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# DETECTION OF MULTIVARIATE OUTLIERS WITH LOCATION SLIPPAGE OR SCALE INFLATION IN LEFT ORTHOGONALLY INVARIANT OR ELLIPTICALLY CONTOURED DISTRIBUTIONS

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**Abstract.** This paper is concerned with two kinds of multiple outlier problems in multivariate regression. One is a multiple location-slippage problem and the other is a multiple scale-inflation problem. A multi-decision rule is proposed. Its optimality is shown for the first problem in a class of left orthogonally invariant distributions and is also shown for the second problem in a class of elliptically contoured distributions. Thus the decision rule is robust against departures from normality. Further the null robustness of the decision statistic which the rule is based on is pointed out in each problem.

*Key words and phrases*: Left orthogonally invariant, elliptically contoured, null robust, maximal invariant, Wijsman's representation theorem, UBIS decision rule.

## 1. Introduction

Statistical theory related to outliers is a rapidly expanding area of research. This can be seen from excellent surveys by Beckman and Cook (1983) and Barnett and Lewis (1984). The problem of outliers with either location slippage or scale inflation can be traced to Thompson (1935), Pearson and Chandra Sekar (1936), Cochran (1941), Paulson (1952), Truax (1953), and Kudô (1956). One of the simplest forms of such problem may be stated as follows: Suppose that $x_1, x_2, \ldots, x_n$ are independent univariate normal observations with unknown mean, $\theta_i$, and common unknown variance, $\sigma^2$. Then we wish to decide if all of the $\theta_i$ are equal, or, if not, which one has slipped. More precisely, we want to test the null hypothesis $H_0$: $\theta_1 = \cdots = \theta_n$ against $n$ alternatives $H_i$: $\theta_1 = \cdots = \theta_i - \delta = \cdots = \theta_n$ $(i = 1, 2, \ldots, n)$ where $\delta > 0$ (or $\delta \neq 0$). For this problem, the decision rule based on the maximum (absolute) studentized residual has been shown to be optimal by Paulson (1952) and Kudô (1956). Recently, by developing the ideas and methods of Grubbs (1950) and Wilks (1963), Butler (1981) treated two kinds of multiple outlier problems in normal multivariate regression, where alternatives involve a multiplicity of spurious observations.

In this paper, we slightly modify Butler's (1981) formulation and attempt a different approach to the multiple outlier problems. Our decision rule proposed here is, indeed, different from Butler's (1981) rules, and hence from Grubbs' (1950) and Wilks' (1963). Butler's (1981) approach is Bayesian decision theoretical, and his interest is in the admissibility of his decision rules. Our approach is a rather traditional one along the lines of Paulson (1952), Kudô (1956), Karlin and Truax ((1960), Sec. 9), Ferguson ((1961), Sec. 3), and Butler ((1983), Sec. 6). The purpose of this paper is to extend their multi-decision optimality results to the multiple outlier problems in multivariate regression in a class of left orthogonally invariant distributions or elliptically contoured distributions. Under some mild conditions without normality, a simpler derivation of the results is provided.

Our results in this paper can be viewed as a robustness property of their multi-decision rules. First, their rules are still optimal in the above class. Second, the null distributions of the decision statistics which their rules are based on under any member of the class remain the same as those under normality. As mentioned in Kariya and Sinha (1985), the former is called optimality robustness and the latter null robustness. Sinha (1984) studied the optimality robustness of an LBI (locally best invariant) test for a multivariate location-slippage outlier model in a similar class. This testing problem with location-slippage alternative differs from our multiple location-slippage problem in the structure of location-slippage. This will be described in Section 3 more explicitly. From another viewpoint, Kimura (1984) investigated the robustness of outlier detection.

In Section 2, in a class of left orthogonally invariant distributions or elliptically contoured distributions, two kinds of multiple outlier problems are formulated and an appropriate decision rule is proposed. One of the problems is a multiple location-slippage problem and the other is a multiple scale-inflation problem. In Section 3, our rule is shown to be UBIS (uniformly best invariant symmetric) for the multiple location-slippage problem in the class of left orthogonally invariant distributions. This result is regarded as an extension of Kudô (1956), Karlin and Truax (1960), and Theorem 1 in Butler (1983). In Section 4, the UBIS property of our rule is also shown for the multiple scale-inflation problem in the class of elliptically contoured distributions. This is regarded as an extension of Ferguson ((1961), Sec. 3) and (6.9) in Theorem 2 of Butler (1983).

In the derivation of the optimality results, Hall and Kudô's (1968) generalized Neyman-Pearson lemma and Wijsman's (1967) representation theorem are used. In Section 5, by applying corollaries in Kariya (1981) to both of the problems, we point out the null robustness of the decision statistic which our rule is based on.

