Ann. Inst. Statist. Math.
Vol. 41, No. 1, 9–12 (1989)

AN APPLICATION OF THE CONVOLUTION INEQUALITY FOR THE FISHER INFORMATION

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(Received April 23, 1987; revised August 30, 1988)

Abstract. A characterization of the normal distribution by a statistical independence on a linear transformation of two mutually independent random variables is proved by using the convolution inequality for the Fisher information.

Key words and phrases: The Fisher information, convolution inequality, statistical independence, characterization of the normal distribution.

1. Introduction

Normal distribution is characterized by an independence of two variables obtained by a linear transformation of two mutually independent random variables. The theorem has a long history going back to Maxwell's investigation, and has been studied by M. Kac, S. Bernstein, and others, as mentioned in the book by Feller (1971). Itoh (1970) gave a proof by using Linnik's (1959) information assuming a stronger condition. Murata and Tanaka (1974) gave a proof using another functional, which is not based on information. Here we give another proof using a convolution inequality for the Fisher information discussed by Stam (1959), Blachman (1965), Brown (1982), Barron (1986) and other authors, which makes the proof clearer than the previous one by Itoh (1970).

The Fisher information is defined as

$$I(Y) = E(g'(Y)/g(Y))^2\,, \tag{1.1}$$

for a random variable Y with density $g(y)$. Obviously,

$$I(Y) = \int \frac{1}{g(y)}\,(g'(y))^2 dy\,. \tag{1.2}$$

Anytime we use this quantity, it will be understood that g satisfies the conditions

(i) $g > 0$ for $-\infty < x < \infty$,

(ii) g' exists,

(iii) the integral (1.2) exists, i.e., $g' \to 0$ rapidly enough for $x \to \pm\infty$,

as given by Stam (1959).

The Fisher information is translation invariant

$$I(Y + c) = I(Y)\,, \tag{1.3}$$

and is not scale invariant

$$I(cY) = I(Y)/c^2\,. \tag{1.4}$$

Let $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$. Then the Fisher information satisfies

$$I(Y_1 + Y_2) \le \alpha_1^2 I(Y_1) + \alpha_2^2 I(Y_2) \tag{1.5}$$

with equality only if Y_1 and Y_2 are independent normal random variables, as given by Stam (1959). By replacing Y_i with $\sqrt{\alpha_i}\, Y_i$ the following result is obtained from properties (1.3), (1.4) and (1.5). If $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, then the Fisher information satisfies

$$I(\sqrt{\alpha_1}\, Y_1 + \sqrt{\alpha_2}\, Y_2) \le \alpha_1 I(Y_1) + \alpha_2 I(Y_2), \tag{1.6}$$

with equality if and only if Y_1 and Y_2 are independent normal random variables with the same variance, as mentioned by Barron (1984).

2. A proof by the convolution inequality

We give a proof of the following well-known theorem, which is given in the book by Feller (1971) in a slightly different form.

THEOREM 2.1. *Suppose that the random variables X_1 and X_2 are independent of each other and that the same is true of the pair Y_1, Y_2, where α is not a multiple of $\pi/2$, then all four variables are normal*

$$Y_1 = X_1 \cos\alpha + X_2 \sin\alpha\,,$$

$$Y_2 = -X_1 \sin\alpha + X_2 \cos\alpha\,.$$

PROOF. First suppose that the distributions for X_1 and X_2 have densities which satisfy (i), (ii) and (iii). Making use of equations (1.3), (1.4) and inequality (1.5), we have

$$I(X_1 \cos \alpha + X_2 \sin \alpha) \le I(X_1) \cos^2\alpha + I(X_2) \sin^2\alpha \,,$$
$$I(-X_1 \sin \alpha + X_2 \cos \alpha) \le I(X_1) \sin^2\alpha + I(X_2) \cos^2\alpha \,.$$

Hence, we have

$$I(Y_1) + I(Y_2) \le I(X_1) + I(X_2) \,.$$

Since

$$X_1 = Y_1 \cos \alpha - Y_2 \sin \alpha \,,$$
$$X_2 = Y_1 \sin \alpha + Y_2 \cos \alpha \,,$$

we have

$$I(X_1) + I(X_2) \le I(Y_1) + I(Y_2) \,.$$

Hence,

$$I(X_1) + I(X_2) = I(Y_1) + I(Y_2) \,,$$

which implies that the four variables X_1, X_2, Y_1 and Y_2 are normal. To avoid the smoothness conditions imposed on X_1 and X_2, let Z_1 and Z_2 be independent standard normal random variables and set $X_1' = X_1 + \beta Z_1$ and $X_2' = X_2 + \beta Z_2$. The independence property assumed for the linear transformation of (X_1, X_2) continues to hold for the transformation of (X_1', X_2'), so by the above proof X_1' and X_2' are independent normal random variables. Taking the limit in distribution as $\beta \to 0$, it is seen that X_1 and X_2 must also be normal.

Remark. The convolution inequalities for the Shannon entropy and the Fisher information have been used in Brown (1982) and Barron (1984, 1986) to prove the central limit theorem. The inequality for the Shannon entropy will also give a proof for our theorem which is simpler than the previous one by Itoh (1970).

Acknowledgements

The author is grateful to Professor T. Cover for discussion on information-theoretic proofs. He is also grateful to the referees for helpful comments.

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