Ann. Inst. Statist. Math.
Vol. 41, No. 1, 19-29 (1989)

A NOTE ON UNIFORM ASYMPTOTIC NORMALITY OF INTERMEDIATE ORDER STATISTICS

MICHAEL FALK

Department of Mathematics, University of Siegen, Hölderlinstr. 3, 5900 Siegen 21, West Germany

(Received July 20, 1987; revised April 25, 1988)

Abstract. It is proved that under fairly general von Mises-type conditions on the underlying distribution, the intermediate order statistics, properly standardized, converge uniformly over all Borel sets to the standard normal distribution. This closes the gap between central order statistics and extremes, where uniform convergence under mild conditions is well-known.

Key words and phrases: Intermediate order statistics, extreme order statistics, von Mises-type conditions, asymptotic normality.

1. Introduction

Let $X_1,\ldots,X_n$ be independent and identically distributed random variables ($\equiv$ iid rvs) with common distribution function ($\equiv$ df) F and denote by $X_{1:n},\ldots,X_{n:n}$ the pertaining order statistics.

If the distribution of $c_n^{-1}(X_{n:n}-d_n)$ tends weakly to some nondegenerate limit G for some choice of constants $c_n>0$, $d_n\in\mathbb{R}$, $n\in\mathbb{N}$, then we know from Gnedenko (1943) that this limit G must be one of the following types with $\alpha>0$

$$(1.1)\qquad \begin{aligned} G_{1,\alpha}(x) &:= \begin{cases} 0 & x\le 0\,, \\ \exp(-x^{-\alpha}) & x>0\,, \end{cases} \\ G_{2,\alpha}(x) &:= \begin{cases} \exp(-(-x)^{\alpha}) & x\le 0\,, \\ 1 & x>0\,, \end{cases} \\ G_3(x) &:= \exp(-e^{-x}) \qquad x\in\mathbb{R}\,. \end{aligned}$$

In addition, Gnedenko (1943) gave necessary and sufficient conditions for F to belong to the domain of attraction ($\equiv\mathscr{D}(G)$) of each of the above limits.

Smirnov (1952) extended this result to $c_n^{-1}(X_{n-k+1:n} - d_n)$ with $k \in \mathbb{N}$ fixed (see Theorem 2.8.1 of Galambos (1987). In the case of intermediate order statistics $X_{n-k+1:n}$, where

$$(1.2) \qquad k = k(n) \underset{n\in\mathbb{N}}{\longrightarrow} \infty, \qquad k/n \underset{n\in\mathbb{N}}{\longrightarrow} 0\,,$$

Chibisov ((1964), Theorem 3) proved that for particular sequences k the set of possible limiting distributions of $c_n^{-1}(X_{n-k+1:n} - d_n)$ consists of distributions of one of the following types with $\alpha > 0$

$$(1.3) \qquad \begin{aligned} H_{1,\alpha}(x) &:= \begin{cases} 1 - \Phi(\alpha \log(-x)) & x < 0\,, \\ 1 & x > 0\,, \end{cases} \\ H_{2,\alpha}(x) &:= \begin{cases} 0 & x \le 0\,, \\ 1 - \Phi(-\alpha \log(x)) & x > 0\,, \end{cases} \\ H_3(x) &:= \Phi(x) \qquad\qquad\qquad\qquad x \in \mathbb{R}\,, \end{aligned}$$

where Φ denotes the df of the standard normal distribution $N_{(0,1)}$. Chibisov (1964) also stated necessary and sufficient conditions for F such that the above weak convergence holds.

Balkema and de Haan ((1978*a*) and Theorem 7.1 in (1978*b*)) proved that for particular F (which are dense in the set of df's) $X_{n-k+1:n}$ may have any limiting distribution if it is suitably standardized and if the sequence k is chosen appropriately.

However, as is pointed out in Smirnov (1967), a (nondegenerate) limiting distribution of $X_{n-k+1:n}$ different from the normal one can only occur if k has an exact preassigned asymptotic behaviour. Assuming only (1.2), Smirnov (1967) gave necessary and sufficient conditions for F such that $X_{n-k+1:n}$ is asymptotically standard normal and he specified the appropriate norming constants. For multivariate extensions we refer to Cooil (1985). Suppose now that the underlying distribution function F satisfies the following condition.