## 2. Problems and our rule

Let $B=(b_1,\ldots,b_n)$ be an $m\times n$ matrix, and define $\mathrm{vec}(B)=(b_1',\ldots,b_n')'$. Let $\mathscr{O}(n)$ denote the set of $n\times n$ orthogonal matrices, $\mathscr{S}(p)$ ($\bar{\mathscr{S}}(p)$) the set of $p\times p$ positive (nonnegative) definite matrices, $Gl(p)$ the set of $p\times p$ nonsingular matrices, $\boldsymbol{R}^{n\times p}$ the set of $n\times p$ matrices, and $dX$ the Lebesgue measure on $\boldsymbol{R}^{n\times p}$.

Now we formulate two kinds of multiple outlier problems in multivariate regression. Consider a random sample of size $n$ from a $p$-dimensional multivariate population, and denote the sample by $X$: $n\times p$.

### 2.1 *Multiple location-slippage problem*

Assume

$$X = C\beta + D\Delta + \varepsilon\ , \tag{2.1}$$

and that the error term $\varepsilon$ has a left $\mathscr{O}(n)$-invariant density of the form:

$$f(\varepsilon|\Sigma) = |\Sigma|^{-n/2}\phi(\Sigma^{-1/2}\varepsilon'\varepsilon\Sigma^{-1/2})\ , \tag{2.2}$$

where $C$: $n\times q$ and $D$: $n\times r$ are known matrices, $\beta$: $q\times p$, $\Delta$ : $r\times p$ and $\Sigma\in\mathscr{S}(p)$ are unknown parameters, and $\phi$ is a function from $\bar{\mathscr{S}}(p)$ into $[0,\infty)$ such that $\int_{\boldsymbol{R}^{n\cdot p}}\phi(X'X)dX=1$ and belongs to a certain class $\Phi$, which is specified in Section 3. Let $\Delta=(\delta_1, \delta_2,\ldots, \delta_r)'$, and for any given $s<r$, let $\Omega(s)=\{\omega|\omega\subsetneqq\{1, 2,\ldots, r\}, \#\omega=s\}$. The problem is to test

$$\begin{aligned}&H_0\colon \delta_i = 0 \quad (i = 1, 2,\ldots, r)\\ &\qquad\text{against } \binom{r}{s} \text{ alternatives } H_\omega\colon \delta_i = \delta\ (i \in \omega) \text{ and } \delta_i = 0\ (i \notin \omega)\ ,\end{aligned} \tag{2.3}$$

where $\omega\in\Omega(s)$, and $\delta\neq0$ is an unknown $p$-vector.

### 2.2 *Multiple scale-inflation problem*

Assume

$$X = C\beta + \varepsilon\ , \tag{2.4}$$

and that the error term $\varepsilon$ has an elliptically contoured density of the form:

$$
\begin{aligned}
&f(\varepsilon|(I_n + D\Delta\Delta'D')\otimes\Sigma) \\
(2.5)\qquad &\quad = |(I_n + D\Delta\Delta'D')\otimes\Sigma)|^{-1/2} \\
&\quad\cdot\ \psi(\mathrm{vec}'\,(\varepsilon')((I_n + D\Delta\Delta'D')\otimes\Sigma)^{-1}\,\mathrm{vec}\,(\varepsilon'))\ ,
\end{aligned}
$$

where $C, D, \beta, \Delta$ and $\Sigma$ are the same as defined above, and $\psi$ is a function from $[0,\infty)$ into $[0,\infty)$ such that $\int_{R^{n\times p}} \psi(\mathrm{tr}X'X)dX=1$ and belongs to a certain class $\Psi$, which is specified in Section 4. The problem here is also to test (2.3).

For $C$ and $D$ in these problems, we assume

$$
(2.6)\qquad
\begin{array}{lll}
\mathrm{rank}\,(C) = q\ , & & \\
d(\omega,\omega) = \text{positive constant} & \text{for all} & \omega \in \Omega(s)\ , \\
d(\omega,\omega') = \text{constant} & \text{for all} & \omega,\ \omega' \in \Omega(s)\ (\omega \neq \omega')\ ,
\end{array}
$$

where $D=(d_1, d_2,\ldots, d_r)$, $d_\omega=\sum_{i\in\omega} d_i$, $P=I_n-C(C'C)^{-1}C'$, and $d(\omega,\omega')=d_\omega'Pd_{\omega'}$. Further $n\geq p+q$ is assumed.

