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ENTROPY LOSS AND RISK OF IMPROVED ESTIMATORS FOR THE GENERALIZED VARIANCE AND PRECISION*

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Abstract. Let the distributions of $X(p\times r)$ and $S(p\times p)$ be $N(\xi, \Sigma\otimes I_r)$ and $W_p(n, \Sigma)$ respectively and let them be independent. The risk of the improved estimator for $|\Sigma|$ or $|\Sigma^{-1}|$ based on X and S under entropy loss $(=d/|\Sigma| -\log(d/|\Sigma|)-1$ or $d|\Sigma|-\log(d|\Sigma|)-1)$ is evaluated in terms of incomplete beta function of matrix argument and its derivative. Numerical comparison for the reduction of risk over the best affine equivariant estimator is given.

Key words and phrases: Stein's truncated estimator, zonal polynomials, incomplete beta function, multivariate linear hypotheses, mixture representation of noncentral Wishart and multivariate beta distributions.

1. Introduction

Suppose that observed random matrix $X(p\times r)$ has normal distribution $N(\xi, \Sigma\otimes I_r)$ and that $S(p\times p)$ has Wishart distribution $W_p(n, \Sigma)$, where $n\geq p$ and ξ, Σ are unknown. Assume that X and S are independent. Shorrock and Zidek (1976) obtained a better estimator of $|\Sigma|$ than the best affine equivariant estimator under squared loss by generalizing Stein (1964) to multivariate case. Sinha and Ghosh (1987) noted that under entropy loss $L(d, |\Sigma|) =d/|\Sigma|-\log(d/|\Sigma|)-1$, the estimator

$$(1.1)\qquad d^*(X, S) = \min\left\{d(S)\,,\frac{(n+r-p)!}{(n+r)!}|S+XX'|\right\},$$

dominates the best affine equivariant estimator $d(S)=\{(n-p)!/n!\}|S|$. Their method of proof is based on Sinha (1976) and did not make use of zonal polynomials of matrix argument. However Shorrock and Zidek (1976) can be applied to get the same result. For estimating the generalized precision $|\Sigma^{-1}|$, Sugiura and Konno (1987) noted that the estimator

*Dedicated to Professor Yukihiro Kodama on his 60th birthday.

$$(1.2) \qquad e^*(X,S) = \max\left\{e(S)\,, \frac{(n+r-2)!}{(n+r-p-2)!}\,|S+XX^t|^{-1}\right\},$$

dominates the best affine equivariant estimator $e(S)=\{(n-2)!/(n-p-2)!\}|S|^{-1}$ in the case of $n\geq p+2$, under entropy loss $L(d, |\Sigma^{-1}|) = d|\Sigma| - \log(d|\Sigma|) - 1$, following Shorrock and Zidek (1976).

In this paper the risk of $d^*(X, S)$ or $e^*(X, S)$ is expressed by an infinite series in terms of incomplete beta functions of matrix argument used by Sugiura and Konno (1987) and its derivative with respect to a parameter. Then numerical computation of the risk of each estimator is performed and the reduction of risk of $d^*(X, S)$ or $e^*(X, S)$ over $d(S)$ or $e(S)$ is checked. As in the case of squared loss discussed by Sugiura and Konno (1987), the reduction of risk is more in estimating the generalized precision than in estimating the generalized variance. However, it is less under entropy loss than under squared loss.

2. Risk of estimators for the generalized variance

We shall first note that risk of the best affine equivariant estimator $d(S)=\{(n-p)!/n!\}|S|$ under entropy loss is given by

$$(2.1) \qquad R(d) = \log\frac{n!}{(n-p)!} - p\log 2 - \sum_{i=1}^{p}\psi\left(\frac{n-i+1}{2}\right),$$

where $\psi(x)=d\log\Gamma(x)/dx$ is digamma function and has a simple form for half-integer argument used also in Sugiura and Fujimoto (1982), Dey and Srinivasan (1986) for calculating risk of the minimax estimator of normal covariance matrix.