(1.4) $F' = f$ exists throughout some left neighborhood of

$$\omega(F) := \sup\{x \in \mathbb{R} \colon F(x) < 1\}\,.$$

There exist sequences $a_n > 0$, $b_n \in \mathbb{R}$ such that $(d/dx)F^n(a_n x + b_n) \underset{n\in\mathbb{N}}{\longrightarrow} G'(x)$ uniformly in x for all finite subintervals in the support of G, where $G \in \{G_1, G_2, G_3\}$.

Under these assumptions on F, Cooil ((1985), Theorem 2.3) established weak convergence of $c_n^{-1}(X_{n-k+1:n} - d_n)$ to $N_{(0,1)}$, where $d_n = F^{-1}(1 - k/n)$

and $c_n = a_{\langle n/m \rangle}/m^{1/2}$, for any sequence k satisfying (1.2). Here $F^{-1}(p) := \inf\{t \in \mathbb{R}: F(t) \geq p\}$, $p \in (0, 1)$, denotes the generalized inverse of F and $\langle x \rangle$ the integral part of $x \in \mathbb{R}$.

Sweeting ((1985), Theorem 1) proved that, if F is ultimately increasing, condition (1.4) is equivalent to the assumption that F satisfies one of the usual von Mises-type conditions stated below. This greatly simplifies the result by Cooil (1985).

Under these von Mises-type conditions on F a much stronger result actually holds. In particular we will prove in the present paper that in this case with $c_n > 0$, $b_n \in \mathbb{R}$,

$$\sup_{B \in \mathscr{B}} |P\{c_n^{-1}(X_{n-k+1:n} - d_n) \in B\} - N_{(0,1)}(B)| \underset{n \in \mathbb{N}}{\longrightarrow} 0 ,$$

where $\mathscr{B}$ denotes the Borel-σ-algebra of $\mathbb{R}$, if and only if

$$\text{(1.5)} \qquad \lim_{n \in \mathbb{N}} c_n / a_n = 1 \qquad \text{and} \qquad \lim_{n \in \mathbb{N}} (d_n - b_n)/a_n = 0 ,$$

where $b_n := F^{-1}(1 - k/n)$, $a_n := k^{1/2}/(nf(b_n))$. Thus, the von Mises-type conditions on F do not only entail asymptotic normality of $X_{n-k+1:n}$ uniformly over all Borel sets for any intermediate sequence k, but they also provide unified and easily wieldable norming constants a_n and b_n.

Respective uniform results for central order statistics were established by Weiss (1969), Ikeda and Matsunawa (1972) and Reiss (1976), whereas the case of extremes was investigated by Pickands (1967), Weiss (1971), Ikeda and Matsunawa (1976), Reiss (1981), de Haan and Resnick (1982), Sweeting (1985) and Falk (1985), among others.

Finally, we mention that the respective results for the lower intermediate order statistics $X_{k:n}$ with k satisfying (1.2), can easily be deduced from our results in the usual way by the equality $X_{k:n} = -Y_{n-k+1:n}$, where $Y_{1:n}, \ldots, Y_{n:n}$ are the order statistics pertaining to the sample $-X_1, \ldots, -X_n$.

2. Main results

First we state the von Mises-type conditions on F which we will deal with in the following. We begin with the case G_3 which occupies the preeminent position.

As mentioned in the introduction, Gnedenko (1943) gave necessary and sufficient conditions for F to belong to $\mathscr{D}(G_3)$ (see Section 2.4 in de Haan (1975) for further details). These conditions are somewhat complex whereas the following sufficient condition, due to von Mises (1936), is often easily wieldable.

Suppose that there exists $x_0 < \omega(F)$ such that F is twice differentiable for all $x \in (x_0, \omega(F))$ with $f'(x) > 0$, where $f = F'$. Then, von Mises' (1936) condition is as follows:

$$\lim_{x \uparrow \omega(F)} (d/dx)[(1 - F(x))/f(x)] = 0 . \tag{2.1}$$

This condition implies the following one which we will deal with in the following (see Theorems 2.7.3 and 2.7.4 in de Haan (1975)). Note that, although a result by Balkema and de Haan (1972) indicates that von Mises' condition (2.1) is already a rather weak one, it is easy to find distributions which satisfy (2.2) below but not (2.1).