To consider the above multiple outlier problems along the lines of Paulson (1952), Kudô (1956), Karlin and Truax ((1960), Sec. 9), Ferguson ((1961), Sec. 3), and Butler ((1983), Sec. 6), our consideration is restricted to the class of invariant symmetric level $\alpha$ decision rules satisfying (2.7) and (2.8) below. Let $\varphi(X)=(\varphi_0(X), (\varphi_\omega(X))_{\omega\in\Omega(s)})$ be a decision rule of choosing among the $1+\binom{r}{s}$ hypotheses in (2.3). A rule $\varphi(X)$ is said to be of level $\alpha$ if

$$
(2.7)\qquad E_{0,\beta,\Sigma}[\varphi_0(X)] \geq 1-\alpha \quad \text{for any } \beta \in \boldsymbol{R}^{q\times p} \text{ and } \Sigma \in \mathscr{S}(p)\ ,
$$

where $E_{0,\beta,\Sigma}[\ \ ]$ is the expectation under $\beta$, $\Sigma$ and $H_0$. Also we say that $\varphi(X)$ is symmetric if

$$
\begin{aligned}
&E_{\omega,\delta,\beta,\Sigma}[\varphi_\omega(X)] \text{ is independent of } \omega \in \Omega(s) \\
(2.8)\qquad &\qquad \text{for any } \delta \in \boldsymbol{R}^p - \{0\}\ ,\ \beta \in \boldsymbol{R}^{q\times p} \text{ and } \Sigma \in \mathscr{S}(p)\ ,
\end{aligned}
$$

where $E_{\omega,\delta,\beta,\Sigma}[\ \ ]$ is the expectation under $\delta$, $\beta$, $\Sigma$ and $H_\omega$. Further, to consider both of the problems via invariance, let the group $G=Gl(p)\times \boldsymbol{R}^{q\times p}$ act on $X$ by: $X\to XA+C\mu$ for $A \in Gl(p)$ and $\mu \in \boldsymbol{R}^{q\times p}$. Then the problems remain invariant under the group $G$.

Define

$$
(2.9)\qquad S = X'PX \quad \text{and} \quad T_\omega = d_\omega'PXS^{-1}X'Pd_\omega\ .
$$

In this paper, for both of the problems, we propose the following decision rule of the form:

$$(2.10)\qquad \begin{aligned} \varphi_0^*(X) &= 1,\, 0 && \text{if} \quad \max_{\omega\in\Omega(s)} T_\omega <,\ > c\,,\\ \varphi_\omega^*(X) &= (1-\varphi_0^*(X))/\kappa(X),\, 0 && \text{if} \quad T_\omega =,\ < \max_{\omega\in\Omega(s)} T_\omega\,, \end{aligned}$$

where $c$ is a constant determined by the level condition $E_{0,\beta,\Sigma}[\varphi_0^*(X)]=1-\alpha$ and $\kappa(X)$ is the number of $\omega$'s for which $\max_{\omega\in\Omega(s)} T_\omega$ is attained. As will be seen in Sections 3 and 4, it follows from invariance that the null distribution of $\max_{\omega\in\Omega(s)} T_\omega$ is independent of $\beta$ and $\Sigma$, and also as will be seen in Section 5, that $\max_{\omega\in\Omega(s)} T_\omega$ is null robust. Therefore the cut-off point $c$ can be determined under normal distribution independently of $\beta$ and $\Sigma$. It is clear that our rule is different from Butler's (1981) rules when $s\geq 2$.

In the following sections, the decision rule $\varphi^*$ defined in (2.10) above is shown to be UBIS (uniformly best invariant symmetric) for each problem, i.e., $\varphi^*$ satisfies

$$(2.11)\qquad \begin{aligned} &\sup_{\varphi\in\mathscr{D}(\alpha)} E_{\omega,\delta,\beta,\Sigma}[\varphi_\omega(X)] = E_{\omega,\delta,\beta,\Sigma}[\varphi_\omega^*(X)]\\ &\qquad \text{for any } \delta\in\boldsymbol{R}^p-\{0\},\ \beta\in\boldsymbol{R}^{q\times p},\ \Sigma\in\mathscr{S}(p) \text{ and } \omega\in\Omega(s)\,, \end{aligned}$$

where $\mathscr{D}(\alpha)$ is the class of invariant symmetric level $\alpha$ decision rules. This equation (2.11) implies that $\varphi^*$ maximizes the probability of making the correct decision under the alternatives.