To give a useful expression for the risk of the improved estimator $d^*(X, S)$, we need the following incomplete beta function of matrix argument and its derivative:

$$(2.2) \qquad \begin{aligned} &I_a^{(p)}(\alpha,\beta;\kappa) \\ &\quad = \frac{(\alpha+\beta)_\kappa}{(\beta)_\kappa C_\kappa(I_p)}\int_{|Z|<a,\,0<Z<I_p} \frac{|Z|^{\alpha-(p+1)/2}|I_p-Z|^{\beta-(p+1)/2}}{B_p(\alpha,\beta)}\,C_\kappa(I_p-Z)dZ\,, \\ &J_a^{(p)}(\alpha,\beta;\kappa) \\ &\quad = \frac{(\alpha+\beta)_\kappa}{(\beta)_\kappa C_\kappa(I_p)} \\ &\quad\quad \times\int_{|Z|<a,\,0<Z<I_p} (\log|Z|)\,\frac{|Z|^{\alpha-(p+1)/2}|I_p-Z|^{\beta-(p+1)/2}}{B_p(\alpha,\beta)}\,C_\kappa(I_p-Z)dZ\,, \end{aligned}$$

where α, $\beta>(p-1)/2$, $0<a<1$ and $C_\kappa(Z)$ stands for zonal polynomial of positive definite matrix Z corresponding to partition $\kappa=\{k_1,\dots,k_p\}$ $(k_1\geq\cdots\geq k_p)$ of k; $Z<I_p$ means I_p-Z is positive definite. The constant factor is so chosen that for $a=1$, $I_a^{(p)}(\alpha,\beta;\kappa)=1$. We note that when $p=1$, these two functions are reduced to

$$
\begin{aligned}
I_a(\alpha,\beta+k) &= \frac{1}{B(\alpha,\beta+k)}\int_0^a z^{\alpha-1}(1-z)^{\beta+k-1}dz\,, \\
J_a(\alpha,\beta+k) &= \frac{1}{B(\alpha,\beta+k)}\int_0^a (\log z) z^{\alpha-1}(1-z)^{\beta+k-1}dz\,.
\end{aligned} \tag{2.3}
$$

For the numerical computation of $I_a^{(p)}(\alpha,\beta;\kappa)$ when $p>1$, the following expression by the latent roots of Z is useful:

$$
\begin{aligned}
I_a^{(p)}(\alpha,\beta;\kappa) = {} & \frac{\pi^{p^2/2}}{\Gamma_p\left(\dfrac{p}{2}\right)}\,\frac{(\alpha+\beta)_\kappa}{(\beta)_\kappa C_\kappa(I_p)} \\
& \times\int_{|L|<a,\ 0<l_p<\cdots<l_1<1}\frac{|L|^{\alpha-(p+1)/2}|I_p-L|^{\beta-(p+1)/2}}{B_p(\alpha,\beta)} \\
& \times C_\kappa(I_p-L)\prod_{1\leq i<j\leq p}(l_i-l_j)dL\,,
\end{aligned}
$$

where $L=\mathrm{diag}(l_1,\dots,l_p)$ and similarly for $J_a^{(p)}(\alpha,\beta;\kappa)$. We now prove the following theorem.

THEOREM 2.1. *Assume that $n\geq p$. Put $\Lambda=\xi'\Sigma^{-1}\xi$ and $a=\{n!(n+r-p)!\}/\{(n-p)!(n+r)!\}$. Then under the entropy loss, the risk of the improved estimator $d^*(X,S)$ is given by*

(i) *when $r\geq p$,*

$$
\begin{aligned}
R(d^*;\Lambda) = \sum_{k=0}^{\infty}\sum_{(\kappa)}\mathrm{etr}\left(-\frac{\Lambda}{2}\right)\frac{C_\kappa\left(\dfrac{\Lambda}{2}\right)}{k!}\Bigg[& I_a^{(p)}\left(\frac{n+2}{2},\frac{r}{2};\kappa\right) \\
& +\left(\prod_{i=1}^{p}\frac{n+r+1-i+2k_i}{n+r+1-i}\right)\left\{1-I_a^{(p)}\left(\frac{n}{2},\frac{r}{2};\kappa\right)\right\} \\
& +(\log a)I_a^{(p)}\left(\frac{n}{2},\frac{r}{2};\kappa\right)-J_a^{(p)}\left(\frac{n}{2},\frac{r}{2};\kappa\right) \\
& -\sum_{i=1}^{p}\psi\left(\frac{n+r-i+1}{2}+k_i\right)-\log\frac{(n+r-p)!}{(n+r)!} \\
& -p\log 2-1\Bigg]\,,
\end{aligned}
$$

