Ann. Inst. Statist. Math.
Vol. 41, No. 1, 31-45 (1989)

A FRAMEWORK FOR POSITIVE DEPENDENCE

GEORGE KIMELDORF[1*] AND ALLAN R. SAMPSON[2**]

[1] *Programs in Mathematical Sciences, University of Texas at Dallas, Richardson, TX 75080, U.S.A.*
[2] *Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA 15260, U.S.A.*

(Received October 30, 1986; revised May 17, 1988)

Abstract. This paper presents, for bivariate distributions, a unified framework for studying and relating three basic concepts of positive dependence. These three concepts are positive dependence orderings, positive dependence properties and measures of positive dependence. The latter two concepts are formally defined and their properties discussed. Interrelationships among these three concepts are given, and numerous examples are presented.

Key words and phrases: Positive dependence, positive dependence ordering, positive dependence property, measure of positive dependence, Fréchet bounds, positive quadrant dependence, measure of association, ordinal contingency table.

1. Introduction

A large body of statistical research has focused on deriving and studying concepts of positive dependence for bivariate distributions. Three different types of positive dependence concepts can be distinguished: positive dependence properties, positive dependence orderings and (numerical) measures of positive dependence.

Roughly speaking, a bivariate distribution is considered to have a specific positive dependence property if larger values of either random variable are probabilistically associated with larger values of the other random variable. Examples of positive dependence properties are positive quadrant dependence, association and totally positive of order 2 (TP_2). Detailed reviews and discussions of a variety of these properties can be found in Barlow and Proschan ((1975), Chapter 4), Marshall and Olkin ((1979), Chapter 12), Tong ((1980), Chapter 5) and Block and Sampson (1984). Interrelationships among specific properties have been examined;

---

*Supported by the National Science Foundation under Grant DMS-8301361.
**Supported by the Air Force Office of Scientific Research under Contract 84-0113. Reproduction in whole or part is permitted for any purpose of the United States Government.

e.g., Block and Ting (1981) and Shaked (1982). However, there has been little work done to establish a general framework for studying positive dependence properties, with the exception of the framework provided by Dabrowska (1981, 1985) for studying regression dependence.

A positive dependence ordering is a comparison of two bivariate distributions to determine whether one distribution is more positively dependent than the other, thus attempting to partially order distributions according to their degree of positive dependence. Examples of such orderings are Tchen's (1980) more concordant ordering (or essentially equivalently Ahmed *et al.*'s (1979) positively quadrant dependence ordering) and Kimeldorf and Sampson's (1987) TP_2 ordering. A general framework for studying positive dependence orderings was provided by Kimeldorf and Sampson (1987). Other frameworks for orderings which are more focused on regression considerations are given by Dabrowska (1981, 1985).

A (numerical) measure of positive dependence, loosely speaking, is a number measuring the degree of positive dependence in a bivariate distribution. Examples of such measures are Pearson's correlation, Kendall's τ and Spearman's ρ. A framework for examining general measures of dependence was provided by Renyi (1959). Schweizer and Wolff (1981) provided axioms for "nonparametric measures" of bivariate dependence; these can also be viewed as axioms for measures of absolute positive dependence. Other axiomatic approaches are discussed by Dabrowska ((1981) and (1985), Section 4).

The primary purpose of this paper is to establish a comprehensive structure for studying the three types of concepts of positive dependence. Specifically, we define and develop the general concept of positive dependence property. Secondarily, we relate the concept of positive dependence property to the idea of positive dependence orderings introduced by Kimeldorf and Sampson (1987), and also study their relationships to measures of association. Finally, this leads us to define and study the concept of a measure of positive dependence.

2. Positive dependence properties

In this section, we introduce the concept of a positive dependence property. A positive dependence property can be viewed as a subset $\mathscr{P}^+$ of the set of all bivariate distributions. Many of the interesting properties previously obtained for specific positive dependence properties can roughly be thought of as closure properties of the subset under different probabilistic operations.

Depending on the context, i.e., whether working with random variables or c.d.f.'s, we write $(X, Y) \in \mathscr{P}^+$ or $F(x,y) \in \mathscr{P}^+$. We call a c.d.f. $F_+(x,y)$ an upper Fréchet bound if $F_+(x,y) = \min(F_+(x,\infty), F_+(\infty,y))$. Similarly, a c.d.f. $F_-(x,y)$ is called a lower Fréchet bound if $F_-(x,y) =$

$\max[F_-(x,\infty)+F_-(\infty,y)-1,0]$. A c.d.f. $F_I(x,y)$ is called an independence c.d.f. if $F_I(x,y)=F_I(x,\infty)F_I(\infty,y)$. If (X,Y) has c.d.f. $F(x,y)$, then we use the notation (X_I, Y_I) to indicate a bivariate random variable with c.d.f. $F(x,\infty)F(\infty,y)$. Further, let $\mathscr{F}_+$, $\mathscr{F}_-$, and $\mathscr{I}$ denote, respectively, the sets of all upper Fréchet bounds, lower Fréchet bounds, and independence c.d.f.'s.

DEFINITION 2.1. A subset $\mathscr{P}^+$ of the family of all bivariate distributions is a positive dependence property (PDP) if it satisfies the following seven conditions.

(C1) $\quad (X,Y)\in\mathscr{P}^+ \quad$ implies $\quad P(X>x, Y>y)\geq P(X>x)P(Y>y),$ for all $\quad x,y$.

(C2) $\quad \mathscr{F}_+\subseteq\mathscr{P}^+$.