Suppose that F has a positive derivative f for all $x \in (x_0, \omega(F))$ such that

$$\lim_{x \uparrow \omega(F)} u(x) = 1 , \tag{2.2}$$

where

$$u(x) := f(x) \int_x^{\omega(F)} (1 - F(t))\,dt/(1 - F(x))^2 , \tag{2.3}$$

$x \in (x_0, \omega(F))$. Then $F \in \mathscr{D}(G_3)$. Moreover, if f is nonincreasing near $\omega(F)$ and $F \in \mathscr{D}(G_3)$, then (2.2) holds (see Theorem 2.7.3 in de Haan (1975)).

Next we turn to the case G_1. Gnedenko (1943) proved that $F \in \mathscr{D}(G_{1,\alpha})$ if and only if $\omega(F) = \infty$ and for any $t > 0$

$$\lim_{x \to \infty} [1 - F(x)]/[1 - F(tx)] = t^\alpha . \tag{2.4}$$

If we assume that F has a positive derivative f near infinity such that for some $\alpha > 0$

$$\lim_{x \to \infty} x f(x)/(1 - F(x)) = \alpha , \tag{2.5}$$

then it is easy to verify that F satisfies (2.4) (see Galambos (1987), p. 102). Furthermore, if f is ultimately nonincreasing and $F \in \mathscr{D}(G_{1,\alpha})$, then (2.5) holds (see Theorem 2.7.1 in de Haan (1975)).

Finally, consider the case G_2. It is proved in Gnedenko (1943) that $F \in \mathscr{D}(G_{2,\alpha})$ if and only if $\omega(F) < \infty$ and for any $t > 0$

$$\lim_{x \downarrow 0} [1 - F(\omega(F) - tx)]/[1 - F(\omega(F) - x)] = t^\alpha . \tag{2.6}$$

For a corrected proof we refer to Theorem 2.3.2 in de Haan (1975).

The following sufficient condition for F to belong to $\mathscr{D}(G_{2,\alpha})$ is stated in Theorem 2.7.2 of de Haan (1975).

Suppose that $\omega(F) < \infty$ and that $F'(x)$ exists for all $x \in (x_0, \omega(F))$ with $f(x) = F'(x) > 0$. If for some $\alpha > 0$

$$\lim_{x \uparrow \omega(F)} (\omega(F) - x) f(x) / (1 - F(x)) = \alpha , \tag{2.7}$$

then $F \in \mathscr{D}(G_{2,\alpha})$. Moreover, if f is non-increasing and $F \in \mathscr{D}(G_{2,\alpha})$, then (2.7) holds. Now we are ready to state our main result.

THEOREM 2.1. *Suppose that F satisfies one of the von Mises-type conditions* (2.2), (2.5) *or* (2.7) *and let $k = k(n) \in \{1, \dots, n\}$, $n \in \mathbb{N}$, satisfy $k \underset{n \in \mathbb{N}}{\longrightarrow} \infty$, $k/n \underset{n \in \mathbb{N}}{\longrightarrow} 0$. Then*

$$\sup_{B \in \mathscr{B}} | P\{c_n^{-1}(X_{n-k+1:n} - d_n) \in B\} - N_{(0,1)}(B) | \underset{n \in \mathbb{N}}{\longrightarrow} 0 ,$$

for any sequences $c_n > 0$, $d_n \in \mathbb{R}$ which satisfy (1.5).

Remarks. (i) It is immediate from the symmetry of the standard normal distribution that the above result also holds for negative c_n with $\lim_{n \in \mathbb{N}} c_n / a_n = -1$.

(ii) Theorem 1 of Smirnov (1967) shows that the distribution of $c_n^{-1}(X_{n-k+1:n} - d_n)$ converges weakly to $N_{(0,1)}$ for some choice of constants $c_n > 0$, $d_n \in \mathbb{R}$, if and only if for any $x \in \mathbb{R}$

$$\lim_{n \in \mathbb{N}} [k + n(F(c_n x + d_n) - 1)] / k^{1/2} = x . \tag{2.8}$$

Consequently, it follows from Theorem 2.1 and Lemma 2.2.3 of Galambos (1987) that if F satisfies one of the von Mises-type conditions (2.2), (2.5) or (2.7), sequences $c_n > 0$, $d_n \in \mathbb{R}$ satisfy (1.5) if and only if they satisfy (2.8).