(ii) *when $r<p$,*

$$R(d^*;\Lambda)=\sum_{k=0}^{\infty}\sum_{(\kappa)}\operatorname{etr}\left(-\frac{\Lambda}{2}\right)\frac{C_\kappa\left(\frac{\Lambda}{2}\right)}{k!}\left[I_a^{(r)}\left(\frac{n+r-p+2}{2},\frac{p}{2};\kappa\right)\right.$$
$$+\left(\prod_{i=1}^{r}\frac{n+r+1-i+2k_i}{n+r+1-i}\right)\left\{1-I_a^{(r)}\left(\frac{n+r-p}{2},\frac{p}{2};\kappa\right)\right\}$$
$$+(\log a)I_a^{(r)}\left(\frac{n+r-p}{2},\frac{p}{2};\kappa\right)-J_a^{(r)}\left(\frac{n+r-p}{2},\frac{p}{2};\kappa\right)$$
$$-\sum_{i=1}^{p}\psi\left(\frac{n+r-i+1}{2}+k_i\right)-\log\frac{(n+r-p)!}{(n+r)!}$$
$$\left.-p\log 2-1\right].$$

COROLLARY 2.1. *When $p=1$, $a=n/(n+r)$ and the risk of $d^*(X,S)$ is given by*

$$\sum_{k=0}^{\infty}\operatorname{etr}\left(-\frac{\Lambda}{2}\right)\frac{(\operatorname{tr}\Lambda/2)^k}{k!}\left[I_a\left(\frac{n+2}{2},\frac{r}{2}+k\right)\right.$$
$$+\frac{n+r+2k}{n+r}\left\{1-I_a\left(\frac{n}{2},\frac{r}{2}+k\right)\right\}$$
$$+(\log a)I_a\left(\frac{n}{2},\frac{r}{2}+k\right)-J_a\left(\frac{n}{2},\frac{r}{2}+k\right)-\psi\left(\frac{n+r}{2}+k\right)$$
$$\left.+\log(n+r)-\log 2-1\right].$$

COROLLARY 2.2. *When $r=1$, $a=(n-p+1)/(n+1)$ and the risk of $d^*(X,S)$ is given by*

$$\sum_{k=0}^{\infty}\exp\left(-\frac{\Lambda}{2}\right)\frac{(\Lambda/2)^k}{k!}\left[I_a\left(\frac{n-p+3}{2},\frac{p}{2}+k\right)+\frac{n+1+2k}{n+1}\right.$$
$$\times\left\{1-I_a\left(\frac{n-p+1}{2},\frac{p}{2}+k\right)\right\}+(\log a)I_a\left(\frac{n-p+1}{2},\frac{p}{2}+k\right)$$
$$-J_a\left(\frac{n-p+1}{2},\frac{p}{2}+k\right)-\psi\left(\frac{n+1}{2}+k\right)$$
$$\left.-\sum_{i=2}^{p}\psi\left(\frac{n+1}{2}-\frac{i-1}{2}\right)+\log\frac{(n+1)!}{(n-p+1)!}-p\log 2-1\right].$$

Corollaries 2.1 and 2.2 are the easiest cases to be computed, since only the usual incomplete beta functions and its derivatives are involved.

PROOF OF THEOREM 2.1. Since the loss function is invariant under the transformation $(X, S)\to(AX, ASA')$ and $(\xi, \Sigma)\to(A\xi, A\Sigma A')$ for any nonsingular matrices A, we may assume that $\Sigma=I$. From Shorrock and Zidek (1976) or by Lemma 2.1 in Sugiura and Konno (1987), put $V=S+XX'$ and $T=(S+XX')^{-1/2}S(S+XX')^{-1/2}$ when $r\geq p$ and $T=I-X'(S+XX')^{-1}X$ when $r<p$. We can write $d^*(X, S)=|V|h(T)$ for $h(T)=\{(n-p)!/n!\}\min\{|T|, a\}$, where V and T are independent for given κ; Under $\Lambda=0$, V has Wishart distribution $W_p(n+r, I)$ and T has a p-variate beta distribution with parameter $(n/2, r/2)$ when $r\geq p$ and an r-variate beta distribution with parameter $((n+r-p)/2, p/2)$ when $p\geq r$. Further the risk of $d^*(X, S)$ is written by

