+n} \left(\prod_{j=1}^n k_j! \right)} .$$

It is seen from Feller (1978) that

$$\Pr[N_n^* = k_j; j = 1, 2, \ldots, n] = \frac{\gamma^{\beta n} \theta^{\beta k(n)}}{(\theta^\beta + \gamma^\beta)^{k(n)+n}} = \prod_{j=1}^n p q^{k_j},$$

where p and q are defined as above. Thus N_n^* are iid as $\Pr[N_n^* = k] = (pq^k)$,

($k = 0, 1, \ldots; n = 1, 2, \ldots$), hence [A] follows.

$N_1^*, N_2^*, \ldots, N_k^*$ being k independent and identically distributed geometric random variables, $\sum_{i=1}^{k} N_i^*$ has a negative binomial distribution as given in [C]. Proofs of [B] and [D] would be parallel to the above.

3. Sampling schemes

For a PL(θ,β), θ unknown and β known, estimation of θ is considered under the following sampling schemes.

[I] *Fixed number of events*

Under this sampling scheme, it is assumed that the given PL process is observed till n events occur. The sample, thus, consists of ($t_1, t_2, \ldots, t_n$) where t_j is the time point of the occurrence of the j-th event; $j = 1, 2, \ldots, n$. The likelihood of the sample is given by

$$(3.1)\quad L_n = (\beta/\theta)^n e^{-(t_n/\theta)^\beta} \left(\prod_{i=1}^{n} t_i \right)^{\beta-1} \Big/ \theta^{n(\beta-1)}, \qquad 0 < t_1 < t_2 < \cdots < t_n < \infty\,.$$

Then $\hat{\theta}_{\mathrm{I}} = t_n/n^{1/\beta}$, is the solution to $\partial \ln L_n/\partial\theta = 0$. Further, it may be verified that $\partial^2 \ln L_n/\partial\theta^2|_{\hat{\theta}_{\mathrm{I}}} < 0$; thus $\hat{\theta}_{\mathrm{I}}$ is the maximum likelihood estimator (mle). The asymptotic variance (Billingsley (1961)) of $\hat{\theta}_{\mathrm{I}}$ is

$$(3.2)\qquad \left\{ E_\theta\left[-\frac{\partial^2 \ln L_n}{\partial\theta^2} \right] \right\}^{-1} = \frac{\theta^2}{n\beta^2}\,.$$

[II] *Random number of events*

For fixed time T, the stochastic process is observed in the interval $[0, T]$. Thus the number of events is a random variable that depends on T, say $n(T)$.

We note $\{n(T) = 0\}$ is equivalent to $\{T_1 > T\}$, where T_1 is the time of occurrence of the first event in PL(θ, β). Now $\Pr[n(T) = 0] = \Pr[T_1 > T] = \exp(-(T/\theta)^\beta)$, which is negligible for large values of T, for all $\theta, \beta < 0$. We assume that T is sufficiently large, so that the sample ($t_1, t_2, \ldots, t_{n(T)}$) is realized. Hence the likelihood is

$$L_n = \frac{\beta^{n(T)} e^{-(T/\theta)^\beta} \left\{ \prod_{i=1}^{n(T)} t_i \right\}^{\beta-1}}{\theta^{\beta n(T)}}, \qquad 0 < t_1 < \cdots < t_{n(T)} < T < \infty\,.$$

Solving $\partial \ln L_n/\partial\theta = 0$, the mle is

$$(3.3)\qquad \hat{\theta}_{\mathrm{II}} = T/[n(T)]^{1/\beta}, \quad \text{and}$$

$$\text{asy Var}(\hat{\theta}_{II}) = \theta^{2+\beta}/(\beta^2 T^\beta)\,. \tag{3.4}$$

Remark. To test the null hypothesis that $(t_1, t_2, \ldots, t_{n(T)})$ is from an HPP the desirable least value of $n(T)$ is three (Cox and Lewis (1966), pp. 45–51) at 5% level of significance. However, the test is valid for all $n(T) \geq 1$.