(C3) $\quad \mathscr{I}\subseteq\mathscr{P}^+$.

(C4) $\quad (X,Y)\in\mathscr{P}^+ \quad$ implies $\quad (\phi(X),Y)\in\mathscr{P}^+$ for all increasing functions ϕ.

(C5) $\quad (X,Y)\in\mathscr{P}^+ \quad$ implies $\quad (Y,X)\in\mathscr{P}^+$.

(C6) $\quad (X,Y)\in\mathscr{P}^+ \quad$ implies $\quad (-X,-Y)\in\mathscr{P}^+$.

(C7) $\quad \{F_n\}\in\mathscr{P}^+ \quad$ and $\quad F_n\xrightarrow{\mathscr{L}}F \quad$ imply $\quad F\in\mathscr{P}^+$, where $\xrightarrow{\mathscr{L}}$ denotes convergence in distribution.

Condition (C1) indicates that any positive dependence property must satisfy the basic intuitive concept that given X is large, Y is more likely to be large than without this knowledge and the analogous statement about X, given that Y is large. Since the upper Fréchet bounds are the most positive dependent distributions possible (see Kimeldorf and Sampson ((1978), Theorem 2)), condition (C2) expresses the requirement that they should always possess every possible positive dependence property. Condition (C3) can be viewed as a boundary condition and (C4) is an increasing monotone invariance condition. Condition (C5) is a symmetry condition which requires the dependence property to treat both random variables symmetrically. Including (C6) along with (C4) and (C5) allows a concordance condition. Condition (C7) is closure under weak convergence.

These seven conditions are logically independent in that if any six of them hold, the seventh need not necessarily hold. For example, choosing $\mathscr{P}=\mathscr{F}_+\cup\mathscr{F}_-\cup\mathscr{I}$, we have a property satisfying (C2)–(C7), but not (C1). Choose $\mathscr{P}=\mathscr{I}$ or $\mathscr{P}=\mathscr{F}$ to have properties satisfying all conditions, except (C2) or (C3), respectively. If $\mathscr{P}=\mathscr{F}_+\cup\mathscr{I}\cup\{N(\rho),\rho\geq 0\}$, where $N(\rho)$ denotes the standardized bivariate normal distribution with correlation ρ, then $\mathscr{P}$ satisfies all conditions but (C4). To describe the remaining three examples, we use the notion of translates (e.g., Mardia (1970), Chapter 4) of a set of bivariate distributions, $\mathscr{G}$, defined by $\{P[\phi(X) \le x, \psi(Y) \le y]: \phi, \psi$ are increasing functions, and $(X, Y) \sim G \in \mathscr{G}\}$. To find $\mathscr{P}$ satisfying all but (C5), let $F(s,t) = P(X \le s, Y \le t)$ be any bivariate c.d.f. with continuous marginals which is positively quadrant dependent (see Example 2.1) and which satisfies $P(X \le s, Y \le t) = P(X \ge -s, Y \ge -t)$ for all s, t and $P(X \le s, Y \le t) \ne P(X \le t, Y \le s)$ for some s, t. Let $\mathscr{T}$ denote the set of translates of such an F and define $\mathscr{P} = \mathscr{I} \cup \mathscr{F}^+ \cup \mathscr{T}$. Then $\mathscr{P}$ satisfies all conditions but (C5). Now let F be the bivariate distribution putting mass $1/2$ uniformly on the square $(0,0)$ to $(1/2, 1/2)$ and mass $1/2$ uniformly on the line segment from $(1/2, 1/2)$ to $(1,1)$; and let $\mathscr{T}$ be the set of all translates of this F. The property $\mathscr{I} \cup \mathscr{F}_+ \cup \mathscr{T}$ satisfies all conditions but (C6). To find a property satisfying (C1)–(C6), but not (C7), let $\mathscr{T}$ be the set of all translates of $\{N(\rho): \rho > 1/2\}$ and consider $\mathscr{I} \cup \mathscr{F}_+ \cup \mathscr{T}$.

The following lemma is immediately derivable from the properties of a PDP.

LEMMA 2.1. (a) *Let $\mathscr{P}^+$ be a PDP. If $(X, Y) \in \mathscr{P}^+$, then $(\phi(X), \psi(Y)) \in \mathscr{P}^+$ for ϕ, ψ both increasing or both decreasing.*

(b) *If $(X, Y) \in \mathscr{P}^+$ and $(-X, Y) \in \mathscr{P}^+$, then $(X, Y) \in \mathscr{I}$.*

Note that the result of Lemma 2.1(b) says that only independent random variables can be both positively and negatively dependent.

In the framework for regression dependence, Dabrowska (1985) considers some conditions similar to (C1)–(C6). But due to the specific nature of the regression situation, her conditions reflect the necessary asymmetries. For example, in place of Lemma 2.1(a), she requires the condition: if (X, Y) has the property, then $(aX + b, \phi(Y))$ has the property for all $a > 0$ and increasing ϕ.

We now consider several illustrative examples.

Example 2.1. (*Positively Quadrant Dependence PDP*) Lehmann (1966) defined $F(x,y)$ to be *positively quadrant dependent* (*PQD*) if $F(x,y) \ge F(x,\infty)F(\infty,y)$ for all x, y. Let $\mathscr{P}^+_{\text{PQD}}$ denote the set of all PQD c.d.f.'s. That $\mathscr{P}^+_{\text{PQD}}$ satisfies (C2) is the well-known Fréchet result (see Johnson and Kotz (1972), pp. 22–23), and that it satisfies (C1) and (C3)–(C7) is obvious.