$$
\begin{aligned}
(2.4)\quad & R(d^*, \Lambda) \\
&= E_\Lambda[|V|h(T) - \log(|V|h(T)) - 1] \\
&= E_\Lambda^\kappa\Bigg[\frac{E_0[|V|C_\kappa(V)|\kappa]\,E_0[h(T)C_\kappa(I-T)|\kappa]}{E_0[C_\kappa(V)|\kappa]\,E_0[C_\kappa(I-T)|\kappa]} \\
&\quad - \frac{E_0[(\log|V|)C_\kappa(V)|\kappa]}{E_0[C_\kappa(V)|\kappa]} - \frac{E_0[(\log h(T))C_\kappa(I-T)|\kappa]}{E_0[C_\kappa(I-T)|\kappa]} - 1\Bigg],
\end{aligned}
$$

where E_0 stands for $E_{\Lambda=0}$ and E_Λ^κ denotes the expectation with respect to random partition κ with probability $(\mathrm{etr}(-\Lambda/2))C_\kappa(\Lambda/2)/k!$. Note that $E_0[|V|C_\kappa(V)|\kappa]/E_0[C_\kappa(V)|\kappa]$ is equal to

$$
(2.5)\qquad \frac{2^p\Gamma_p\left(\dfrac{n+r+2}{2}\right)\left(\dfrac{n+r+2}{2}\right)_\kappa}{\Gamma_p\left(\dfrac{n+r}{2}\right)\left(\dfrac{n+r}{2}\right)_\kappa}.
$$

When $r\geq p$, $E_0[h(T)C_\kappa(I-T)|\kappa]/E_0[C_\kappa(I-T)|\kappa]$ is shown to be

$$
\begin{aligned}
(2.6)\quad & \frac{(n-p)!}{n!}\,\frac{\left(\dfrac{n+r}{2}\right)_\kappa}{\left(\dfrac{n+r+2}{2}\right)_\kappa}\,\frac{B_p\left(\dfrac{n+2}{2}, \dfrac{r}{2}\right)}{B_p\left(\dfrac{n}{2}, \dfrac{r}{2}\right)}\,I_a^{(p)}\left(\frac{n+2}{2}, \frac{r}{2}; \kappa\right) \\
&+ \frac{(n+r-p)!}{(n+r)!}\left\{1 - I_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}; \kappa\right)\right\},
\end{aligned}
$$

giving the first term within the blanket in R.H.S. of (2.4) as

$$(2.7)\qquad I_a^{(p)}\left(\frac{n+2}{2}, \frac{r}{2}; \kappa\right) + \left(\prod_{i=1}^{p} \frac{n+r+1-i+2k_i}{n+r+1-i}\right)\left\{1 - I_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}; \kappa\right)\right\}$$

The second term $E_0[(\log|V|)C_\kappa(V)|\kappa]/E_0[C_\kappa(V)|\kappa]$ is equal to

$$(2.8)\qquad p\log 2 + \sum_{i=1}^{p} \psi\left(\frac{n+r-i+1}{2} + k_i\right),$$

which can be obtained by differentiating the identity

$$\int_{W>0} |W|^{\alpha-(p+1)/2} \operatorname{etr}(-W) C_\kappa(W) dW = \Gamma_p(\alpha)(\alpha)_\kappa C_\kappa(I),$$

with respect to α. The third term in R.H.S. of (2.4) is similarly obtained when $r \geq p$ as

$$(2.9)\qquad \begin{aligned} &\frac{E_0[(\log h(T))C_\kappa(I-T)|\kappa]}{E_0[C_\kappa(I-T)|\kappa]} \\ &= \left\{\log \frac{(n-p)!}{n!}\right\} I_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}; \kappa\right) + J_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}; \kappa\right) \\ &\quad + \left\{\log \frac{(n+r-p)!}{(n+r)!}\right\}\left\{1 - I_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}; \kappa\right)\right\}. \end{aligned}$$

Combined with (2.7), (2.8) and (2.9), we get the desired formula in (i) in Theorem 2.1. The case of $r<p$ is similarly obtained.