## 3. Optimality result for the multiple location-slippage problem

In this section, we discuss via invariance the multiple location-slippage problem in the class of left $\mathscr{O}(n)$-invariant distributions with densities of the form (2.2) where $\phi$ is assumed to belong to the class

$$(3.1)\qquad \begin{aligned} \Phi = \{\phi\colon \overline{\mathscr{S}}(p) \to [0,\infty)\,|\,&\phi \text{ is strictly convex on } \overline{\mathscr{S}}(p),\\ &\text{and } \phi(RVR')=\phi(V) \text{ for all } V\in\overline{\mathscr{S}}(p) \text{ and } R\in\mathscr{O}(p)\}\,. \end{aligned}$$

To do so, let $Q$ be an $n\times n$ orthogonal matrix such that $Q'PQ=\begin{pmatrix} I_{n-q} & 0\\ 0 & 0\end{pmatrix}$, and let $Y=(I_{n-q},0)Q'X$. Then, for $S$ in (2.9), $S=Y'Y$, and a maximal invariant statistic and a maximal invariant parameter under $G=Gl(p)\times\boldsymbol{R}^{q\times p}$ are, respectively, $W=YS^{-1}Y'$ and $\eta=\delta'\Sigma^{-1}\delta$. To derive the distribution of the maximal invariant $W$, we first consider the marginal density of $Y$.

LEMMA 3.1. *The marginal density of $Y$ under $H_\omega$ is given by*

$$(3.2)\qquad \tilde{f}(Y|M_\omega,\Sigma) = |\Sigma|^{-(n-q)/2}\,\tilde{\phi}(\Sigma^{-1/2}(Y-M_\omega)'(Y-M_\omega)\Sigma^{-1/2})\,,$$

*where* $M_\omega=(I_{n-q},0)Q'd_\omega\delta'$, $\tilde{\phi}(V)=\int_{R^{q\times p}}\phi(V+Z'Z)dZ$, *and* $dZ$ *is the Lebesgue measure on* $\boldsymbol{R}^{q\times p}$. *Further,* $\tilde{\phi}$ *is strictly convex on* $\bar{\mathscr{S}}(p)$, *and it satisfies*

$$\tilde{\phi}(RVR')=\tilde{\phi}(V) \quad \text{for all } V\in\bar{\mathscr{S}}(p) \text{ and } R\in\mathscr{O}(p)\,. \tag{3.3}$$

Since $W$ is also a maximal invariant under the group $Gl(p)$ acting on $Y$ by: $Y\to YA$ for $A\in Gl(p)$, using Wijsman's (1967) representation theorem yields

LEMMA 3.2. *Let* $P^W_{\omega,\eta}$ *be the distribution of* $W$ *under* $\eta$ *and* $H_\omega$. *Then the density of* $W$ *under* $\eta$ *and* $H_\omega$ *with respect to* $P^W_{0,0}$, *evaluated at* $W=w(X)$, *is given by*

$$\begin{aligned}&\frac{dP^W_{\omega,\eta}}{dP^W_{0,0}}(w(X))\\&=\frac{\int_{Gl(p)}\tilde{\phi}(AA'-\eta^{1/2}T_\omega^{1/2}(a_1e_1'+e_1a_1')+\eta d(\omega)e_1e_1')|A'A|^{k/2}dA}{\int_{Gl(p)}\tilde{\phi}(AA')|A'A|^{k/2}dA},\end{aligned}\tag{3.4}$$

*where* $d(\omega)\equiv d(\omega,\omega)$, $k=n-p-q$, $e_1=(1,0,\ldots,0)'\in\boldsymbol{R}^p$, $a_1$ *is the first column of* $A$, *and* $dA$ *is the Lebesgue measure on* $\boldsymbol{R}^{p\times p}$.

PROOF. In order to apply Wijsman's theorem, it is sufficient to show that $\mathscr{Y}=\{Y\colon (n-q)\times p\,|\,\text{rank }(Y)=p\}$ is a Cartan $Gl(p)$-space because $\boldsymbol{R}^{(n-q)\times p}-\mathscr{Y}$ has measure 0. For any $Y\in\mathscr{Y}$, since $Y$ is of maximal rank, $YA=Y$ implies $A=I_p$. Hence it follows from Theorem 1.1.3 in Palais (1961) that $\mathscr{Y}$ is a Cartan $Gl(p)$-space (see Kariya (1985), pp. 53–58). Therefore we have

$$\frac{dP^W_{\omega,\eta}}{dP^W_{0,0}}(w(Y))=\frac{\int_{Gl(p)}\tilde{f}(YA|M_\omega,\Sigma)|A'A|^{(n-q)/2}dv(A)}{\int_{Gl(p)}\tilde{f}(YA|0,\Sigma)|A'A|^{(n-q)/2}dv(A)},\tag{3.5}$$

where $v$ is a left invariant measure on $Gl(p)$. Take $dv(A)=|A'A|^{-p/2}dA$. Let $N_{\omega,\eta}$ be the numerator of (3.5). From (3.2), $N_{\omega,\eta}$ is written as

$$N_{\omega,\eta}=c_1\int_{Gl(p)}\tilde{\phi}(\Sigma^{-1/2}(YA-M_\omega)'(YA-M_\omega)\Sigma^{-1/2})|A'A|^{k/2}dA\,,\tag{3.6}$$

where $c_1=|\Sigma|^{-(n-q)/2}$. The substitution of $Y=(I_{n-q},0)Q'X$ into (3.6) yields

$$N_{\omega,\eta}=c_1\int_{Gl(p)}\tilde{\phi}(\Sigma^{-1/2}(XA-d_\omega\delta')'P(XA-d_\omega\delta')\Sigma^{-1/2})|A'A|^{k/2}dA\,.\tag{3.7}$$