Example 2.2. (*TP_2 PDP*) Block *et al.* (1982) define F to be TP_2 if $P(x_1 < X \le x_2, y_1 < Y \le y_2) \cdot P(x_3 < X \le x_4, y_3 < Y \le y_4) \ge P(x_1 < X \le x_2, y_3 < Y \le y_4) \cdot P(x_3 < X \le x_4, y_1 < Y \le y_2)$ for all $x_1 \le x_2 \le x_3 \le x_4$, $y_1 \le y_2 \le$

$y_3 \le y_4$. Let $\mathscr{P}^+_{\mathrm{TP}_2}$ denote the set of all TP_2 c.d.f.'s. Barlow and Proschan (1975) show $\mathscr{P}^+_{\mathrm{TP}_2}$ satisfies (C1) and Nguyen and Sampson (1982) show $\mathscr{P}^+_{\mathrm{TP}_2}$ satisfies (C2). That $\mathscr{P}^+_{\mathrm{TP}_2}$ satisfies (C3), (C5) and (C7) is obvious. Condition (C4) follows because $x_1 \le x_2 \le x_3 \le x_4$ implies $\phi(x_1) \le \phi(x_2) \le \phi(x_3) \le \phi(x_4)$ and (C6) follows by the symmetry of the TP_2 condition and by choosing $-x_4 \le -x_3 \le -x_2 \le -x_1$ and $-y_4 \le -y_3 \le -y_2 \le -y_1$.

Example 2.3. (*Association PDP*) Esary *et al.* (1967) define X, Y to be associated random variables if $\mathrm{Cov}\,(f(X, Y), g(X, Y)) \ge 0$ for all increasing f, g for which the covariance exists. Let $\mathscr{P}^+_{\mathrm{ASSOC}}$ denote the set of c.d.f.'s for all associated random variables. Since $\mathscr{P}^+_{\mathrm{TP}_2} \subseteq \mathscr{P}^+_{\mathrm{ASSOC}} \subseteq \mathscr{P}^+_{\mathrm{PQD}}$ (e.g., Barlow and Proschan (1975), Chapter 4), it follows that (C1), (C2) and (C3) are satisfied. Conditions (C4), (C5) and (C6) are obvious, and (C7) follows from property (P5) of Esary *et al.* (1967).

Example 2.4. (*Weak PQD*) Let $\mathscr{P}^+_{\mathrm{W}} \equiv \mathscr{F}_+ \cup \mathscr{I}$. Obviously $\mathscr{P}^+_{\mathrm{W}}$ satisfies Conditions (C1)–(C6). Condition (C7) follows essentially from Theorem 3 of Kimeldorf and Sampson (1978).

Example 2.5. (*Nonnegative Covariance*) Let $\mathscr{C}$ be the set of all c.d.f.'s with nonnegative covariance. Then $\mathscr{C}$ is not a PDP, because it fails to satisfy (C1) and (C4). To see the latter, let (X, Y) put mass .45 on $(0, .5)$, mass .45 on $(.5, 1)$ and mass .1 on $(1, 0)$. Then $\mathrm{Cov}\,(X, Y) = .005625$. Let $\phi(X) = 0$, if $X \le 1/2$, and 1, otherwise. Then $\mathrm{Cov}\,(\phi(X), Y) = -.0675$.

In establishing the conditions which specify a PDP, a number of other conditions were considered, but not included in Definition 2.1 for a variety of reasons. However, several of these are worth noting in their own right.

DEFINITION 2.2. A set $\mathscr{P}$ of c.d.f.'s satisfies the *generalized monotone invariance condition* if for independent pairs (X_1, Y_1) and (X_2, Y_2)

$$(X_1, Y_1) \in \mathscr{P} \quad \text{and} \quad (X_2, Y_2) \in \mathscr{P} \quad \text{imply} \quad (\phi(X_1, X_2), \psi(Y_1, Y_2)) \in \mathscr{P}, \tag{2.1}$$

for any functions $\phi(u, v)$ and $\psi(u, v)$, which are increasing in each argument.

DEFINITION 2.3. A set $\mathscr{P}$ of c.d.f.'s having the same pair of marginal distributions satisfies the *mixture condition*, if

$$F_1, F_2 \in \mathscr{P} \quad \text{implies} \quad \alpha F_1 + (1 - \alpha) F_2 \in \mathscr{P}, \tag{2.2}$$

for all $\alpha \in (0,1)$. The mixture condition can also be viewed as requiring the convexity of $\mathscr{P}$.

DEFINITION 2.4. A set $\mathscr{P}$ of c.d.f.'s satisfies the *normal-agreeing condition* if

$$\mathscr{N}_+ \subseteq \mathscr{P}, \tag{2.3}$$

where $\mathscr{N}_+$ denotes the set of all bivariate normal c.d.f.'s with nonnegative correlation.

$\mathscr{P}^+_{\mathrm{PQD}}$ satisfies the generalized monotone invariance condition (see Theorem 3.1 of Kimeldorf and Sampson (1987)), obviously the mixture condition and the normal-agreeing condition (see Slepian (1962)). $\mathscr{P}^+_{\mathrm{TP}_2}$ satisfies the normal-agreeing condition (see Chapter 4 of Barlow and Proschan (1975)), but does not satisfy the generalized monotone invariance condition (see Example 3.1 of Kimeldorf and Sampson (1987)), and does not satisfy the mixture condition as may be seen from the following example.