Our numerical study in Section 4 shows that the minimum value of $R(d^*, \Lambda)$ is obtained when $\Lambda=0$ given by

$$\begin{aligned} R(d^*, 0) = {} & I_a^{(p)}\left(\frac{n+2}{2}, \frac{r}{2}\right) + (\log a - 1) I_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}\right) - J_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}\right) \\ & - \sum_{i=1}^{p} \psi\left(\frac{n+r-i+1}{2}\right) - \log \frac{(n+r-p)!}{(n+r)!} - p\log 2, \end{aligned}$$

for $r \geq p$ and

$$\begin{aligned} R(d^*, 0) = {} & I_a^{(r)}\left(\frac{n+r-p+2}{2}, \frac{p}{2}\right) + (\log a - 1) I_a^{(r)}\left(\frac{n+r-p}{2}, \frac{p}{2}\right) \\ & - J_a^{(r)}\left(\frac{n+r-p}{2}, \frac{p}{2}\right) - \sum_{i=1}^{p} \psi\left(\frac{n+r-i+1}{2}\right) \\ & - \log \frac{(n+r-p)!}{(n+r)!} - p\log 2, \end{aligned}$$

for $r<p$, where $I_a^{(p)}(\alpha,\beta)=I_a^{(p)}(\alpha,\beta;0)$ and similarly $J_a^{(p)}(\alpha,\beta)=J_a^{(p)}(\alpha,\beta;0)$. The maximum rate of reduction of risk $\{R(d)-R(d^*,\Lambda)\}/R(d)$ is obtained at $\Lambda=0$. Although these findings are intuitively obvious as in Sinha (1976), we were unable to give rigorous proof.

3. Risk of estimators for the generalized precision

For estimating the generalized precision $|\Sigma^{-1}|$ under entropy loss $L(d, |\Sigma^{-1}|)=d|\Sigma|-\log(d|\Sigma|)-1$, risk of the best affine equivariant estimator $e(S)=\{(n-2)!/(n-p-2)!\}|S^{-1}|$ is given by

$$R(e) = \sum_{i=1}^{p} \psi\left(\frac{n-i+1}{2}\right) + p\log 2 - \log\frac{(n-2)!}{(n-p-2)!}\,.$$

By the same argument as in the proof of Theorem 2.1, we get the following mixture representation of risk of the improved estimator $e^*(X, S)$ given by (1.2).

THEOREM 3.1. *Assume that $n>p+1$. Put $\Lambda=\xi^t\Sigma^{-1}\xi$ and $a=\{(n-2)!\times(n+r-p-2)!\}/\{(n-p-2)!(n+r-2)!\}$. Then under the entropy loss ($=d|\Sigma|-\log(d|\Sigma|)-1$), the risk of $e^*(X, S)$ has the following expression.*

(i) *when $r\geq p$,*

$$\begin{aligned} R(e^*,\Lambda) = \sum_{k=0}^{\infty}\sum_{(\kappa)} \operatorname{etr}\left(-\frac{\Lambda}{2}\right)\frac{C_\kappa\left(\frac{\Lambda}{2}\right)}{k!}\Bigg[& I_a^{(p)}\left(\frac{n-2}{2},\frac{r}{2};\kappa\right) \\ & + \left(\prod_{i=1}^{p}\frac{n+r-1-i}{n+r-1-i+2k_i}\right)\left\{1-I_a^{(p)}\left(\frac{n}{2},\frac{r}{2};\kappa\right)\right\} \\ & - (\log a) I_a^{(p)}\left(\frac{n}{2},\frac{r}{2};\kappa\right) + J_a^{(p)}\left(\frac{n}{2},\frac{r}{2};\kappa\right) \\ & + \sum_{i=1}^{p}\psi\left(\frac{n+r-i+1}{2}+k_i\right) - \log\frac{(n+r-2)!}{(n+r-p-2)!} \\ & + p\log 2 - 1\Bigg], \end{aligned}$$

