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ON THE E- AND MV-OPTIMALITY OF BLOCK DESIGNS HAVING $k \geq v$

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Abstract. In this paper we consider the problem of determining and constructing E- and MV-optimal block designs to use in experimental settings where v treatments are applied to experimental units occurring in b blocks of size k, $k \geq v$. It is shown that some of the well-known methods for constructing E- and MV-optimal unequally replicated designs having $v \geq k$ fail to yield optimal designs in the case where $v < k$. Some sufficient conditions are derived for the E- and MV-optimality of block designs having $v < k$ and methods for constructing designs satisfying these sufficient conditions are given.

Key words and phrases: Incidence matrix, C-matrix, eigenvalue, E-optimality, MV-optimality.

1. Introduction

Let d denote a block design having v treatments arranged in b blocks of size k. The incidence matrix of d, denoted by N_d, is a $v \times b$ matrix whose entries n_{dij} give the number of times treatment i occurs in block j. When $k = v\alpha + t$ ($\alpha \geq 0$, $0 \leq t \leq v-1$, both integers), then a design d is said to be binary if $n_{dij} = \alpha$ or $\alpha + 1$. The i-th row sum of N_d is denoted by r_{di} and the matrix $N_d N_d'$ (N_d' denotes the transpose of N_d) is called the concurrence matrix of d and its entries are denoted by λ_{dij}. If d has $N_d N_d'$ with all of its diagonal elements equal to one value and all of its off-diagonal elements equal to another value, then d is called a balanced block design (BBD). A (binary) BBD with $k < v$ is a (binary) balanced incomplete block design (BIBD).

The model assumed here for analyzing the data from a given design d is the two-way additive model. This model specifies that an observation Y_{ijm} obtained after applying treatment i to an experimental unit in block j can be expressed as

$$(1.1)\quad Y_{ijm} = \alpha_i + \beta_j + E_{ijm}\,, \quad 1 \leq i \leq v\,, \quad 1 \leq j \leq b\,, \quad 0 \leq m \leq n_{dij}\,,$$

where α_i=the effect of treatment i, β_j=the effect of block j, and E_{ijm} is a random variable having variance σ^2 and expectation zero. All observations are assumed to be uncorrelated. Under this model and using T_d and B_d to denote vectors of treatment totals and block totals, the reduced normal equations for the least squares estimates of the treatment effects can be written as

$$C_d\hat{\alpha} = T_d - (1/k)N_dB_d \,, \tag{1.2}$$

where

$$C_d = \operatorname{diag}(r_{d1},\ldots, r_{dv}) - (1/k)N_dN_d' \,, \tag{1.3}$$

$\hat{\alpha}=(\hat{\alpha}_1,\ldots, \hat{\alpha}_v)'$ is any solution to equations (1.2) and diag $(r_{d1},\ldots, r_{dv})$ denotes a $v\times v$ diagonal matrix. The matrix C_d is called the C-matrix of d and is known to be positive semi-definite with zero row sums.

In this paper we shall only be considering designs which are connected, i.e., designs for which all linear combinations $\sum_{i=1}^{v} l_i\alpha_i$ of the treatment effects having $\sum_{i=1}^{v} l_i=0$ (called treatment contrasts) are estimable under model (1.1). Such designs have C-matrices of rank $(v-1)$. We shall use $D(v, b, k)$ to denote the class of all connected block designs having v treatments arranged in b blocks of size k and $M(v, b, k)$ to denote the subclass of designs in $D(v, b, k)$ whose C-matrices have maximal trace.

The primary purpose of this paper is to consider the determination and construction of optimal block designs in classes $D(v, b, k)$ where $v<k$. The optimality criteria considered here for selecting an optimal design in $D(v, b, k)$ are the E- and MV-optimality criteria.