Example 2.6. Let F_1 assign mass $1/3$ to each of the points $(1,1)$, $(2,2)$, $(3,3)$ and let F_2 assign mass $1/9$ to each of the points in $\{1,2,3\} \times \{1,2,3\}$, so that F_1 is an upper Fréchet bound and F_2 is an independence distribution. Then $F_1 \in \mathscr{P}^+_{\mathrm{TP}_2}$, $F_2 \in \mathscr{P}^+_{\mathrm{TP}_2}$, but $(F_1 + F_2)/2 \notin \mathscr{P}^+_{\mathrm{TP}_2}$.

$\mathscr{P}^+_{\mathrm{ASSOC}}$ satisfies the generalized monotone invariance condition (this follows from Esary *et al.* (1967)) and the normal-agreeing condition (see Chapter 4 of Barlow and Proschan (1975) or more generally Pitt (1982)). It is obvious that $\mathscr{P}^+_{\mathrm{W}}$ does not satisfy any of the three conditions.

If we remove from Definition 2.3 the requirement that the distributions have the same set of marginals, then $\mathscr{P}^+_{\mathrm{TP}_2}$, $\mathscr{P}^+_{\mathrm{ASSOC}}$ and $\mathscr{P}^+_{\mathrm{PQD}}$ all fail to satisfy the mixture condition. To see this, let $F_1(x,y) = xy$ and $F_2(x,y) = x^2y^{1/2}$ on $(0,1)\times(0,1)$ with $\alpha = 1/2$. Although F_1 and F_2 are TP_2, the resulting mixture is not TP_2. In fact, this mixture is not even PQD. To verify this fact, note that as $x \to 0^+$, $2P(X \le x, Y \le y) = xy^{1/2}(y^{1/2} + x) = xy + o(x) < x(y + y^{1/2})/2 + o(x) = (x + x^2)(y + y^{1/2})/2 = 2P(X \le x)P(Y \le y)$, for $0 < y < 1$. Krishnaiah *et al.* (1985) provide a counterexample for the non-convexity of $\mathscr{P}^+_{\mathrm{PQD}}$ when the marginals differ.

3. Obtaining PDP's from PDO's

Kimeldorf and Sampson (1987) discuss properties which an ordering on bivariate distributions should have, if that ordering describes how positively dependent one pair of random variables is relative to another pair. In particular, they introduce the concept of a positive dependence ordering.

DEFINITION 3.1. (Kimeldorf and Sampson (1987)) A relation $\ll$ on the family of all bivariate distributions is a positive dependence ordering (PDO) if it satisfies the following nine properties:

(P0) $F \ll G$ implies $F(x,\infty) = G(x,\infty)$ and $F(\infty,y) = G(\infty,y)$,

(P1) $F \ll G$ implies $F(x,y) \leq G(x,y)$ for all x, y,

(P2) $F \ll G$ and $G \ll H$ imply $F \ll H$,

(P3) $F \ll F$,

(P4) $F^- \ll F \ll F^+$,

where F^+ and F^- are, respectively, the upper and lower Fréchet bounds corresponding to $F(x,y)$.

(P5) $(X, Y) \ll (U, V)$ implies $(\phi(X), Y) \ll (\phi(U), V)$,

for all increasing functions ϕ, where the notation $(X, Y) \ll (U, V)$ means that the relation $\ll$ holds between the corresponding bivariate distributions $F_{X,Y}$ of (X, Y) and $F_{U,V}$ of (U, V), i.e., $F_{X,Y} \ll F_{U,V}$.

(P6) $(X, Y) \ll (U, V)$ implies $(-U, V) \ll (-X, Y)$.

(P7) $(X, Y) \ll (U, V)$ implies $(Y, X) \ll (V, U)$.

(P8) $F_n \ll G_n$, $F_n \xrightarrow{\mathscr{L}} F$, $G_n \xrightarrow{\mathscr{L}} G$ implies $F \ll G$,

where F_n, F, G_n, G all have the same pair of marginals.

It follows immediately from (P1) that

(P9) $F \ll G$ and $G \ll F$ imply $F = G$.

We conclude from (P2), (P3) and (P9) that the relation $\ll$ induces a partial ordering on the family of all bivariate distributions. The notation $G \gg F$ indicates that G (or a pair of random variables whose distribution is G) is more positively dependent than F (or a pair of random variables whose distribution is F).

The motivation of these properties is essentially given in Kimeldorf and Sampson (1987). Proceeding as we did for PDP's, we could similarily demonstrate the logical independence of these nine conditions.

We now discuss the relationship between PDO's and PDP's, showing that every PDO yields a PDP.

THEOREM 3.1. *Let* $\preccurlyeq$ *be a PDO and define*

$$\mathscr{P}^+_{\succcurlyeq} = \{F: F(x,y) \succcurlyeq F(x,\infty)F(\infty,y) \text{ for all } x, y\} . \tag{3.1}$$

Then $\mathscr{P}^+_{\succcurlyeq}$ *is a PDP.*

PROOF.
(C1): This follows directly from (P1).

(C2): Let $F^+ \in \mathscr{F}_+$. Then by property (P4), $F^+ \succcurlyeq F_{\mathrm{I}}$, so that $F^+ \in \mathscr{P}^+_{\succcurlyeq}$.

(C3): By (P3), $F_{\mathrm{I}} \in \mathscr{P}^+_{\succcurlyeq}$.