Transforming $A$ into $S^{1/2}A\Sigma^{-1/2}$, we obtain

$$N_{\omega,\eta} = c_2 \int_{Gl(p)} \tilde{\Phi}(A'A - \tau W_\omega' A - A' W_\omega \tau' + d(\omega)\tau\tau')|A'A|^{k/2} dA\ , \tag{3.8}$$

where $c_2 = |S|^{-(n-q)/2}$, $W_\omega = S^{-1/2} X' P d_\omega$ and $\tau = \Sigma^{-1/2}\delta$. Let $R_1$ and $R_2$ be $p \times p$ orthogonal matrices with $\tau/\|\tau\|$ and $W_\omega/\|W_\omega\|$ as their first columns respectively. Transforming $A$ into $R_1' A' R_2$ and using (3.3), we find

$$\begin{aligned} N_{\omega,\eta} = c_2 \int_{Gl(p)} \tilde{\Phi}(AA' - \eta^{1/2} T_\omega^{1/2}(a_1 e_1' + e_1 a_1') \\ + \eta d(\omega) e_1 e_1')|A'A|^{k/2} dA\ . \end{aligned} \tag{3.9}$$

Finally taking the ratio of $N_{\omega,\eta}$ and $N_{0,0}$, we get (3.4). □

Our main result is the following.

THEOREM 3.1. *For the multiple location-slippage problem, the rule* $\varphi^*$ *in* (2.10) *is UBIS in the sense of* (2.11).

PROOF. First we show that $\varphi^*$ is symmetric in power. Under $H_\omega$, (2.1) can be written as $X = C\beta + d_\omega \delta' + \varepsilon$. Then, for $S$ in (2.9), $S = d(\omega,\omega)\delta\delta' + 2\delta d_\omega' P\varepsilon + \varepsilon' P \varepsilon$, and $d_{\omega'}' PX = d(\omega,\omega')\delta' + d_{\omega'}' P\varepsilon$. Thus, from (2.6), it is sufficient to show that the joint distribution of $(D_{\omega_1}' P\varepsilon, \varepsilon' P\varepsilon)$ is equal to the joint distribution of $(D_{\omega_2}' P\varepsilon, \varepsilon' P\varepsilon)$ for any $\omega_1, \omega_2 \in \Omega(s)$, where $D_{\omega_1}$ and $D_{\omega_2}$ are $n \times \binom{r}{s}$ matrices with $d_{\omega_1}$ and $d_{\omega_2}$ as their first columns and with $\{d_{\omega'} | \omega' \in \Omega(s), \omega' \neq \omega_1\}$ and $\{d_{\omega'} | \omega' \in \Omega(s), \omega' \neq \omega_2\}$ as their remainders, respectively. Since

$$\begin{aligned} ((I_{n-q}, 0) Q' D_{\omega_1})'(I_{n-q}, 0) Q' D_{\omega_1} &= D_{\omega_1}' P D_{\omega_1} = D_{\omega_2}' P D_{\omega_2} \\ &= ((I_{n-q}, 0) Q' D_{\omega_2})'(I_{n-q}, 0) Q' D_{\omega_2}\ , \end{aligned}$$

there exists an $(n-q) \times (n-q)$ orthogonal matrix $R$ such that $(I_{n-q}, 0) Q' D_{\omega_2} = R(I_{n-q}, 0) Q' D_{\omega_1}$. Let $U = Q\begin{pmatrix} R & 0 \\ 0 & I_q \end{pmatrix} Q'$. Then

$$PD_{\omega_2} = UPD_{\omega_1} \quad \text{and} \quad UPU' = P\ . \tag{3.10}$$

It follows from $\mathscr{L}(U'\varepsilon) = \mathscr{L}(\varepsilon)$ that $\mathscr{L}(D_{\omega_1}' P\varepsilon, \varepsilon' P\varepsilon) = \mathscr{L}(D_{\omega_2}' P\varepsilon, \varepsilon' P\varepsilon)$.