[III] *Poisson sampling*

The notion of PS has been explained in detail by Kingman (1963), Basawa (1974) and Baba (1982). Let $\tau' = (T_1', T_2', \ldots)$ be a Poisson sequence of known intensity λ. We observe the given PL process till ω Poisson events occur. The sample, thus, consists of ω Poisson counts $(n_1, n_2, \ldots, n_\omega)$. The corresponding likelihood is

$$\begin{aligned} \Pr[N_1 = n_1, N_2 = n_2, \ldots, N_\omega = n_\omega] \\ &= L_\omega \\ &= \frac{\lambda^\omega}{\left(\prod_{i=1}^{\omega} n_i!\right)\theta^{\beta n(\omega)}} \left\{ \prod_{j=2}^{\omega} \sum_{i_j=0}^{n_j} (-1)^{n_j - i_j} \binom{n_j}{i_j} \Big/ [\beta(n(j) - i_j) + j - 1] \right\} \\ &\quad \times \int_0^\infty e^{-(T_\omega'/\theta)^\beta - \lambda T_\omega'} T_\omega'^{\beta n(\omega) + \omega - 1} dT_\omega'\,, \end{aligned} \tag{3.5}$$

where $n(j) = n_1 + n_2 + \cdots + n_j$; $j = 1, 2, \ldots, \omega$. Now,

$$\frac{\partial \ln L_\omega}{\partial \theta} = \{-\beta n(\omega)/\theta\} + \frac{\partial}{\partial \theta} \ln \int_0^\infty e^{-(T_\omega'/\theta)^\beta - \lambda T_\omega'} T_\omega'^{\beta n(\omega) + \omega - 1} dT_\omega'\,. \tag{3.6}$$

If $\beta \neq 1$, as mentioned in Section 1, to solve the likelihood equation (3.6) one has to use numerical methods.

[IV] *PLP sampling of Type I*

The sample, according to this scheme consists of ω PL-counts $(n_1, n_2, \ldots, n_\omega)$. Note that, from (2.5), for fixed ω,

$$\Pr\left[\sum_{i=1}^{\omega} n_i = 0\right] = (\theta^\beta/(\theta^\beta + \gamma^\beta))^\omega\,.$$

Thus, for large values of ω, the samples having each $n_i = 0$ occur with negligible probability. Hence for a large value of ω the likelihood is

$$L_\omega = \Pr[N_j = n_j; j = 1, 2, \ldots, \omega]\,.$$

Using (2.3), we get

(3.7)
$$\frac{\partial \ln L_\omega}{\partial \theta} = \frac{\beta\omega}{\theta} - \frac{(n(\omega)+\omega)\beta\theta^{\beta-1}}{(\theta^\beta+\gamma^\beta)},$$

where $n(\omega) = \sum_{i=1}^{\omega} n_i$. Solving $\partial \ln L_\omega / \partial\theta = 0$, the mle is

(3.8)
$$\hat{\theta}_{\mathrm{IV}} = \left[\frac{\omega}{n(\omega)}\right]^{1/\beta}, \quad \text{(we assume that } n(\omega) \neq 0) .$$

Then, using (2.5),

(3.9)
$$\text{asy Var}\,(\theta_{\mathrm{IV}}) = \frac{(\theta^\beta+\gamma^\beta)\,\theta^2}{\omega\gamma^\beta\beta^2} .$$

[V] *PLP sampling of Type II*

In PLP sampling of Type II, the number of events to be observed in PL(θ, β), say ω, is fixed. Hence the number of events to be observed in the dual count process are random. Thus, the likelihood is

$$L_\omega = \Pr[N_j^* = n_j;\ j = 1, 2, \ldots, \omega] .$$

Using (2.2) in the above likelihood

$$\frac{\partial \ln L_\omega}{\partial \theta} = \frac{\beta n(\omega)}{\theta} - \frac{(n(\omega)+\omega)\beta\theta^{\beta-1}}{\theta^\beta+\gamma^\beta},$$

where $n(\omega) = \sum_{i=1}^{\omega} n_i$. Solving $\partial \ln L_\omega / \partial\theta = 0$, the mle is,

(3.10)
$$\hat{\theta}_{\mathrm{V}} = \gamma\left[\frac{n(\omega)}{\omega}\right]^{1/\beta} .$$

Using (2.4) in the asymptotic variance formula, we get

(3.11)
$$\text{asy Var}\,(\hat{\theta}_{\mathrm{V}}) = \frac{(\theta^\beta+\gamma^\beta)\theta^2}{\omega\theta^\beta\beta^2} .$$