Examples. (i) The standard normal distribution satisfies (2.1) and hence (2.2). This is immediate from $1 - \Phi(x) \underset{x \to \infty}{\sim} \varphi(x)/x$, where $\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$. Thus, by Theorem 2.1 with $F = \Phi$ and k satisfying (1.2)

$$\sup_{B \in \mathscr{B}} | P\{n k^{-1/2} \varphi(\Phi^{-1}(1 - k/n))(X_{n-k+1:n} - \Phi^{-1}(1 - k/n)) \in B\} - N_{(0,1)}(B) | \underset{n \in \mathbb{N}}{\longrightarrow} 0 .$$

Moreover, choose d_n as the solution of the equation

$$\varphi(d_n)/d_n = k/n , \tag{2.9}$$

and put $c_n = 1/(k^{1/2}d_n)$. Note that these norming constants are completely analogous to the appropriate norming constants for $X_{n:n}$ (see, for example, Hall (1979)). Then it is easy to see that c_n, d_n satisfy (2.8) if

$$\lim_{n\in\mathbb{N}} k^{1/2}/\log(n) = 0 . \tag{2.10}$$

Consequently, if (1.2) and (2.10) are satisfied

$$\sup_{B\in\mathscr{B}} |P\{k^{1/2}d_n(X_{n-k+1:n} - d_n) \in B\} - N_{(0,1)}(B)| \underset{n\in\mathbb{N}}{\longrightarrow} 0 . \tag{2.11}$$

Note that condition (2.10) coincides nearly exactly with the rate of uniform approximation of the joint distribution of the k upper extremes, equally standardized, from a normal distribution by its appropriate limit as established in Falk ((1986), Example 2.33). This indicates that condition (2.10) is essential for the asymptotic normality of $k^{1/2}d_n(X_{n-k+1:n} - d_n)$.

(ii) The standard Exponential distribution with df $F(x) = 1 - \exp(-x)$, $x > 0$, obviously satisfies (2.1). Consequently, we obtain in this case

$$\sup_{B\in\mathscr{B}} |P\{k^{1/2}(X_{n-k+1:n} - \log(n/k)) \in B\} - N_{(0,1)}(B)| \underset{n\in\mathbb{N}}{\longrightarrow} 0 . \tag{2.12}$$

(iii) The standard Pareto distribution with df $F_\alpha(x) = 1 - x^{-\alpha}$, $x \geq 1$, $\alpha > 0$, satisfies (2.5). Thus, if $F = F_\alpha$

$$\sup_{B\in\mathscr{B}} |P\{\alpha n^{-1/\alpha}k^{1/2+1/\alpha}(X_{n-k+1:n} - \alpha(n/k)^{1/\alpha}) \in B\} - N_{(0,1)}(B)| \underset{n\in\mathbb{N}}{\longrightarrow} 0 . \tag{2.13}$$

(iv) The triangular distribution on the interval $(-1, 1)$ with Lebesgue density $f_\Delta(x) = 1 - |x|$, $x \in (-1, 1)$, satisfies (2.7) with $\alpha = 2$. In this case we obtain

$$\sup_{B\in\mathscr{B}} |P\{(2n)^{1/2}(X_{n-k+1:n} - 1 + (2k/n)^{1/2}) \in B\} - N_{(0,1)}(B)| \underset{n\in\mathbb{N}}{\longrightarrow} 0 . \tag{2.14}$$

PROOF OF THEOREM 2.1. Our proof is based on the uniform approximation of the distribution of the k-th largest order statistic from the uniform distribution by the normal distribution and the probability integral transformation theorem (e.g., Proposition 2.10 of Reiss (1981)).

Denote by $U_{1:n}, \ldots, U_{n:n}$ the order statistics pertaining to a sample of independent and uniformly on $(0, 1)$ distributed rvs. The proof of Proposition 2.10 of Reiss (1981) shows that $h_{k,n}(x) \underset{n\in\mathbb{N}}{\longrightarrow} \varphi(x)$, $x \in \mathbb{R}$, where $h_{k,n}$ denotes the Lebesgue density of the df $H_{k,n}$ of $(n/k^{1/2})(U_{n-k+1:n} - (1 - k/n))$.

Let now $X_{n-k+1:n}$ be the k-th largest order statistic in an iid sample of size n generated according to F. We may assume the particular representation $X_{n-k+1:n} = F^{-1}(U_{n-k+1:n})$.