DEFINITION 1.1. For $d \in D(v, b, k)$, let $0=z_{d0}<z_{d1}\leq\cdots\leq z_{d,v-1}$ denote the eigenvalues of C_d. Then $d^* \in D(v, b, k)$ is said to be E-optimal if for any other $d \in D(v, b, k)$,

$$z_{d^*1} \geq z_{d1} \,.$$

DEFINITION 1.2. A design $d^* \in D(v, b, k)$ is said to be MV-optimal if for any other $d \in D(v, b, k)$,

$$\max_{i\neq j} \operatorname{Var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) \leq \max_{i\neq j} \operatorname{Var}_d(\hat{\alpha}_i - \hat{\alpha}_j) \,,$$

where $\operatorname{Var}_d(\hat{\alpha}_i-\hat{\alpha}_j)$ denotes the variance of the least squares estimate $\hat{\alpha}_i-\hat{\alpha}_j$ of $\alpha_i-\alpha_j$ derived under d.

A number of results are known concerning the E- and MV-optimality of block designs in classes $D(v, b, k)$ where $v\geq k$, e.g., see Takeuchi (1961), Kiefer

(1975), Cheng (1980), Jacroux (1980, 1982, 1983*a*, 1983*b*) and Constantine (1981, 1982). All of the above results concerning the E- and MV-optimality of designs in classes $D(v, b, k)$ where $k \leq v$ and bk/v is an integer have straightforward extensions to classes $D(v, b, k)$ where $k \geq v$ and bk/v is an integer. For example, it follows from the results of Kiefer (1975) that if $d^* \in M(v, b, k)$ is a BBD, then d^* is both E- and MV-optimal in $D(v, b, k)$. However, a number of results which have been proved concerning the E- and MV-optimality of block designs in classes $D(v, b, k)$ where $k \leq v$ and bk/v is not an integer do not necessarily have straightforward extensions to classes $D(v, b, k)$ where $k \geq v$ and bk/v is not an integer. For example, it is shown in Jacroux (1980, 1983*b*) and Constantine (1981) that by adding blocks to and deleting blocks from various BIBD's, new E- and MV-optimal block designs having treatments unequally replicated can be obtained. It is shown by an example in Section 2 that such methods of constructing E- and MV-optimal block designs from BIBD's cannot necessarily be extended in a natural way to BBD's in $M(v, b, k)$ and classes $D(v, b, k)$ where $k > v$. In this paper, we derive several sufficient conditions for designs to be E- and MV-optimal in classes $D(v, b, k)$ where $k > v$ and characterize the C-matrices of certain designs which satisfy the sufficient conditions given. Examples are also given to illustrate usage of the results obtained.

2. Main results

In this section we give our main results. We begin by giving some notation which is used throughout the sequel. For a given class of designs $D(v, b, k)$ and using $[x]$ to denote the greatest integer not exceeding some real number $x \geq 0$, we shall let

$$
\begin{array}{lll}
 & r = [bk/v] , & \\
 & bk = vr + p , & 0 \leq p \leq v-1 , \\
 & \alpha = [r/b] = [k/v] , & \\
(2.1) & r = b\alpha + s , & 0 \leq s \leq b-1 , \\
 & k = v\alpha + t , & 0 \leq t \leq v-1 , \\
 & \lambda = [(rk - s(\alpha+1)^2 - (b-s)\alpha^2)/(v-1)] , & \\
 & rk - s(\alpha+1)^2 - (b-s)\alpha^2 = (v-1)\lambda + q , & 0 \leq q \leq v-2 .
\end{array}
$$

In terms of this notation, we note that

$$
(2.2) \qquad M(v, b, k) = \{d \in D(v, b, k) \mid n_{dij} = \alpha \text{ or } \alpha + 1\} .
$$

We now present a lemma which can be used to bound the smallest nonzero eigenvalue of C_d.

LEMMA 2.1. *Suppose $d \in D(v, b, k)$ and let M denote a set containing*

m, $2 \leq m \leq v-1$, subscripts corresponding to treatments in d. Then the following statements concerning z_{d1} can be made.

(i) $z_{d1} \leq v c_{dii}/(v-1)$ *for* $i = 1, \ldots, v$.