(ii) *when $r<p$,*

$$\begin{aligned} R(e^*,\Lambda) = \sum_{k=0}^{\infty}\sum_{(\kappa)} \operatorname{etr}\left(-\frac{\Lambda}{2}\right)\frac{C_\kappa\left(\frac{\Lambda}{2}\right)}{k!}\Bigg[& I_a^{(r)}\left(\frac{n+r-p-2}{2},\frac{p}{2};\kappa\right) \\ & + \left(\prod_{i=1}^{r}\frac{n+r-1-i}{n+r-1-i+2k_i}\right)\left\{1-I_a^{(r)}\left(\frac{n+r-p}{2},\frac{p}{2};\kappa\right)\right\} \end{aligned}$$

$$- (\log a) I_a^{(r)}\left(\frac{n+r-p}{2}, \frac{p}{2}; \kappa\right) + J_a^{(r)}\left(\frac{n+r-p}{2}, \frac{p}{2}; \kappa\right)$$
$$+ \sum_{i=1}^{p} \psi\left(\frac{n+r-i+1}{2} + k_i\right) - \log \frac{(n+r-2)!}{(n+r-p-2)!}$$
$$+ p \log 2 - 1\Big].$$

COROLLARY 3.1. *When $p=1$, $a=(n-2)/(n+r-2)$ and the risk of $e^*(X, S)$ is given by*

$$\sum_{k=0}^{\infty} \operatorname{etr}\left(-\frac{\Lambda}{2}\right) \frac{(\operatorname{tr}\Lambda/2)^k}{k!} \left[I_a\left(\frac{n-2}{2}, \frac{r}{2} + k\right) \right.$$
$$+ \frac{n+r-2}{n+r-2+2k} \left\{1 - I_a\left(\frac{n}{2}, \frac{r}{2} + k\right)\right\}$$
$$- (\log a) I_a\left(\frac{n}{2}, \frac{r}{2} + k\right) + J_a\left(\frac{n}{2}, \frac{r}{2} + k\right) + \psi\left(\frac{n+r}{2} + k\right)$$
$$\left. - \log(n+r-2) + \log 2 - 1 \right].$$

COROLLARY 3.2. *When $r=1$, $a=(n-p-1)/(n-1)$ and the risk of $e^*(X, S)$ is given by*

$$\sum_{k=0}^{\infty} \exp\left(-\frac{\Lambda}{2}\right) \frac{(\Lambda/2)^k}{k!} \left[I_a\left(\frac{n-p-1}{2}, \frac{p}{2} + k\right) + \frac{n-1}{n-1+2k} \right.$$
$$\times \left\{1 - I_a\left(\frac{n-p+1}{2}, \frac{p}{2} + k\right)\right\}$$
$$- (\log a) I_a\left(\frac{n-p+1}{2}, \frac{p}{2} + k\right)$$
$$+ J_a\left(\frac{n-p+1}{2}, \frac{p}{2} + k\right) + \psi\left(\frac{n+1}{2} + k\right)$$
$$+ \sum_{i=2}^{p} \psi\left(\frac{n-i+2}{2}\right)$$
$$\left. - \log \frac{(n-1)!}{(n-p-1)!} + p \log 2 - 1 \right].$$

Corollaries 3.1 and 3.2 are used for the numerical computation in Section 4. Our numerical study shows that the minimum value of $R(e^*, \Lambda)$ is obtained at $\Lambda=0$, giving, for $r \geq p$

$$R(e^*, 0) = I_a^{(p)}\left(\frac{n-2}{2}, \frac{r}{2}\right) - (\log a + 1) I_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}\right)$$
$$+ J_a^{(p)}\left(\frac{n}{2}, \frac{r}{2}\right) + \sum_{i=1}^{p} \psi\left(\frac{n+r-i+1}{2}\right)$$
$$- \log \frac{(n+r-2)!}{(n+r-p-2)!} + p \log 2 ,$$

and for $r<p$

$$R(e^*, 0) = I_a^{(r)}\left(\frac{n+r-p-2}{2}, \frac{p}{2}\right) - (\log a + 1) I_a^{(r)}\left(\frac{n+r-p}{2}, \frac{p}{2}\right)$$
$$+ J_a^{(r)}\left(\frac{n+r-p}{2}, \frac{p}{2}\right) + \sum_{i=1}^{p} \psi\left(\frac{n+r-i+1}{2}\right)$$
$$- \log \frac{(n+r-2)!}{(n+r-p-2)!} - p \log 2 .$$

The maximum rate of reduction of risk $\{R(e) - R(e^*, \Lambda)\}/R(e)$ is obtained at $\Lambda = 0$.