[VI] *Periodic sampling scheme*

We observe the given PL process at fixed time points $\{a, 2a, \ldots, na\}$, $a > 0$, and count the number of events in $((j-1)a, ja]$; $j = 1, 2, \ldots, n$ where 'a' is a preselected positive real number. Thus, the sample consists of $\{x(0,a), x(a,2a), \ldots, x((n-1)a, na)\}$ where $x((j-1)a, ja)$ denotes the number of events in $((j-1)a,\ ja]$; $j = 1,\ 2, \ldots, n$. It may be shown that, the probability of non-occurrence of any event in $(0, na]$ is negligible for large values of n. Hence for a large value of n, the likelihood is,

$$L_n = \Pr[X(0,a) = k_1,\ X(a,2a) = k_2, \ldots, X((n-1)a,\ na) = k_n]$$
$$= \frac{\exp(-na/\theta)^{\beta} a^{\beta k(n)} [2^{\beta} - 1]^{k_2} \cdots [n^{\beta} - (n-1)^{\beta}]^{k_n}}{\left(\prod_{j=1}^{n} k_j!\right) \theta^{\beta k(n)}} .$$

Solving $\partial \ln L_n / \partial\theta = 0$, the mle is

$$\hat{\theta}_{\mathrm{VI}} = \frac{na}{[k(n)]^{1/\beta}} . \tag{3.12}$$

Following a similar procedure as in the previous sampling schemes, we get

$$\text{asy Var}(\hat{\theta}_{\mathrm{IV}}) = \frac{\theta^{\beta+2}}{\beta^2 (an)^{\beta}} . \tag{3.13}$$

4. Comparison

In this section, we compare the estimator obtained by PLP sampling of Type I (sampling scheme [IV]) with those obtained by sampling schemes [I], [II] and [VI] as regards (1) asymptotic variance and (2) cost factor.

The sampling scheme [IV] would be better than the sampling scheme [I] in the sense of asymptotic variance if

$$\omega \geq n[1 + (\theta/\gamma)^{\beta}] . \tag{4.1}$$

Similarly, sampling scheme [IV] would be better than sampling schemes [II] and [VI] in the sense of asymptotic variance if

$$\omega \geq \left(\frac{1}{\theta^{\beta}} + \frac{1}{\gamma^{\beta}}\right) T^{\beta} , \tag{4.2}$$

and

$$\omega \geq \left(\frac{1}{\theta^{\beta}} + \frac{1}{\gamma^{\beta}}\right) (na)^{\beta} , \tag{4.3}$$

respectively. The above inequalities are obtained by using (3.2), (3.4) and (3.13).

To answer, when the sampling scheme [IV] would be better costwise than the other sampling schemes viz. [I], [II] and [VI], the following assumptions are made.

[i] The parameters of the sampling schemes are chosen in such a way that all of them have the same asymptotic variance.

[ii] C_0 = per unit observation cost by sampling scheme [I].

[iii] C_1 = observation cost per unit time by sampling scheme [II].

[iv] C_2 = per unit observation cost by sampling scheme [IV].

[v] C_3 = observation cost per unit time by sampling scheme [VI].

Then using inequalities (4.1), (4.2) and (4.3), we get

$$\frac{C_2}{C_0} \leq 1/[1+(\theta/\gamma)^{\beta}], \tag{4.4}$$

$$\frac{C_2}{C_1} \leq 1\bigg/\left[\left(\frac{1}{\theta^{\beta}}+\frac{1}{\gamma^{\beta}}\right)T^{\beta-1}\right], \tag{4.5}$$

and

$$\frac{C_2}{C_3} \leq 1\bigg/\left[\left(\frac{1}{\theta^{\beta}}+\frac{1}{\gamma^{\beta}}\right)(na)^{\beta-1}\right]. \tag{4.6}$$

For specified values of γ, β, C_1 and C_2 if the preliminary investigations suggest that most likely θ is less than θ_0 (θ_0 is known), then the economical advantage of using PLP sampling scheme of Type I may be decided using the equations (4.4) to (4.6). To illustrate this fact in Table 2 we have given the maximum feasible values of C_2/C_1 for different values of θ when $\beta = 2$, $T = 1$ for $\gamma = 0.1, 0.5, 1, 1.5$ and 2.