(ii) $z_{d1} \leq (1/m) \sum_{i \in M} c_{dii} - (2/m(m-1)) \sum_{i \in M} \sum_{\substack{j \in M \\ j \neq i}} c_{dij}$.

(iii) $z_{d1} \leq (1/m(v-m)) \left(\sum_{i \in M} c_{dii} + \sum_{i \in M} \sum_{\substack{j \in M \\ j \neq i}} c_{dij} \right)$.

Comment. Part (i) of Lemma 2.1 is from Kiefer (1958), Chakrabarti (1963) and Whittinghill (1984). Part (ii) is from Whittinghill (1984). Part (iii) is from Jacroux (1983*a*). Constantine (1981) presented (i) and (ii) for equal k only.

The following lemma, from Jacroux (1983*b*), bounds the smallest variance of an elementary treatment difference.

LEMMA 2.2. *Let $d \in D(v, b, k)$ be arbitrary. Then for any i and j, $i \neq j$,*

$$\mathrm{Var}_d(\hat{\alpha}_i - \hat{\alpha}_j) \geq 4/(c_{dii} + c_{djj} - 2c_{dij}) .$$

Using Lemma 2.1, one can prove the following theorem which is analogous to Theorem 2.2 of Jacroux (1983*a*).

THEOREM 2.1. *Let $D(v, b, k)$ be such that $v \leq (v-p)(v-q)$. If $d^* \in D(v, b, k)$ is any design such that*

$$z_{d^*1} = \{rk - s(\alpha+1)^2 - (b-s)\alpha^2 + \lambda\}/k ,$$

then d^* is E-optimal in $D(v, b, k)$.

Using Theorem 2.1, we obtain the following corollary which can be proved using arguments similar to those used in Constantine (1981) and Jacroux (1982).

COROLLARY 2.1. *Let $\bar{d} \in M(v, b, k)$ be a BBD. If we add w blocks of size k to $\bar{d}$ such that $0 < wt < v$ and each treatment appears in each of the w blocks at least α times, then the new design $\hat{d}$ is E-optimal if*

(i) $wt = v-1$ *(for any $\alpha \geq 0$),*

(ii) $wt < v-1$, $wt\alpha < v-1$, *and* $v \leq (v-wt)(v-wt\alpha)$.

If we remove w blocks of size k from $\bar{d}$ such that the extra-replicated treatments from each removed block form disjoint sets, then the new design $\hat{d}$ is E-optimal if

(iii) $\alpha = 0$ *and* $v/k^2 \leq w \leq v/k$,

(iv) $\alpha > 0$, $[(wt-1)\alpha + t - 1]$ *is not divisible by* $v-1$, *and* $v \leq wt[(wt-v)\alpha + t]$.

Comment. Part (i) was proved another way by Whittinghill (1984). Part (ii) was proved for $\alpha=0$ by Constantine (1981) and Jacroux (1982) for $wk<v$.

From Corollary 2.1 we see that under certain conditions new E-optimal block designs having treatments unequally replicated can be obtained by adding and deleting blocks from various BBD's. However, as the next two examples illustrate, such methods of construction cannot generally be applied to arbitrary BBD's in $M(v, b, k)$.

Example 2.1. Consider the class of designs $D(7, 7, k)$ where $k=7\alpha+3$ and the class of BBD's given by

$$N_{\bar{d}} = \begin{bmatrix} \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha & \alpha \\ \alpha+1 & \alpha & \alpha & \alpha & \alpha & \alpha+1 & \alpha+1 \\ \alpha & \alpha+1 & \alpha & \alpha+1 & \alpha & \alpha+1 & \alpha \\ \alpha & \alpha+1 & \alpha & \alpha & \alpha+1 & \alpha & \alpha+1 \\ \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha+1 \\ \alpha & \alpha & \alpha+1 & \alpha & \alpha+1 & \alpha+1 & \alpha \end{bmatrix}.$$