Second we show the UBI property of $\varphi^*$. By the definition of $\varphi^*$ in (2.10), it is easy to see that $\varphi^*$ is a function of the maximal invariant $W$. From Lemma 3.2, the density of $W$ under $H_\omega$ is given by (3.4). Let $N_\eta(T_\omega^{1/2})$ be the numerator of (3.4). Since transforming $A$ into $-A$ leaves $N_\eta(T_\omega^{1/2})$ the same and $\tilde{\Phi}$ is strictly convex, it follows from the argument as in Kariya ((1981), p. 1274) that $N_\eta(T_\omega^{1/2})$ is a strictly monotone increasing function of $T_\omega$. Therefore there exists some $c_\eta^*$ such that

$$\{\max_{\omega\in\Omega(s)} T_\omega \leqq c\} = \left\{\max_{\omega\in\Omega(s)} \frac{dP^W_{\omega,\eta}}{dP^W_{0,0}} \leqq c^*_\eta\right\}. \tag{3.11}$$

By Theorem 1 in Hall and Kudô (1968), $\varphi^*$ is best for each fixed $\eta$. Since $\varphi^*$ does not depend on $\eta$, $\varphi^*$ is uniformly best. Hence the proof is completed. □

We remark that the class of left $\mathcal{O}(n)$-invariant distributions with densities of the form (2.2) where $\phi \in \Phi$ in (3.1) includes the multivariate normal distribution, the multivariate $t$-distribution, the multivariate Cauchy distribution, the contaminated normal distribution, the continuous normal mixture as in Sinha (1984), and the matrix variate $t$-distribution as in Kariya ((1981), p. 1272). Thus Theorem 3.1 is an extension of Kudô (1956), Karlin and Truax (1960), and Theorem 1 in Butler (1983). The following three special cases are worthy of notice.

(i) When $r=n$, $s=1$, $D=I_n$, and $\mathcal{L}(X)=N_{n\times p}\ (C\beta+\Delta,\ I_n\otimes\Sigma)$, the problem is reduced to the same as Butler's ((1983), Theorem 1). Note that Butler ((1983), Sec. 6) gives a weight to each alternative $H_i$ instead of considering $C$ which satisfies (2.6).

(ii) In addition to (i), suppose $q=1$ and $C=\mathbf{1}=(1,\dots,1)'\in\boldsymbol{R}^n$. Then the UBIS rule (2.10) is based on

$$\max_{i=1,2,\dots,n} (X_i-\bar{X})'S^{-1}(X_i-\bar{X}), \tag{3.12}$$

where $X_i$ is the $i$-th column of $X'$, $\bar{X}=(1/n)\sum_{i=1}^n X_i$, and $S=\sum_{i=1}^n (X_i-\bar{X})(X_i-\bar{X})'$. This is the same as obtained by Karlin and Truax (1960), in particular when $p=1$, by Kudô (1956).

(iii) When $p\geq 1$, $q=1$, $r=n-1$, $s=1$, $C=\mathbf{1}$, and $D$ is the $n\times(n-1)$ matrix such as

$$\begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 1 & 0 & & & \cdot \\ 0 & 1 & 1 & \cdot & & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & & 0 & \cdot & \cdot & 0 \\ \cdot & & & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

by letting $\mathbf{1}\beta+D\Delta=(\theta_1,\theta_2,\dots,\theta_n)'$, the alternatives in (2.3) are expressed as

$$H_i\colon \theta_1=\cdots=\theta_{i-1}=\theta_i-\delta=\theta_{i+1}-\delta=\theta_{i+2}=\cdots=\theta_n,$$

for $i=1,2,\dots,n-1$. Then the UBIS rule (2.10) is based on

$$(3.13) \qquad \max_{i=1,2,\dots,n-1} (X_i + X_{i+1} - 2\bar{X})'S^{-1}(X_i + X_{i+1} - 2\bar{X}) .$$

The alternative of the form $\bigcup_{i=1}^{n-1} H_i$ cannot be treated in the multivariate location-slippage outlier model of Schwager and Margolin (1982) and Sinha (1984).

## 4. Optimality result for the multiple scale-inflation problem

We discuss now the multiple scale-inflation problem in the class of elliptically contoured distributions with densities of the form (2.5) where $\psi$ is assumed to belong to the class

$$(4.1) \qquad \Psi = \{\psi:[0,\infty) \to [0,\infty) \mid \psi \text{ is strictly monotone decreasing}\} .$$

With $Y$ and $W$ as defined in Section 3, a maximal invariant statistic under $G = Gl(p)\times \boldsymbol{R}^{q\times p}$ is $W$. A maximal invariant parameter is $\lambda=\delta'\delta$. The marginal density of $Y$ is the following.