4. Numerical results

To compute the risk of the improved estimators under entropy loss when $p=1$ or $r=1$, we need to evaluate the function $I_a(\alpha, \beta)$ and $J_a(\alpha, \beta)$. We found that the following infinite power series was useful:

$$J_a(\alpha, \beta) = \left(\log a - \frac{1}{\alpha}\right) I_a(\alpha, \beta) + \frac{a^\alpha (1-a)^\beta}{\alpha B(\alpha, \beta)}$$
$$\cdot \left\{\left(\frac{1}{\alpha+\beta} - \frac{1}{\alpha+1}\right) \frac{\alpha+\beta}{\alpha+1} a + \left(\frac{1}{\alpha+\beta} + \frac{1}{\alpha+\beta+1}\right.\right.$$
$$\left.\left. - \frac{1}{\alpha+1} - \frac{1}{\alpha+2}\right) \frac{(\alpha+\beta)(\alpha+\beta+1)}{(\alpha+1)(\alpha+2)} a^2 + \cdots \right\} .$$

This is obtained by differentiating the following power series of incomplete beta function (Abramowitz and Stegun (1964), p. 944) with respect to α

$$\int_0^a t^{\alpha-1}(1-t)^{\beta-1} dt$$
$$= \frac{a^\alpha (1-a)^\beta}{\alpha} \left\{1 + \frac{\alpha+\beta}{\alpha+1} a + \frac{(\alpha+\beta)(\alpha+\beta+1)}{(\alpha+1)(\alpha+2)} a^2 + \cdots \right\} .$$

Tables 1 and 2 give the values of risk of the improved estimator $d^*(X, S)$ defined by (1.1) when $p=1$ based on Corollary 2.1 and when $r=1$ based on

Table 1. Risk of $d^*(X, S)$ when $n=5$ (upper) and $n=10$ (lower) for $p=1$.

r	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
1	.207 (3.0%)	.207	.208	.211
	.102 (1.7%)	.102	.102	.103
2	.199 (6.7%)	.200	.201	.206
	.099 (4.1%)	.099	.100	.101
3	.192 (9.8%)	.193	.195	.202
	.097 (6.5%)	.097	.098	.100
5	.183 (14.4%)	.183	.185	.193
	.093 (10.4%)	.093	.094	.097
10	.169 (20.7%)	.169	.171	.177
	.086 (17.2%)	.086	.087	.090

Table 2. Risk of $d^*(X, S)$ when $n=10$ and $r=1$.

p	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
2	.213 (2.4%)	.214	.214	.216
3	.339 (2.8%)	.339	.340	.344
4	.483 (3.2%)	.483	.484	.489
5	.651 (3.5%)	.651	.653	.659
6	.853 (3.9%)	.854	.855	.863
7	1.108 (4.3%)	1.109	1.111	1.120
8	1.454 (4.8%)	1.454	1.456	1.467
9	1.989 (5.5%)	1.990	1.992	2.004
10	3.164 (6.2%)	3.164	3.167	3.179

Corollary 2.2. The maximum rate of reduction of risk $100\times\{R(d)-R(d^*, 0)\}/R(d)$ is shown in the parentheses. We can see from Table 1 that the rate of reduction increases as r increases and that it decreases as $\lambda=\mathrm{tr}\Lambda$ increases. The maximum rate of reduction is 3.0% when $n=5$ and $p=r=1$, which is compared with 1.7% under the squared loss in Sugiura and Konno (1987). From Table 2, we can see that the rate of reduction increases very slowly as p increases with fixed n and $r=1$. This tendency can not be seen under squared loss in Sugiura and Konno (1987).

To obtain the risk of $d^*(X, S)$ when $p=2$ or $r=2$, based on Theorem 2.1, we made use of computer program for zonal polynomials due to Sugiyama (1979), giving Tables 3 and 4. From Table 3 we can see that the risk is monotonically increasing with respect to each latent root of Λ.

Table 4 shows the increase of the maximum rate of reduction for increasing r or increasing p. The tendency is the same as that in Tables 1 and 2.