By Corollary 2.1 (ii) it follows that when $\alpha=0$ or 1, a design obtained by adding one or two blocks containing all treatments at least α times will be E-optimal in the appropriate class $D(7, b, k)$ where $b=8$ or 9 and $k=3$ or 10. However, when $\alpha \geq 2$, a design obtained by adding a single block containing all treatments at least α times will not in general be E-optimal in $D(7, 8, k)$. For example, when $\alpha=2$, a design d obtained by adding a single block containing all treatments at least twice will have $z_{d1}=321/17$. But the design given by

$$N_d = \begin{bmatrix} 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 3 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \end{bmatrix},$$

has $z_{d1}=322/17$ and is E-optimal by Theorem 2.1. Similarly, it can be shown that for any $\alpha \geq 2$, no design obtained by adding a single block of any kind to a BBD of the type given by $N_{\bar{d}}$ will be E-optimal in $D(7, 8, k)$.

Example 2.2. Consider the class of designs $D(7, 7, k)$ where $k=7\alpha+4$ and the class of BBD's given by

$$N_{\bar{d}} = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \\ \alpha & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha+1 & \alpha+1 \\ \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha \\ \alpha+1 & \alpha & \alpha+1 & \alpha & \alpha+1 & \alpha & \alpha+1 \\ \alpha+1 & \alpha & \alpha+1 & \alpha+1 & \alpha & \alpha+1 & \alpha \\ \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha \\ \alpha+1 & \alpha+1 & \alpha & \alpha+1 & \alpha & \alpha & \alpha+1 \end{bmatrix}.$$

By Corollary 2.1, when $\alpha=0$, a design obtained by dropping a single block from $N_{\bar{d}}$ will be E-optimal in $D(7, 6, 4)$. However, when $\alpha \geq 1$, a design obtained by dropping a single block from $N_{\bar{d}}$ may not be E-optimal in $D(7, 6, k)$. For example, when $\alpha=1$, the design d^* obtained from $N_{\bar{d}}$ by dropping block one has $z_{d^*1}=97/11$. However, the design given by

$$N_d = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 \end{bmatrix},$$

has $z_{d1}=98/11$ and is E-optimal in $D(7, 6, 11)$ by Theorem 2.1. Similarly, it can be shown that for any $\alpha \geq 1$, no design obtained by dropping a single block from a BBD of the type given above by $N_{\bar{d}}$ will be E-optimal in $D(7, 6, k)$.

As can be seen from Examples 2.1 and 2.2, in certain cases, one can obtain additional E-optimal designs having treatments unequally replicated by adding and deleting blocks from various BBD's. However, when $\bar{d} \in M(v, b, k)$ is a BBD, one cannot necessarily produce additional E-optimal designs having treatments unequally replicated by simply adding blocks to or deleting blocks from $\bar{d}$. Thus methods of constructing E-optimal designs which can be applied in situations where $v \geq k$ do not, in general, have natural extensions to situations where $v<k$. For the remainder of this paper, we consider the determination and construction of optimal designs in classes $D(v, b, k)$ where $v<k$. We begin by proving a lemma.

LEMMA 2.3. *Let $d \in D(v, b, k)$ and let z_{d1} be the minimum nonzero eigenvalue of C_d. Then for $i \neq j$ we have*

$$z_{d1} \leq (r_{di} + r_{dj})/2 ,$$

with the inequality being equality if and only if $n_{dix}=n_{djx}$ for $x=1,\ldots, b$ (which in turn implies $r_{di}=r_{dj}$).

PROOF. By Lemma 2.1 (ii), for any $i \neq j$,

$$\begin{aligned} z_{d1} &\leq (c_{dii} + c_{djj} - 2c_{dij})/2 \\ &= (r_{di} + r_{dj})/2 - (1/2k) \sum_{x=1}^{b} (n_{dix} - n_{djx})^2 \leq (r_{di} + r_{dj})/2 \,, \end{aligned}$$

with the last inequality clearly being equality if and only if $n_{dix} = n_{djx}$ for $x = 1, \ldots, b$ and $r_{di} = r_{dj}$.