(C4): If $(X, Y) \succcurlyeq (X_{\mathrm{I}}, Y_{\mathrm{I}})$, then (P5) implies $(\phi(X), Y) \succcurlyeq (\phi(X_{\mathrm{I}}), Y_{\mathrm{I}}) = (\phi(X)_{\mathrm{I}}, Y_{\mathrm{I}})$ which yields that $(\phi(X), Y) \in \mathscr{P}^+_{\succcurlyeq}$.

(C5): This follows from (P7).

(C6): If $(X, Y) \succcurlyeq (X_{\mathrm{I}}, Y_{\mathrm{I}})$, then by (P6) and (P7) $(-X, -Y) \succcurlyeq (-X_{\mathrm{I}}, -Y_{\mathrm{I}}) = ((-X)_{\mathrm{I}}, (-Y)_{\mathrm{I}})$, so that $(-X, -Y) \in \mathscr{P}^+_{\succcurlyeq}$.

(C7): If $F_n \succcurlyeq F_{\mathrm{I}}$ and $F_n \xrightarrow{\mathscr{L}} F$, then by (P8) $F \succcurlyeq F_{\mathrm{I}}$, which implies $F \in \mathscr{P}^+_{\succcurlyeq}$.

Note that if $F \in \mathscr{P}^+_{\succcurlyeq}$ and $G \succcurlyeq F$, then by property (P1), $G \in \mathscr{P}^+_{\succcurlyeq}$.

Kimeldorf and Sampson (1987) define a TP_2 ordering for bivariate distributions F and G with the same marginals. The distribution G is said to be more TP_2 than F if for all intervals $I_1 < I_2$ and $J_1 < J_2$,

$$\begin{aligned} F(I_1, J_1)F(I_2, J_2)G(I_1, J_2)G(I_2, J_1) \\ \le G(I_1, J_1)G(I_2, J_2)F(I_1, J_2)G(I_2, J_1) , \end{aligned} \tag{3.2}$$

where $F(I_i, J_j)$ and $G(I_i, J_j)$ denote the probabilities assigned by F and G, respectively, to the rectangle $I_i \times J_j$, and $I < J$ means $x \in I$, $y \in J$ imply $x < y$. Denote this ordering by $G \succcurlyeq_{\mathrm{T}} F$. Also considered by Kimeldorf and Sampson (1987) is the weak Fréchet ordering where for any pair F, G of bivariate distributions, we write $G \succcurlyeq_{\mathrm{W}} F$ if and only if either $F = G$, or G is the upper Fréchet bound corresponding to F, or F is the lower Fréchet bound corresponding to G. Kimeldorf and Sampson (1987) show that $\succcurlyeq_{\mathrm{T}}$, $\succcurlyeq_{\mathrm{W}}$ are PDO's, and also that the more PQD ordering $\succcurlyeq_{\mathrm{PQD}}$ (see Tchen (1980)), is a PDO. Thus by Theorem 3.1, the corresponding dependence concepts defined by (3.1) must be PDP's. It is straightforward to prove the following theorem.

THEOREM 3.2. (a) $\mathscr{P}^+_{\geqslant_T} = \mathscr{P}^+_{TP_2}$, (b) $\mathscr{P}^+_{\geqslant_W} = \mathscr{P}^+_W$, *and* (c) $\mathscr{P}^+_{\geqslant_{PQD}} = \mathscr{P}^+_{PQD}$.

We note that just as $\geqslant_W$ can be viewed as the weakest PDO, $\mathscr{P}^+_W$ is the weakest PDP, since by (C2) and (C3), if $\mathscr{P}^+$ is any other PDP, then $\mathscr{P}^+_W \subseteq \mathscr{P}^+$.

The interesting question remains: Is there a PDO $\geqslant_A$ such that $\mathscr{P}^+_{\geqslant_A} = \mathscr{P}^+_{\text{ASSOC}}$? There are two orderings related to the concept of association which have been recently introduced. Kimeldorf and Sampson (1984) and Hollander *et al.* (1985) define $(U, V) \geqslant_a (X, Y)$ if

$$\text{Cov}\,(\phi(X, Y), \psi(X, Y)) \leq \text{Cov}\,(\phi(U, V), \psi(U, V))\,, \tag{3.3}$$

for all increasing ϕ, ψ. Schriever (1985) defines $(U, V) \geqslant_A (X, Y)$ if there exist functions $K_1(a, b)$, $K_2(a, b)$ such that

(i) K_1, K_2 are monotone in each argument,

(ii) $K_1(a_1, b_1) < K_1(a_2, b_2)$, $K_2(a_1, b_1) > K_2(a_2, b_2)$ imply $a_1 < a_2$, $b_1 > b_2$, and

(iii) (U, V) has the same distribution as $(K_1(X, Y), K_2(X, Y))$.

Suppose $(X, Y) \in \mathscr{P}^+_{\text{ASSOC}}$. Then it is direct to show that $(U, V) \geqslant_a (X, Y)$ implies $(U, V) \in \mathscr{P}^+_{\text{ASSOC}}$, and also $(U, V) \geqslant_A (X, Y)$ implies $(U, V) \in \mathscr{P}^+_{\text{ASSOC}}$ (Schriever (1985), Proposition 4.1.2). However, there are elements (X, Y) of $\mathscr{P}^+_{\text{ASSOC}}$ which do not satisfy $(X, Y) \geqslant_A (X_I, Y_I)$ (Schriever (1985), p. 66) and also elements which do not satisfy $(X, Y) \geqslant_a (X_I, Y_I)$, so that neither of these orderings fully "generates" $\mathscr{P}^+_{\text{ASSOC}}$. Additionally, we note that $\geqslant_a$ is not a PDO, because as the following example demonstrates, $\geqslant_a$ violates (P4).