Suppose that for any $x \in \mathbb{R}$

$$a_n x + b_n \xrightarrow[n \in \mathbb{N}]{} \omega(F) , \tag{2.15}$$

from below (denoted by ↑) and

$$f(b_n + \theta a_n x)/f(b_n) \xrightarrow[n \in \mathbb{N}]{} 1 , \tag{2.16}$$

uniformly for $\theta \in (0, 1)$ where a_n, b_n are defined in (1.5) and $f = F'$ (in the upper tail of F). The df $F_{k,n}$ of $(F^{-1}(U_{n-k+1:n}) - b_n)/a_n$ is given by

$$\begin{aligned} F_{k,n}(x) &= P\{U_{n-k+1:n} \leq F(a_n x + b_n)\} \\ &= H_{k,n}\{(n/k^{1/2})(F(a_n x + b_n) - (1 - k/n))\} \\ &= H_{k,n}\{(n/k^{1/2})(F(a_n x + b_n) - F(b_n))\} , \end{aligned}$$

if n is large. By (2.15) we may differentiate $F_{k,n}(x)$, and by using a Taylor expansion and (2.16) we obtain

$$\begin{aligned} F'_{k,n}(x) &= h_{k,n}\{(n/k^{1/2})(F(a_n x + b_n) - F(b_n))\} f(a_n x + b_n)/f(b_n) \\ &= h_{k,n}\{x f(b_n + \theta a_n x)/f(b_n)\} f(a_n x + b_n)/f(b_n) \\ &\xrightarrow[n \in \mathbb{N}]{} \varphi(x) . \end{aligned}$$

Scheffé's lemma now implies the assertion of Theorem 2.1 for the particular choice of constants a_n and b_n. For general $c_n > 0$, $d_n \in \mathbb{R}$ satisfying (1.5), the assertion of Theorem 2.1 follows from the representation

$$\begin{aligned} c_n x + d_n &= a_n x + b_n + a_n x\{(c_n/a_n - 1) + (d_n - b_n)/a_n\} \\ &= a_n x(1 + o(1)) + b_n , \end{aligned}$$

and the above arguments.

Consequently, it remains to show that the von Mises-type conditions (2.2), (2.5) and (2.7) imply (2.15) and (2.16). This will be verified in the following.

(A) First we consider the case that F satisfies (2.2). If $\omega(F) = \infty$, then (2.15) is clearly true whenever $a_n/b_n \xrightarrow[n \in \mathbb{N}]{} 0$. If, however, $\omega(F) < \infty$, then the convergence $a_n/b_n \xrightarrow[n \in \mathbb{N}]{} 0$ is not sufficient to ensure that $a_n x + b_n$ converges to $\omega(F)$ from below. Therefore, we need an extra argument for the case

$\omega(F) < \infty$. As a first step we prove

$$\lim_{n\in\mathbb{N}} a_n/(\omega(F) - b_n) = 0 \quad \text{if} \quad \omega(F) < \infty . \tag{2.17}$$

PROOF. We have

$$\begin{aligned} &a_n/(\omega(F) - b_n) \\ &\quad = k^{1/2}/\{nf(b_n)(\omega(F) - b_n)\} \\ &\quad = (1 - F(b_n))/\{k^{1/2}f(b_n)(\omega(F) - b_n)\} \\ &\quad = \left[\int_{b_n}^{\omega(F)}(1 - F(t))dt/\{k^{1/2}(\omega(F) - b_n)(1 - F(b_n))\}\right] \\ &\qquad \cdot\left[(1 - F(b_n))^2/\{f(b_n)\int_{b_n}^{\omega(F)}(1 - F(t))dt\}\right] \\ &\quad \leq 1/(k^{1/2}u(b_n)) \underset{n\in\mathbb{N}}{\longrightarrow} 0 , \end{aligned}$$

by (2.2).

As a consequence we obtain in the case $\omega(F) < \infty$ from (2.17) and the equality $a_nx + b_n = \omega(F) - (\omega(F) - b_n)\{1 - a_nx/(\omega(F) - b_n)\}$ that for any $x \in \mathbb{R}$

$$a_nx + b_n \underset{n\in\mathbb{N}}{\uparrow} \omega(F) \quad \text{if} \quad \omega(F) < \infty . \tag{2.18}$$

Next we show that (2.18) also holds in the case $\omega(F) = \infty$. This will be immediate from

$$\lim_{n\in\mathbb{N}} a_n/b_n = 0 \quad \text{if} \quad \omega(F) = \infty . \tag{2.19}$$

Note that in the case $\omega(F) < \infty$ this convergence follows from (2.17).