Corresponding to Tables 1 and 2, the risk of the improved estimator $e^*(X, S)$ defined in (1.2) is shown in Tables 5 and 6. They are computed by

Table 3. Risk of $d^*(X, S)$ when $p=r=2$ and $n=10$ for $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2)$.

λ_2	$\lambda_1=0$	$\lambda_1=1$	$\lambda_1=2$	$\lambda_1=5$	$\lambda_1=10$
0	.207 (5.2%)	.208	209	.212	.217
1		.209	.210	.213	.217
2			.211	.214	.217
5				.217	.218
10					.218

Table 4. Risk of $d^*(X, S)$ at $\Lambda=0$ when $n=10$ and the maximum rate of reduction of risk $100\times\{R(d)-R(d^*, 0)\}/R(d)$.

p	$r=2$	r	$p=2$
1	.0990 (4.1%)	3	.2017 (7.7%)
2	.2071 (5.2%)	5	.1928 (11.8%)
3	.3284 (5.8%)	10	.1787 (18.2%)
4	.4668 (6.4%)	20	.1710 (21.8%)
5	.6282 (6.8%)		
6	.8225 (7.3%)		
7	1.0677 (7.8%)		
8	1.3983 (8.4%)		
9	1.9126 (9.1%)		
10	3.0416 (9.9%)		

Table 5. Risk of $e^*(X, S)$ when $n=5$ (upper) and $n=10$ (lower) for $p=1$.

r	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
1	.281 (5.7%)	.282	.284	.291
	.117 (2.3%)	.117	.118	.119
2	.261 (12.3%)	.263	.266	.279
	.113 (5.6%)	.113	.114	.117
3	.246 (17.5%)	.247	.251	.267
	.109 (8.8%)	.110	.111	.114
5	.224 (24.8%)	.226	.229	.246
	.103 (14.1%)	.103	.105	.109
10	.196 (34.2%)	.197	.199	.212
	.092 (23.2%)	.092	.093	.099

Corollaries 3.1 and 3.2. We can see that the risk of $e^*(X, S)$ has the same tendency as that of $d^*(X, S)$. Each maximum rate of reduction of risk shown in the parentheses is higher than that of $d^*(X, S)$. However it is not as much as under the squared loss, except for the case of larger p. For instance the maximum rate of reduction for $e^*(X, S)$ under entropy loss is 5.7% when $n=5$

Table 6. Risk of $e^*(X, S)$ when $n=10$ and $r=1$.

p	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
2	.247 (3.4%)	.248	.249	.252
3	.396 (4.2%)	.397	.398	.404
4	.570 (5.0%)	.571	.573	.582
5	.781 (5.9%)	.782	.785	.797
6	1.046 (7.2%)	1.048	1.051	1.067
7	1.407 (9.3%)	1.408	1.413	1.434
8	1.969 (13.6%)	1.971	1.977	2.006

Table 7. Risk of $e^*(X, S)$ when $p=r=2$ and $n=10$ for $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2)$.

λ_2	$\lambda_1=0$	$\lambda_1=1$	$\lambda_1=2$	$\lambda_1=5$	$\lambda_1=10$
0	.237 (7.5%)	.237	.239	.245	.252
1		.239	.241	.247	.253
2			.243	.249	.254
5				.252	.255
10					.256

Table 8. Risk of $e^*(X, S)$ at $\Lambda=0$ when $n=10$ and the maximum rate of reduction of risk $100\times\{R(e)-R(e^*, 0)\}/R(e)$.

p	$r=2$	r	$p=2$
1	.1131 (5.6%)	1	.2473 (3.4%)
2	.2367 (7.5%)	2	.2367 (7.5%)
3	.3766 (8.9%)	3	.2273 (11.2%)
4	.5382 (10.3%)	5	.2117 (17.3%)
5	.7301 (12.0%)	10	.1874 (26.8%)
6	.9661 (14.2%)	20	.1619 (36.7%)
7	1.2770 (17.6%)		
8	1.7308 (24.1%)		

and $p=1$, but corresponding value under squared loss in Sugiura and Konno (1987) is 12.5%.

Corresponding to Tables 3 and 4, the risk of $e^*(X, S)$ for the generalized precision is computed by Theorem 3.1 and is given in Tables 7 and 8. The reduction of risk is larger than that of $d^*(X, S)$.

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