Example 3.1. Let (X, Y) correspond to the upper Fréchet bound where both marginals are uniform distributions on $[0, 1]$, so that $(X, Y) \in \mathscr{P}^+_{\text{ASSOC}}$. Let $A = \{(s, t): s \geq 2^{-1/2} \text{ or } t \geq 2^{-1/2}\}$. Clearly, A is an upper set, so that $\phi(s, t) \equiv \psi(s, t) \equiv I_A(s, t)$, where I_A denotes the indicator function of the set A, are both increasing functions. Direct calculation shows $\text{Cov}\,(\phi(X, Y), \psi(X, Y)) = .2071$ and $\text{Cov}\,(\phi(X_I, Y_I), \psi(X_I, Y_I)) = .2500$, thus violating (3.3).

The issue of how to use a PDO $\geqslant$ to obtain concepts of negative dependence is discussed by Kimeldorf and Sampson (1984).

4. Measures of positive dependence

The subject of how to measure numerically the association between random variables has been addressed extensively in the statistical literature. The more specific area of developing measures of positive association for ordinal contingency tables has also been widely examined. Good reviews of these areas are given by Goodman and Kruskal (1959) and Agresti (1984). Renyi (1959) and Hall (1970) have provided axiomatic frameworks that a non-ordinal measure of association or relationship should satisfy. Generalizations of Renyi's axioms have been considered by Rödel (1970) and Höschel (1976). No such coherent framework appears generally to exist for measures of positive association. Schweizer and Wolff (1981) have provided a modification of Renyi's axioms to handle measures of dependence that are based upon ranks. In some sense, their axioms begin to approach the issue of measures of positive association. Dabrowska (1985) considers general properties of asymmetric measures of positive association dealing with regression concepts, and Scarsini (1984) deals with some aspects of the issue in the context of the $\geqslant_{\mathrm{PQD}}$ ordering. Underlying the research of these authors and others is what we believe to be a fundamental observation: that without specifying a corresponding PDO, any discussion of measures of positive association is problematic. This is due in part to the obvious difficulties in attempting to represent the dependence structure of a bivariate distribution by a single number (see Kowalczyk and Pleszczynska (1977)). Also it is often unclear exactly what dependence concept a specific measure of positive association is attempting to describe.

DEFINITION 4.1. Let $\leqslant$ be an arbitrary PDO and m be a finite real-valued function defined on the set of all bivariate distributions. The function m is *a measure of positive dependence* (*MPD*) *concurring with* $\leqslant$ if

$$F \leqslant G \quad \text{implies} \quad m(F) \leq m(G)\,. \tag{4.1}$$

In the specific situation they were considering, other authors have used differing terms for this concept. In the context of regression dependence, Dabrowska (1981) calls m "monotone with respect to $\leqslant$;" in the context of $\geqslant_{\mathrm{A}}$, Schriever (1985) terms m to "preserve the ordering $\leqslant$;" and Scarsini (1984) in dealing with $\geqslant_{\mathrm{PQD}}$ calls m "consistent with $\geqslant$."

If (X, Y) has a c.d.f. F, we sometimes write $m(X, Y)$ instead of $m(F)$.

Clearly, for each PDO $\leqslant$, m is not unique, since $\phi(m)$ is an MPD concurring with $\leqslant$ for any increasing function ϕ. Moreover, there are instances (see Example 4.1) where m_1 and m_2 are both MPD's concurring with $\leqslant$, and yet m_1 is not a monotone function of m_2.

In this section, employing the definition of a PDO and the concept of a concurring MPD, we discuss various conditions like Renyi's that might be meaningful to require for a measure of positive association. We illustrate these conditions with various examples and examine some of their inherent difficulties.

Example 4.1. The following are MPD's concurring with $\leqslant_{\mathrm{PQD}}$: Pearson's correlation, Kendall's τ, Spearman's ρ, Blomqvist's q (see Tchen (1976), Corollary IV, 1.b), and Kimeldorf *et al.*'s (1982) CMC (see Schriever (1985), Example 4.2.3). Note that, in general, none of these MPD's are monotone functions of any other.

Example 4.2. Let $m_{\mathrm{W}}(F) = 1$ if $F \in \mathscr{F}_+$, -1 if $F \in \mathscr{F}_-$, and 0, otherwise. Then m_{W} is an MPD concurring with $\geqslant_{\mathrm{W}}$.

Based upon an examination of Renyi's (1959) and Schweizer and Wolff's (1981) conditions, it would seem that the following conditions should be included in any specification of conditions for a measure of positive association.

(4.2) $$-1 \leq m(F) \leq 1 .$$

(4.3) If $Y = \phi(X)$ a.s. for some increasing ϕ, then $m(F) = 1$.

(4.4) If $Y = \phi(X)$ a.s. for some decreasing ϕ, then $m(F) = -1$.

(4.5) $m(F) = 0$ if and only if $F \in \mathscr{I}$.

In the context of conditions (4.2), (4.3) and (4.4), which can be viewed as norming conditions, we argue that condition (4.5) is inappropriate in that a natural continuity condition is violated (see Lemma 4.2).