PROOF OF (2.19). Put for $x \in \mathbb{R}$, $U(x) := \int_x^{\omega(F)}(1 - F(t))dt/(1 - F(x))$. Then, by (2.2), $U(x)$ is differentiable if x is large with $U'(x) = u(x) - 1 \underset{x\to\infty}{\longrightarrow} 0$. Now,

$$\begin{aligned} a_n/b_n &= (1 - F(b_n))/(k^{1/2}b_nf(b_n)) \\ &= (1/u(b_n))\,U(b_n)/(k^{1/2}b_n) . \end{aligned}$$

Fix x_1 large. Then, by Taylor's formula

$$\begin{aligned} U(b_n)/(k^{1/2}b_n) &= (U(b_n) - U(x_1))/(k^{1/2}b_n) + U(x_1)/(k^{1/2}b_n) \\ &= U'(x_1 + \theta(b_n - x_1))(b_n - x_1)/(k^{1/2}b_n) \\ &\quad + U(x_1)/(k^{1/2}b_n) \underset{n\in\mathbb{N}}{\longrightarrow} 0\,, \end{aligned}$$

$\theta \in (0, 1)$, which completes the proof of (2.19).

Next we show the validity of (2.16) for arbitrary $\omega(F)$, i.e., for any $x \in \mathbb{R}$

$$\lim_{n\in\mathbb{N}} f(b_n + \theta a_n x)/f(b_n) = 1 \qquad \text{uniformly for} \qquad \theta \in (0, 1)\,.$$

PROOF. Put again $U(x) = \int_x^{\omega(F)} (1 - F(t))\,dt/(1 - F(x))$. Then,

$$\begin{aligned} f(b_n + \theta a_n x)/f(b_n) &= \{u(b_n + \theta a_n x)/u(b_n)\}\{U(b_n)/U(b_n + \theta a_n x)\} \\ &\quad \cdot \{(1 - F(b_n + \theta a_n x))/(1 - F(b_n))\}\,. \end{aligned}$$

Now, by (2.17) and (2.19), $b_n + \theta a_n x$ converges to $\omega(F)$ from below uniformly for $\theta \in (0, 1)$. Thus, by Taylor's formula if n is large

$$\begin{aligned} U(b_n + \theta a_n x)/U(b_n) &= 1 + U'(b_n + \tilde{\theta} a_n x)\theta a_n x/U(b_n) \\ &\underset{n\in\mathbb{N}}{\longrightarrow} 1, \qquad \tilde{\theta} \in (0, 1)\,, \end{aligned}$$

uniformly for $\theta \in (0, 1)$, since $U'(x) \underset{x\uparrow\omega(F)}{\longrightarrow} 0$ and $a_n/U(b_n) = u(b_n)/k^{1/2} \underset{n\in\mathbb{N}}{\longrightarrow} 0$.

Moreover, by Theorem 2.4.3 in Galambos (1987)

$$\begin{aligned} &(1 - F(b_n + \theta a_n x))/(1 - F(b_n)) \\ &\quad = [1 - F(b_n + (\theta x u(b_n)/k^{1/2})U(b_n))]/(1 - F(b_n)) \\ &\quad \underset{n\in\mathbb{N}}{\longrightarrow} 1\,, \end{aligned}$$

uniformly for $\theta \in (0, 1)$. This completes the proof of (2.16).

(B) Next we consider the case that F satisfies (2.5). First note that in this case

$$\lim_{n\in\mathbb{N}} k^{1/2} a_n/b_n = 1/\alpha\,, \tag{2.20}$$

i.e., we have in particular that $\lim_{n\in\mathbb{N}} a_n/b_n = 0$.

Moreover, (2.4) and (2.5) imply

$$\lim_{n\in\mathbb{N}} f(c_n)/f(d_n) = 1 \,, \tag{2.21}$$

if $c_n, d_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$, $c_n/d_n \underset{n\in\mathbb{N}}{\longrightarrow} 1$. (2.15) and (2.16) now follow from (2.20) and (2.21).

(C) Finally, suppose that F satisfies (2.7). Recall that in this case in particular $\omega(F) < \infty$. Moreover, (2.7) immediately implies

$$\lim_{n\in\mathbb{N}} a_n/(\omega(F) - b_n) = 0 \,. \tag{2.22}$$

Consequently, we obtain from the equality $a_n x + b_n = \omega(F) - (\omega(F) - b_n) \cdot\{1 - a_n x/(\omega(F) - b_n)\}$, $x \in \mathbb{R}$, that

$$a_n x + b_n \underset{n\in\mathbb{N}}{\uparrow} \omega(F) \,.$$

It remains to show (2.16).