For instance, if $\theta = 6$ and $\gamma = 0.5$ the corresponding 'table value' 0.25 indicates that, the PLP sampling scheme is economically better than the complete observation in $(0, T)$ if C_2/C_1 is less than 0.25.

5. Example

An example given below illustrates the usage of PLP sampling scheme of Type II.

Observations were simulated from PL(θ, β), where $\theta = 2.778$ and $\beta = 0.5$. These values were selected, because, Crow (1974) had used these values to illustrate the mle method for PL process. The number of events of PL(γ, β) between the successive events of PL(θ, β) were collected until a predetermined number, ω of events in PL(θ, β) occurred. Data were simulated for $\gamma = 0.8, 1.5, 2.0, 3.0$ and $\omega = 25$.

To obtain the mles, $\hat{\theta}$, under the above sampling scheme, 15 samples were generated from each PL(γ, β). These estimates along with their sampling variances are given in Table 1.

Table 1. mle of θ using PLP sampling scheme of Type II*.

γ	0.8	1.5	2.0	3.0
$\hat{\theta}$	1.776213	2.074860	2.466933	1.855680
Var ($\hat{\theta}$)	0.179920	0.417476	0.836035	0.772777

*Based on 15 samples with true value of $\theta = 2.778$.

Table 2. Maximum feasible values of C_2/C_1 ($\beta = 2$, $T = 1$).

$\gamma \backslash \theta$	1	2	3	4	5	6	7	8	9	10	11	12
0.1	0.01	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01
0.5	.20	.24	.24	.25	.25	.25	.25	.25	.25	.25	.25	.25
1.0	.50	.80	.90	.94	.96	.97	.98	.98	.99	.99	.99	.99
1.5	.69	1.44	1.80	1.97	2.06	2.11	2.15	2.17	2.19	2.20	2.20	2.20
2	.8	2	2.76	3.20	3.44	3.60	3.69	3.76	3.81	3.83	3.83	3.94

Acknowledgement

The authors thank the referees for helpful comments.

REFERENCES

Ascher, H. E. (1981). Weibull distribution vs "Weibull process", *Proc. Annual Reliability and Maintainibility Symposium*, 426–431.

Baba, Y. (1982). Maximum likelihood estimation of parameters in birth and death processes by Poisson sampling, *J. Oper. Res. Soc. Japan*, **25**, 99–111.

Bain, L. J. (1978). *Statistical Analysis of Reliability and Life Testing Models, Theory and Methods*, Marcel Dekker, New York and Basel.

Basawa, I. V. (1974). Maximum likelihood estimation of parameters in renewal and Markov renewal process, *Austral. J. Statist.*, **16**, 33–43.

Basawa, I. V. and Prakasa Rao, B. L. S. (1980). *Statistical Inference for Stochastic Processes*, Academic Press, London, New York, Toronto, Sydney, San Francisco.

Billingsley, P. (1961). *Statistical Inference for Markov Processes*, Univ. Chicago Press.

Cox, D. R. (1970). *Renewal Theory*, Methuen and Co. Ltd., London.

Cox, D. R. and Isham, V. (1980). *Point Processes*, Chapman & Hall, London-New York.

Cox, D. R. and Lewis, P. A. W. (1966). *The Statistical Analysis of Series of Events*, Methuen and Co. Ltd., London.

Crow, L. H. (1974). *Reliability Analysis for Complex, Repairable Systems, Reliability and Biometry*, (eds. F. Proschan and R. J. Serfling), 379–410, SIAM, Philadelphia.

Feller, W. (1978). *An Introduction to probability Theory and Its Applications*, Vol. I, Wiley, Eastern Ltd.

Kingman, J. F. C. (1963). Poisson counts for random sequences of events, *Ann. Math. Statist.*, **34**, 1217–1232.

McDunnough, P. and Wolfson, D. B. (1980). Fixed versus random sampling of certain continuous parameter processes, *Austral. J. Statist.*, **22**, 40–49.

Schuh, H. J. and Tweedie, R. L. (1979). Parameter estimation using transform estimation in time evolving models, *Math. Biosci.*, **45**, 37–67.