By property (P4), every MPD m concurring with $\leqslant$ satisfies

(4.6) $$m(F_-) \leq m(F) \leq m(F_+) .$$

Thus, there is no apparent loss of generality if we were to require

(4.7) $$m(F_+) = 1 \quad \text{for all} \quad F_+ \in \mathscr{F}_+$$

and

(4.8) $$m(F_-) = -1 \quad \text{for all} \quad F_- \in \mathscr{F}_- .$$

LEMMA 4.1. *If m is an MPD concurring with $\leqslant$ and satisfying* (4.7) *and* (4.8), *then m satisfies* (4.2), (4.3) *and* (4.4).

PROOF. Obvious.

In addition, we could require the following continuity condition. If $\{F_n\}$ is a sequence of distributions such that $F_n \preccurlyeq F$ or $F \preccurlyeq F_n$ for all n, then

$$F_n \xrightarrow{\mathscr{L}} F \quad \text{implies} \quad m(F_n) \to m(F) . \tag{4.9}$$

Consider a PDO, such as $\succcurlyeq_{\mathrm{PQD}}$, which satisfies the following mixture property

$$F \preccurlyeq G \quad \text{implies} \quad F \preccurlyeq \alpha F + (1 - \alpha)G \preccurlyeq G , \tag{4.10}$$

for $0 \le \alpha \le 1$. For such PDO's we now show that if (4.7) and (4.8) hold, both (4.5) and (4.9) cannot hold.

LEMMA 4.2. *Suppose* $\preccurlyeq$ *is a PDO satisfying* (4.10) *and* m *is an MPD concurring with* $\preccurlyeq$ *and satisfying* (4.7) *and* (4.8). *If* m *satisfies* (4.9), *then* m *cannot satisfy* (4.5).

PROOF. Fix α, $0 \le \alpha \le 1$. Let $F_\alpha = \alpha F_+ + (1 - \alpha)F_-$ and let $\alpha_n \downarrow \alpha$. Then by (4.10), $F_\alpha \preccurlyeq F_{\alpha_n} = (1 - \alpha_n)/(1 - \alpha)F_\alpha + [1 - (1 - \alpha_n)/(1 - \alpha)]F_+ \to F_\alpha$, and by (4.9), $m(F_{\alpha_n}) \downarrow m(F_\alpha)$. Thus, $m(F_\alpha)$ is continuous in α from the right, and a similar argument shows it to be continuous from the left. By (4.7) and (4.8), we also have $m(F_1) = m(F_+) = 1$ and $m(F_0) = m(F_-) = -1$. By the intermediate value theorem, there exists α_0 such that $m(F_{\alpha_0}) = 0$. However, $F_{\alpha_0} \notin \mathscr{I}$.

Renyi (1959) and Schweizer and Wolff (1981) individually require additional properties. Two versions of such properties are

$$m(X, Y) = m(\phi(X), y) , \tag{4.11}$$

for all increasing ϕ, and

$$m(N_\rho) = \rho , \tag{4.12}$$

where N_ρ denotes the standardized bivariate normal distributions with correlation ρ.

The condition (4.12) is related to and in some sense generated by the normal-agreeing property for PDO's (Definition 3.3 of Kimeldorf and Sampson (1987)), namely

$$N(\rho_1) \preccurlyeq N(\rho_2) \quad \text{if and only if} \quad \rho_1 \le \rho_2 .$$

If $\preccurlyeq$ has this property, and m is an MPD concurring with $\preccurlyeq$ and satisfying the continuity condition (4.9), then $m(N_\rho)$ is a continuous, increasing function $\eta(\rho)$ in ρ. Thus $\eta^{-1}(m)$ is an MPD concurring with $\ll$ and satisfying (4.12).

While measures of association, in general, do not fully describe the degree of positive dependence, it can be fully described on subsets of c.d.f.'s which are totally ordered by a PDO. For $\mathscr{N}_+$, ρ fully determines which distributions are more positive dependent than others. Let $\mathscr{P}$ be a collection of distributions for which $F, G \in \mathscr{P}$ implies either $F \ll G$ or $G \ll F$. Then it becomes feasible to require a measure of association to agree totally with $\ll$ on $\mathscr{P}$. In such special cases, one may want to modify the concept of an MPD to require that $F \ll G$ if and only if $m(F) \leq m(G)$. It would then follow that $m(F) = m(G)$ if and only if $F = G$.

We conclude this section by presenting a conceptual approach to generating MPD's that concur with a given PDO. This approach has been considered by Nguyen and Sampson (1987) for contingency tables using the PDQ PDO. Let μ be a positive measure suitably defined on the set of all bivariate c.d.f.'s. Relative to μ, define a measure of association m by

$$m(F) = \mu\{G: G \ll F\}\,. \tag{4.13}$$

Intuitively, m gives an indication of the "size" of the set of all c.d.f.'s less extreme than F. If m is relatively large, F should be "close" to F_+; and if m is relatively small, F should be close to F_-.

THEOREM 4.1. *If $m(F)$ is defined by* (4.13), *then m is an MPD concurring with* $\ll$.

PROOF. Let $F_1 \ll F_2$. Then by property (P2) $\{G: G \ll F_1\} \subseteq \{G: G \ll F_2\}$, so that $m(F_1) = \mu\{G: G \ll F_1\} \leq \mu\{G: G \ll F_2\} = m(F_2)$.

Finally, note that $m(F_+)$ is the measure of all bivariate c.d.f.'s with the same marginals as F_+, and $m(F_-) = \mu\{F_-\}$.