In various classes $D(v, b, k)$ where $v < k$, it is possible to establish the E-optimality of a given design using Theorem 2.1. This is illustrated in Examples 2.1 and 2.2. However, as the next example illustrates, in certain classes $D(v, b, k)$, it is physically impossible to find a design satisfying the conditions of Theorem 2.1.

Example 2.3. Consider the class of designs $D(7, 8, 39)$. For this class of designs, $r=44$, $\alpha=5$, $p=4$, $\lambda=245$, $q=2$, $s=4$ and $t=4$. Since $p=4$, it follows that for any $d \in D(7, 8, 39)$, there must exist at least two treatments i and j such that $r_{di} + r_{dj} \leq 2r$. Thus it follows from Lemma 2.3 that $z_{d1} \leq (r_{di} + r_{dj})/2 \leq r = 44$. But we also have that

$$(rk - s(\alpha + 1)^2 - (b - s)\alpha^2 + \lambda)/k = 1717/39 > r \,.$$

Hence we see that it would be impossible to find a design satisfying the conditions of Theorem 2.1.

Comment. One could also show that it is impossible to find a design satisfying the conditions of Theorem 2.1 by using Lemma 3.4 of Kiefer (1958) to show that $z_{d1} \leq 45$.

With Example 2.3 in mind, we give the following sufficient conditions for a design to be E-optimal in $D(v, b, k)$.

THEOREM 2.2. *Let $D(v, b, k)$ be such that $p \leq v-2$. If $d^* \in D(v, b, k)$ has $z_{d^*1} = r$, then d^* is E-optimal in $D(v, b, k)$.*

PROOF. Since $p \leq v-2$, it follows that for any design $d \in D(v, b, k)$, there must exist at least two treatments i and j such that $r_{di} + r_{dj} \leq 2r$. It then follows from Lemma 2.3 that d^* is E-optimal in $D(v, b, k)$.

While Theorem 2.2 provides a sufficient condition for a design to be E-optimal in $D(v, b, k)$, it does not indicate how to construct designs satisfying the given conditions. In the next theorem, we characterize one class of C-matrices corresponding to designs which satisfy the conditions of Theorem 2.2.

COROLLARY 2.2. *Let $D(v, b, k)$ be such that $p \leq v-2$ and suppose d^* has its treatments labeled and replicated so that $r_{d^*1} = \cdots = r_{d^*, v-\bar{p}} = r$, $r_{d^*, v-\bar{p}+1} = \cdots = r_{d^*v} = r+z$, where $\bar{p} \leq p$ and $z \geq 1$ is an integer. Further suppose that kC_{d^*} can be written as*

$$kC_{d^*} = \begin{bmatrix} rkI_{v-\bar{p}} - \lambda_{11}J_{v-\bar{p},v-\bar{p}} & -\lambda_{12}J_{v-\bar{p},\bar{p}} \\ -\lambda_{12}J_{\bar{p},v-\bar{p}} & kC_{d^*22} \end{bmatrix}, \tag{2.3}$$

where $\lambda_{11} = s(\alpha+1)^2 + (b-s)\alpha^2$, $\lambda_{12} = \{rk - (v-\bar{p})\lambda_{11}\}/\bar{p} \geq \lambda_{11}$, C_{d^*22} denotes the lower right hand $\bar{p} \times \bar{p}$ submatrix of C_{d^*}, and $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$ is positive semi-definite. Then d^* has $z_{d^*1} = r$ and d^* is E-optimal in $D(v, b, k)$.

PROOF. From the form of C_{d^*}, it follows that C_{d^*} has eigenvalues 0 of multiplicity one, r of multiplicity $v-\bar{p}-1$, $v\lambda_{12}/k$ of multiplicity one, and that the remaining $\bar{p}-1$ nonzero eigenvalues of C_{d^*} are the same as the $\bar{p}-1$ largest eigenvalues of C_{d^*22}. The fact that $\lambda_{12} \geq \lambda_{11} = s(\alpha+1)^2 + (b-s)\alpha^2$ guarantees that $v\lambda_{12}/k \geq r$ and the fact that $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$ is positive semi-definite guarantees that the $\bar{p}-1$ nonzero eigenvalues of C_{d^*} corresponding to the $\bar{p}-1$ largest eigenvalues of C_{d^*22} are all at least as large as r.