Define the df F^* by $F^*(x) := F(\omega(F) - x^{-1})$, $x > 0$. Then F^* satisfies the von Mises-type condition (2.5) with $f^*(t) := (F^*)'(t) = f(\omega(F) - t^{-1})t^{-2}$ for t large enough. Thus, for any $x \in \mathbb{R}$

$$\begin{aligned} f(b_n + \theta a_n x)/f(b_n) = \{&f^*(\omega(F) - b_n - \theta a_n x)/f^*(\omega(F) - b_n)\} \\ &\cdot \{(\omega(F) - b_n)/(\omega(F) - b_n - \theta a_n x)\}^2 \,. \end{aligned}$$

(2.16) now follows from (2.21) and (2.22). This completes the case that F satisfies (2.7).

Acknowledgements

The author is grateful to R.-D. Reiss and the referee for valuable suggestions.

REFERENCES

Balkema, A. A. and de Haan, L. (1972). On R. von Mises' condition for the domain of attraction of exp $(-e^{-x})$, *Ann. Math. Statist.*, **43**, 1352–1354.

Balkema, A. A. and de Haan, L. (1978*a*). Limit distributions for order statistics I, *Theory Probab. Appl.*, **23**, 77–92.

Balkema, A. A. and de Haan, L. (1978*b*). Limit distributions for order statistics II, *Theory Probab. Appl.*, **23**, 341–358.

Chibisov, D. M. (1964). On limit distributions for order statistics, *Theory Probab. Appl.*, **9**, 142–148.

Cooil, B. (1985). Limiting multivariate distributions of intermediate order statistics, *Ann. Probab.*, **13**, 469–477.

de Haan, L. (1975). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*, 3rd ed., Mathematical Centre Tracts, Vol. 32, Amsterdam.

de Haan, L. and Resnick, S. I. (1982). Local limit theorems for sample extremes, *Ann. Probab.*, **10**, 396–413.

Falk, M. (1985). *Uniform Convergence of Extreme Order Statistics*, Habilitationsschrift, University of Siegen.

Falk, M. (1986). Rates of uniform convergence of extreme order statistics, *Ann. Inst. Statist. Math.*, **38**, 245–262.

Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*, 2nd ed., Krieger, Melbourne, Florida.

Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une série aléatoire, *Ann. Math.*, **44**, 423–453.

Hall, P. (1979). On the rate of convergence of normal extremes, *J. Appl. Probab.*, **16**, 433–439.

Ikeda, S. and Matsunawa, T. (1972). On the uniform asymptotic joint normality of sample quantiles, *Ann. Inst. Statist. Math.*, **24**, 33–52.

Ikeda, S. and Matsunawa, T. (1976). Uniform asymptotic distribution of extremes, *Essays in Probability and Statistics*, (eds. S. Ikeda *et al.*), 419–432, Shinko Tsusho, Tokyo.

Pickands, J. III (1967). Sample sequences of maxima, *Ann. Math. Statist.*, **38**, 1570–1574.

Reiss, R.-D. (1976). Asymptotic expansions for sample quantiles, *Ann. Probab.*, **4**, 249–258.

Reiss, R.-D. (1981). Uniform approximation to distributions of extreme order statistics, *Adv. in Appl. Probab.*, **13**, 533–547.

Smirnov, N. V. (1952). Limit distributions for the terms of a variational series, *Amer. Math. Soc. Transl.*, **67**, 82–143.

Smirnov, N. V. (1967). Some remarks on limit laws for order statistics, *Theory Probab. Appl.*, **12**, 337–339.

Sweeting, T. J. (1985). On domains of uniform local attraction in extreme value theory, *Ann. Probab.*, **13**, 196–205.

von Mises, R. (1936). La distribution de la plus grande de *n* valeurs, reprinted in Selected Papers II, *Amer. Math. Soc.*, Providence, Rhode Island, 1954, 271–294.

Weiss, L. (1969). The asymptotic joint distribution of an increasing number of sample quantiles, *Ann. Inst. Statist. Math.*, **21**, 257–263.

Weiss, L. (1971). Asymptotic inference about a density function at an end of its range, *Naval Res. Logist. Quart.*, **18**, 111–114.