Acknowledgement

The authors wish to thank Professor Dorota Dabrowska for several helpful conversations.

REFERENCES

Agresti, A. (1984). *Analysis of Ordinal Categorical Data*, Wiley, New York.

Ahmed, A., Langberg, N., Léon, R. and Proschan, F. (1979). Partial ordering of positive quadrant dependence with applications, Tech. Report, Florida State University.

Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*, Holt, Rinehart and Winston, New York.

Block, H. W. and Sampson, A. R. (1984). Inequalities on distributions: bivariate and multivariate, *Encyclopedia of Statistical Sciences*, Vol. 4, (eds. N. Johnson and S. Kotz), Wiley, New York.

Block, H. W. and Ting, M. (1981). Some concepts of multivariate dependence, *Comm. Statist. A—Theory Methods*, **10**, 742–762.

Block, H. W., Savits, T. and Shaked, M. (1982). Some concepts of negative dependence, *Ann. Probab.*, **10**, 765–772.

Dabrowska, D. (1981). Regression-based orderings and measures of stochastic dependence, *Math. Operationsforsch. v. Statist., Ser. Statistics*, **12**, 317–325.

Dabrowska, D. (1985). Descriptive parameters of location, dispersion and stochastic dependence, *Statistics*, **1**, 63–88.

Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables with applications, *Ann. Math. Statist.*, **44**, 1466–1474.

Goodman, L. A. and Kruskal, W. H. (1959). Measures of association for cross-classifications, Part II, *J. Amer. Statist. Assoc.*, **54**, 123–163.

Hall, W. J. (1970). On characterizing dependence in joint distributions, *Essays in Probability and Statistics*, 339–376, University of North Carolina Press.

Hollander, M., Proschan, F. and Sconing, J. (1985). Measures of dependence for evaluating information in censored models, Tech. Report No. 85-180, Florida State University.

Höschel, H. P. (1976). Ein axiomatischen Zugang zu Abhängigkeitsmassen für Zufallsuektoren und die Spurkorrelation, *Math. Operationsforsch. v. Statist., Ser. Statistics*, 7, 789–802.

Johnson, N. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions*, Wiley, New York.

Kimeldorf, G. and Sampson, A. (1978). Monotone dependence, *Ann. Statist.*, **6**, 895–903.

Kimeldorf, G. and Sampson, A. (1984). Positive dependence properties and measures of positive dependence, Tech. Report 84-01, Series in Reliability and Statistics, University of Pittsburgh.

Kimeldorf, G. and Sampson, A. (1987). Positive dependence orderings, *Ann. Inst. Statist. Math.*, **39**, 113–128.

Kimeldorf, G., May, J. and Sampson, A. (1982). Concordant and discordant monotone correlations and their evaluations by nonlinear optimization, *Studies in the Management Sciences* (19): *Optimization in Statistics*, (eds. Zanakis and Rustagi), 117–130, North-Holland, Amsterdam.

Kowalczyk, T. and Pleszczynska, E. (1977). Monotone dependence functions of bivariate distributions, *Ann. Statist.*, **5**, 1221–1227.

Krishnaiah, P. R., Rao, M. B. and Subramanyan, K. (1985). Structure theorem on bivariate positively quadrant dependent distributions and tests for independence in two-way contingency tables, Tech Report No. 48, Center for Multivariate Analysis, University of Pittsburgh.

Lehmann, E. (1966). Some concepts of dependence, *Ann. Math. Statist.*, **37**, 1137–1153.

Mardia, K. V. (1970). *Families of Bivariate Distributions*, Hafner Publishing Co., Darien, Connecticut.

Marshall, A. and Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York.

Nguyen, T. T. and Sampson, A. (1982). Extreme points of bivariate discrete distributions with fixed marginals, Tech. Report, Department of Mathematics and Statistics, University of Pittsburgh.

Nguyen, T. T. and Sampson, A. (1987). Testing for positive quadrant dependence in ordinal contingency tables, *Naval Research Logistics Quarterly*, **34**, 859–877.

Pitt, L. D. (1982). Positively correlated normal variables are associated, *Ann. Probab.*, **10**, 469–499.

Renyi, A. (1959). On measures of dependence, *Acta Math. Acad. Sci. Hungar.*, **10**, 441–451.

Rödel, E. (1970). Abhangigkeitsmasse für Zufallsgrossen mit Werten in einem separablem Hilbertraum, *Math. Operationsforsch. v. Statist., Ser. Statistics*, **2**, 117–139.

Scarsini, M. (1984). On measures of concordance, *Stochastica*, **8**, 201–218.

Schriever, B. F. (1985). *Order Dependence*, Doctoral Dissertation, Free University of Amsterdam.

Schweizer, B. and Wolff, E. F. (1981). On nonparametric measures of dependence for random variables, *Ann. Statist.*, **9**, 879–885.

Shaked, M. (1982). A general theory of some positive dependence notions, *J. Multivariate Anal.*, **12**, 199–218.

Slepian, D. (1962). The one-sided barrier problem for Gaussian noise, *Bell System Tech. J.*, **41**, 463–501.

Tchen, A. (1976). Inequalities for distributions with given marginals, Tech. Report No. 19, Department of Statistics, Stanford University.

Tchen, A. (1980). Inequalities for distributions with given marginals, *Ann. Probab.*, **8**, 814–827.

Tong, Y. L. (1980). *Probability Inequalities in Multivariate Distributions*, Academic Press, New York.