COROLLARY 2.3. *Let $D(v, b, k)$ be such that $p \leq v-2$ and suppose $d^* \in D(v, b, k)$ has kC_{d^*} which can be written in the same form as (2.3) with $\lambda_{11} = s(\alpha+1)^2 + (b-s)\alpha^2$ and has $\lambda_{d^*ij} \geq \lambda_{12}$ for all $i, j = v-\bar{p}+1, \ldots, v$, $i \neq j$. Also suppose in (2.3) that $\lambda_{12} \geq \lambda_{11}$. Then $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$ is positive semi-definite and d^* is E-optimal in $D(v, b, k)$.*

PROOF. Assume kC_{d^*} can be written as in (2.3) and consider

$$\begin{aligned} T &= kC_{d^*} - rkI_v + \lambda_{12}J_{v,v} \\ &= \begin{bmatrix} (\lambda_{12} - \lambda_{11})J_{v-\bar{p},v-\bar{p}} & 0 \\ 0 & kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}} \end{bmatrix}. \end{aligned}$$

It then follows from the form of T and the fact that $C_{d^*}J_{v,1} = 0$ that

$$(kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}})J_{\bar{p},1} = (v\lambda_{12} - rk)J_{\bar{p},1},$$

and that $(v\lambda_{12} - rk) \geq 0$ is an eigenvalue of $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$. But we also see that $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$ has constant row sums $v\lambda_{12} - rk \geq 0$ with all of its off-diagonal entries non-positive (since by assumption $\lambda_{d^*ij} \geq \lambda_{12}$ for all i, $j = v-\bar{p}+1, \ldots, v$, $i \neq j$). Thus each diagonal element of $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$ is nonnegative and it is at least as large in magnitude as the sum of the absolute values of the off-diagonal elements in the corresponding row. Hence $kC_{d^*22} - rkI_{\bar{p}} + \lambda_{12}J_{\bar{p},\bar{p}}$ is positive semi-definite, and the result follows from

Corollary 2.2.

COROLLARY 2.4. *Let $D(v, b, k)$ be such that $p \leq v-2$ and suppose $d^* \in D(v, b, k)$ has*

$$kC_{d^*} = \begin{bmatrix} rkI_{v-\bar{p}} - \lambda_{11}J_{v-\bar{p},v-\bar{p}} & -\lambda_{12}J_{v-\bar{p},\bar{p}} \\ -\lambda_{12}J_{\bar{p},v-\bar{p}} & ((r+z)k - \bar{\lambda}_{22} + \lambda_{22})I_{\bar{p}} - \lambda_{22}J_{\bar{p},\bar{p}} \end{bmatrix},$$

where λ_{11} and λ_{12} satisfy the conditions of Corollary 2.2, $z \geq 1$ is an integer, and $\bar{\lambda}_{22}$, λ_{22} are such that $\bar{\lambda}_{22} = \lambda_{d,v-\bar{p}+1,v-\bar{p}+1} = \cdots = \lambda_{d,v,v}$ and $zk \geq \bar{\lambda}_{22} - \lambda_{22}$. Then d^* has $z_{d^*1} = r$ and is E-optimal in $D(v, b, k)$.

PROOF. Let $kC_{d^*22} = ((r+z)k - \bar{\lambda}_{22} + \lambda_{22})I_{\bar{p}} - \lambda_{22}J_{\bar{p},\bar{p}}$ in Corollary 2.2 and observe that the $\bar{p}-1$ largest eigenvalues of kC_{d^*22} are all equal to $(r+z)k - \bar{\lambda}_{22} + \lambda_{22}$. The condition that $zk \geq \bar{\lambda}_{22} - \lambda_{22}$ then insures that $((r+z)k - \bar{\lambda}_{22} + \lambda_{22})/k \geq r$, and the result follows from Corollary 2.2.

In the following two examples, we give illustrations as to how Corollary 2.4 can be applied.