LEMMA 4.1. *The marginal density of $Y$ under $H_\omega$ is given by*

$$(4.2) \qquad \tilde{f}(Y|0, \Gamma_\omega \otimes \Sigma) = |\Gamma_\omega \otimes \Sigma|^{-1/2} \tilde{\psi}(\mathrm{vec}'(Y')(\Gamma_\omega \otimes \Sigma)^{-1}\mathrm{vec}(Y')) ,$$

*where $\Gamma_\omega = I_{n-q} + \lambda(I_{n-q}, 0)Q'd_\omega d_\omega' Q(I_{n-q}, 0)'$, $\tilde{\psi}(v) = \int_{\boldsymbol{R}^{q\times p}} \psi(v + \mathrm{tr} Z'Z)dZ$, and $dZ$ is the Lebesgue measure on $\boldsymbol{R}^{q\times p}$. Further, $\tilde{\psi}$ is strictly monotone decreasing.*

LEMMA 4.2. *Let $P^W_{\omega,\lambda}$ be the distribution of $W$ under $\eta$ and $H_\omega$. Then the density of $W$ under $\lambda$ and $H_\omega$ with respect to $P^W_{0,0}$, evaluated at $W = w(X)$, is given by*

$$(4.3) \qquad \frac{dP^W_{\omega,\lambda}}{dP^W_{0,0}}(w(X)) = \frac{\int_{Gl(p)} \tilde{\psi}\left(\mathrm{tr}A'A - \frac{\lambda}{1+\lambda d(\omega)} T_\omega a_1' a_1\right) |A'A|^{k/2} dA}{(1+\lambda d(\omega))^{p/2} \int_{Gl(p)} \tilde{\psi}(\mathrm{tr}A'A)|A'A|^{k/2}dA} ,$$

*where $d(\omega) \equiv d(\omega,\omega)$, $k = n-p-q$, $a_1$ is the first column of $A$, and $dA$ is the Lebesgue measure on $\boldsymbol{R}^{p\times p}$.*

PROOF. By applying Wijsman's theorem as in Lemma 3.2, we get

$$(4.4) \qquad \frac{dP^W_{\omega,\lambda}}{dP^W_{0,0}}(w(Y)) = \frac{\int_{Gl(p)} \tilde{f}(YA|0, \Gamma_\omega \otimes \Sigma)|A'A|^{k/2}dA}{\int_{Gl(p)} \tilde{f}(YA|0, I_{n-q} \otimes \Sigma)|A'A|^{k/2}dA} .$$

Let $N_{\omega,\lambda}$ be the numerator of (4.4). From (4.2), $N_{\omega,\lambda}$ is written as

$$(4.5)\qquad N_{\omega,\lambda} = c_\omega(\lambda)\int_{Gl(p)} \tilde{\psi}(\mathrm{vec}'((YA)')(\Gamma_\omega \otimes \Sigma)^{-1}\mathrm{vec}((YA)'))\,|A'A|^{k/2}dA\ ,$$

where $c_\omega(\lambda)=|\Gamma_\omega \otimes \Sigma|^{-1/2}$. Let $b_\omega=(I_{n-q}, 0)Q'd_\omega$ and $V_\omega$ be an $(n-q)\times(n-q)$ orthogonal matrix with $b_\omega/\|b_\omega\|$ as its first column. Then $V_\omega'\Gamma_\omega V_\omega=I_{n-q}+\lambda d(\omega)e_1e_1'$ where $e_1=(1, 0,\dots, 0)' \in \boldsymbol{R}^{n-q}$. Letting $Z_\omega=V_\omega'Y$, we have

$$(4.6)\qquad \begin{aligned} N_{\omega,\lambda} = c_\omega(\lambda)\int_{Gl(p)} &\tilde{\psi}(\mathrm{vec}'((Z_\omega A)') \\ &\cdot ((I_{n-q} + \lambda d(\omega)e_1e_1')\otimes\Sigma)^{-1}\mathrm{vec}((Z_\omega A)'))\,|A'A|^{k/2}dA\ . \end{aligned}$$

After calculation, substituting $Z_\omega=V_\omega'(I_{n-q}, 0)Q'X$ into (4.6) and transforming $A$ into $S^{1/2}A\Sigma^{-1/2}$, we find

$$(4.7)\qquad N_{\omega,\lambda} = c_\omega'(\lambda)\int_{Gl(p)} \tilde{\psi}\left(\mathrm{tr}AA' - \frac{\lambda}{1+\lambda d(\omega)}\, W_\omega'AA'W_\omega\right)|A'A|^{k/2}dA\ ,$$

where $c_\omega'(\lambda)=|S|^{-(n-q)/2}(1+\lambda d(\omega))^{-p/2}$ and $W_\omega=S^{-1/2}\,X'Pd_\omega$. Let $U$ be a $p\times p$ orthogonal matrix with $W_\omega/\|W_\omega\|$ as its first column. Transforming $A$ into $A'U$ yields

$$(4.8)\qquad N_{\omega,\lambda} = c_\omega'(\lambda)\int_{Gl(p)} \tilde{\psi}\left(\mathrm{tr}A'A - \frac{\lambda}{1+\lambda d(\omega)}\, T_\omega a_1'a_1\right)|A'A|^{k/2}dA\ .$$

Finally taking the ratio of $N_{\omega,\lambda}$ and $N_{0,0}$, we obtain (4.3). □

Now we will verify our second main theorem.