Example 2.4. Consider the class of designs $D(7, 8, k)$ where $k = 7\alpha + 4$ and the design d^* having incidence matrix

$$N_{d^*} = \begin{bmatrix} \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \\ \alpha & \alpha+1 & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \\ \alpha & \alpha & \alpha+1 & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \\ \alpha & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \end{bmatrix}.$$

It is then easy to verify that for any value of $\alpha \geq 3$, the conditions of Corollary 2.4 are satisfied and that d^* is E-optimal in $D(7, 8, k)$.

Example 2.5. Consider the class of designs $D(7, 8, k)$ where $k = 7\alpha + 3$, $\alpha \geq 1$, and the design d^* given by

$$N_{d^*} = \begin{bmatrix} \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha+1 & \alpha+1 & \alpha+1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha-1 & \alpha & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \\ \alpha & \alpha-1 & \alpha & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \\ \alpha & \alpha & \alpha-1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 & \alpha+1 \end{bmatrix}.$$

It is then easy to verify that for any value of $\alpha \geq 4$, the conditions of Corollary 2.4 are satisfied and that d^* is E-optimal in $D(7, 8, k)$.

Comment. In all previous cases known to the authors where a widely used optimality criterion such as A-, D-, or E-optimality has been used to find an optimal design in $D(v, b, k)$ and where an optimal design has actually been determined, there has always existed at least one optimal design in $M(v, b, k)$, i.e., there has always existed at least one optimal design with its C-matrix having maximal trace. However, in Example 2.5 an E-optimal design $d^* \in (7, 8, k)$ cannot be in $M(7, 8, k)$ for any value of $\alpha \geq 4$. This is because an E-optimal design d^* must satisfy the conditions of Theorem 2.2 and a necessary condition for this to occur is that whenever treatments i and j have $r_{d^*i} = r_{d^*j} = r$, $n_{d^*ix} = n_{d^*jx}$ for $x = 1, \ldots, b$. But since $v - p = 4 > t = 3$, this can never happen if $d^* \in M(7, 8, k)$. Thus an E-optimal design in $D(7, 8, k)$ where $k = 7\alpha + 3$ cannot be in $M(7, 8, k)$ for any value of $\alpha \geq 4$. The authors have found that Example 2.5 is typical of what happens in any class $D(v, b, k)$ where $k = v\alpha + t$ and $v - p > t$, i.e., it is not difficult to show that in such classes when α is sufficiently large, an E-optimal design $d^* \in D(v, b, k)$ having $z_{d^*1} = r$ cannot be in $M(v, b, k)$.

Comment. The examples given here are such that the E-optimal designs given have C-matrices of the form given in Corollary 2.2, Corollary 2.3 or Corollary 2.4. The reason for this is of course that designs having C-matrices of this form are the easiest to construct. There are various numerical conditions that one can prove in order to guarantee that a design has a C-matrix of the general form given in (2.3) (though the design may not satisfy all of the conditions of Corollary 2.2). For example one set of numerical sufficient conditions for a design to have a C-matrix of the form given in (2.3) is that $s(t-v)/\bar{p}$ and $(b-s)t/\bar{p}$ are integers. To see this, we note that a design d^* satisfying Corollary 2.2 must have an incidence matrix of the form

$$N_{d^*} = \begin{bmatrix} (\alpha + 1)J_{v-\bar{p},s} & \alpha J_{v-\bar{p},b-s} \\ N_{d^*21} & N_{d^*22} \end{bmatrix}.$$

Now, if $(sk - (v-\bar{p})s(\alpha+1))/\bar{p}$ is an integer and $((b-s)k - (v-\bar{p})(b-s)\alpha)/\bar{p}$ is an integer, then it is easy to see that it is possible to assign treatments $v - \bar{p} + 1, \ldots, v$ to blocks in N_{d^*21} and N_{d^*22} so that the row sums in N_{d^*21} and N_{d^*22} are constant. Thus the design will have $\lambda_{d^*ij} = \lambda_{12}$ for all $i = 1, \ldots, v - \bar{p}$, $j = v - \bar{p} + 1, \ldots, v$. Now, the fact that $(sk - (v-\bar{p})s(\alpha+1))/\bar{p}$ is an integer implies that $s(t-v)/\bar{p}$ is an integer and $((b-s)k - (v-\bar{p})(b-s)\alpha)/\bar{p}$ being an integer implies that $(b-s)t/\bar{p}$ is an integer. Whether or not a design d^* satisfying the numerical conditions given above satisfies all of the conditions of Corollary 2.2 depends upon whether $v\{rk - (v-\bar{p})\lambda_{11}\}/\bar{p} \geq rk$ and just how treatments $v - \bar{p} + 1, \ldots, v$ are assigned to blocks in N_{d^*21} and N_{d^*22}.

LEMMA 2.4. *Let $D(v, b, k)$ be such that $p \leq v-2$. Then for any design $d \in D(v, b, k)$,*

$$\max_{i \neq j} \mathrm{Var}_d(\hat{\alpha}_i - \hat{\alpha}_j) \geq 2/r .$$

PROOF. Since $p \leq v-2$, it follows that d must have at least two treatments, say treatments i and j, such that $r_{di}+r_{dj} \leq 2r$. By Lemma 2.2,

$$\begin{aligned}\mathrm{Var}_d(\hat{\alpha}_i - \hat{\alpha}_j) &\geq 4/(c_{dii} + c_{djj} - 2c_{dij}) \\ &= 4k \Big/ \left(r_{di}k - \sum_{x=1}^{b} n_{dix}^2 + r_{dj}k - \sum_{x=1}^{b} n_{djx}^2 + 2 \sum_{x=1}^{b} n_{ix}n_{jx} \right) \\ &\geq 4k \Big/ \left(2rk - \sum_{x=1}^{b} (n_{dix} - n_{djx})^2 \right) \geq 4k/2rk = 2/r .\end{aligned}$$

THEOREM 2.3. *Let $D(v, b, k)$ be such that $p \leq v-2$. If $d^* \in D(v, b, k)$ has $z_{d^*1}=r$, then*

$$\max_{i \neq j} \mathrm{Var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) = 2/r .$$

PROOF. Let $d^* \in D(v, b, k)$ have $z_{d^*1}=r$ and let $C_{d^*}^+$ denote the Moore-Penrose inverse of C_{d^*}. Then $C_{d^*}^+$ has eigenvalue 0 with $C_{d^*}^+ J_{v1}=0$ and nonzero eigenvalues $1/z_{d^*1} \geq \cdots \geq 1/z_{d^*,v-1}$. Also, if $l'\alpha$ is estimable and $l'\hat{\alpha}$ is the least squares estimate of $l'\alpha$ under d^*, then

$$\mathrm{Var}_{d^*}(l'\hat{\alpha}) = l'C_{d^*}^+ l = l'l(l'C_{d^*}^+ l/l'l) \leq l'l(1/z_{d^*1}) .$$

Thus for any treatments i and j, we have that

$$\mathrm{Var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) \leq 2(1/r) = 2/r ,$$

hence by Lemma 2.4 and the previous inequality that

$$2/r \leq \max_{i \neq j} \mathrm{Var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) \leq 2/r ,$$

and the result follows.

COROLLARY 2.5. *Suppose $d^* \in D(v, b, k)$ satisfies the conditions given in Corollary 2.2, Corollary 2.3, or Corollary 2.4. Then d^* is both E-optimal and MV-optimal in $D(v, b, k)$.*

PROOF. If $d^* \in D(v, b, k)$ satisfies the conditions of Corollary 2.2, Corollary 2.3, or Corollary 2.4, then $z_{d^*1}=r$ and d^* is E-optimal by Corollary 2.2 and MV-optimal by Theorem 2.3.

We note that the designs given in Examples 2.4 and 2.5 satisfy the conditions of Corollary 2.4, hence they are E- and MV-optimal by Corollary 2.5. We also note that the comments made following Example 2.5 hold for designs satisfying Corollary 2.5.

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