THEOREM 4.1. *For the multiple scale-inflation problem, the rule* $\varphi^*$ *in* (2.10) *is UBIS in the sense of* (2.11).

PROOF. First we show that $\varphi^*$ is symmetric in power. Since $S=Y'Y$ and $d_\omega'PX=d_\omega'Q\,(Y', 0)'$, it is sufficient to show that the joint distribution of $(D_{\omega_1}'Q(Y',0)', Y'Y)$ under $H_{\omega_1}$, $\mathscr{L}_{\omega_1}(D_{\omega_1}'Q(Y',0)', Y'Y)$, is equal to the joint distribution of $(D_{\omega_2}'Q(Y',0)', Y'Y)$ under $H_{\omega_2}$, $\mathscr{L}_{\omega_2}(D_{\omega_2}'Q(Y',0)', Y'Y)$, for any $\omega_1, \omega_2 \in \Omega(s)$, where $D_{\omega_1}$ and $D_{\omega_2}$ are the same as those in Theorem 3.1. Since there exists an $(n-q)\times(n-q)$ orthogonal matrix $R$ as in Theorem 3.1, the density of $R'Y$ under $H_{\omega_2}$ is equal to the density of $Y$ under $H_{\omega_1}$. Hence

$$\begin{aligned} \mathscr{L}_{\omega_2}(D_{\omega_2}'Q(Y',0)', Y'Y) &= \mathscr{L}_{\omega_2}(D_{\omega_1}'Q(Y'R,0)', Y'RR'Y) \\ &= \mathscr{L}_{\omega_1}(D_{\omega_1}'Q(Y',0)', Y'Y)\ . \end{aligned}$$

The proof will be completed if we show the UBI property of $\varphi^*$. It is clear that the numerator of (4.3) is a strictly monotone increasing function of $T_\omega$. The rest of the proof is parallel to that of Theorem 3.1. □

The class of elliptically contoured distributions with densities of the form (2.5) where $\psi \in \Psi$ in (4.1) includes those distributions stated in Section 3 except for the matrix variate $t$-distribution. Some special cases of Theorem 4.1 have been treated in the literature. When $r=n$, $s=1$, $D=I_n$, and $\mathscr{L}(X)=N_{n\times p}(C\beta, (I_n+\Delta\Delta')\otimes\Sigma)$, the problem (2.10) is reduced to the same as (6.9) in Theorem 2 of Butler (1983). In the case where $q=1$ and $C=\mathbf{1}=(1, 1, \ldots, 1)' \in \boldsymbol{R}^n$ in addition, Ferguson ((1961), Sec. 3) has shown that the rule based on (3.12) is UBIS. Thus Theorem 4.1 is an extension of Ferguson ((1961), Sec. 3) and (6.9) in Theorem 2 of Butler (1983).

## 5. Null robustness

To use our UBIS rule $\varphi^*$ in (2.10) in practice, it is required to determine the cut-off point $c$. In both of the problems, the cut-off point $c$ does not depend on $\phi$ or $\psi$, and can be determined under normal distribution. To verify this, it is sufficient to show that $t(X)=\max_{\omega\in\Omega(s)} T_\omega$ satisfies the conditions in Corollaries 1.1 and 1.2 of Kariya (1981):

$$t(X - C\mu) = t(X) \quad \text{for all} \quad \mu \in \boldsymbol{R}^{q\times p}\,. \tag{5.1}$$

$$t(XA) = t(X) \quad \text{for all} \quad A \in \mathscr{S}(p)\,. \tag{5.2}$$

Hence the distribution of $\max_{\omega\in\Omega(s)} T_\omega$ under the null hypothesis $H_0$ remains the same in the class of left $\mathscr{O}(n)$-invariant distributions or elliptically contoured distributions, i.e., the null distribution of $\max_{\omega\in\Omega(s)} T_\omega$ is equal to the distribution of $\max_{\omega\in\Omega(s)} T_\omega$ under the assumption $\mathscr{L}(X)=N_{n\times p}(0, I_n\otimes I_p)$ in each problem.

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