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SECOND ORDER ASYMPTOTIC OPTIMALITY OF ESTIMATORS FOR A DENSITY WITH FINITE CUSPS

MASAFUMI AKAHIRA*

Department of Mathematics, University of Electro-Communications, Chofu, Tokyo 182, Japan

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Abstract. We consider i.i.d. samples from a continuous density with finite cusps. Then we obtain the bound for the second order asymptotic distribution of all asymptotically median unbiased estimators. Further we get the second order asymptotic distribution of a bias-adjusted maximum likelihood estimator, and we see that it is not generally second order asymptotically efficient.

Key words and phrases: Second order asymptotically median unbiased estimator, second order asymptotic distribution, Edgeworth expansion, maximum likelihood estimator.

1. Introduction

Recently higher order asymptotic efficiency of estimators has been studied by Pfanzagl and Wefelmeyer (1978), Ghosh *et al.* (1980), and Akahira and Takeuchi (1981) among others, under suitable regularity conditions.

In non-regular situations where the regularity conditions do not necessarily hold, asymptotic optimality has been discussed by Weiss and Wolfowitz (1968), Prakasa Rao (1968), Akahira and Takeuchi (1981, 1985), Ibragimov and Has'minskii (1981), Jurečková (1981), Akahira (1982, 1987), Takeuchi and Akahira (1983), Antoch (1984), Pfanzagl and Wefelmeyer (1985), and others, in particular cases.

In this paper, we have independently and indentically distributed random variables according to a continuous density, with finite cusps, which includes both of regular and non-regular features. We shall obtain the bound for the second order asymptotic distribution of all second order asymptotically median unbiased estimators. Further, we shall get the second order asymptotic distribution of a bias-adjusted maximum likelihood estimator, and see that it is not generally second order asymptotically efficient.

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*Now at Institute of Mathematics, University of Tsukuba, Ibaraki 305, Japan.

2. Notations and assumptions

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independently and identically distributed (i.i.d.) real random variables with a density function $f(x, \theta)$ with respect to the Lebesgue measure. We assume that θ is a real-valued parameter and consider a location parameter family. Then we denote $f(x, \theta)$ by $f(x-\theta)$. We assume the following conditions.

(A.1) $f(x)$ is continuous in $x \in R^1$, and for m points $s_1, \ldots, s_m$, there exist $\lim_{x \to s_j \pm 0} f'(x)$ and $\lim_{x \to s_j \pm 0} f''(x)$ $(j=1, \ldots, m)$, for other all x, $f(x)$ is three times differentiable with respect to x, and $f(x) > 0$ for all x.

(A.2) The amount I of Fisher information is positive and finite, i.e.,

$$0 < I = E_0\left[\left\{ \frac{d \log f(X)}{dx} \right\}^2\right] = \int_{-\infty}^{\infty} \left\{ \frac{d \log f(x)}{dx} \right\}^2 f(x) dx < \infty .$$

(A.3) There exist $J = E_0[l^{(1)}(X) l^{(2)}(X)]$, $K = E_0[\{l^{(1)}(X)\}^3]$ and $L_3 = E_0[l^{(3)}(X)]$, where $l^{(i)}(x) = d^i \log f(x) / dx^i$ $(i=1, 2, 3)$.

For example we see that the piecewise exponential distributions satisfy the conditions (A.1) to (A.3).

We denote by $P_{\theta,n}$ the n-fold direct products of probability measure P_θ with the density $f(x-\theta)$.

An estimator $\hat{\theta}_n$ is called second order asymptotically median unbiased (AMU) estimator if for any $\vartheta \in R^1$, there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \sqrt{n} \left| P_{\theta,n}\{\hat{\theta}_n \leq \theta\} - \frac{1}{2} \right| = 0 ;$$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \sqrt{n} \left| P_{\theta,n}\{\hat{\theta}_n \geq \theta\} - \frac{1}{2} \right| = 0 .$$

For $\hat{\theta}_n$ second order AMU, $G_0(t, \theta) + n^{-1/2} G_1(t, \theta)$ is defined to be the second order asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ (or $\hat{\theta}_n$ for short) if for each $t \in R^1$ and each $\theta \in R^1$

$$\lim_{n\to\infty} \sqrt{n} \left| P_{\theta,n}\{\sqrt{n}(\hat{\theta}_n - \theta) \leq t\} - G_0(t, \theta) - n^{-1/2} G_1(t, \theta) \right| = 0 .$$

A second order AMU estimator $\hat{\theta}_n^*$ is called to the second order asymptotically efficient if its second order asymptotic distribution attains uniformly the bound $F_n^*(t, \theta)$ for the second order asymptotic distribution of all second order AMU estimators in the sense that for any second order AMU estimator $\hat{\theta}_n$

$$\overline{\lim_{n\to\infty}}\ \sqrt{n}[P_{\theta,n}\{\sqrt{n}(\hat{\theta}_n-\theta)\le t\}-F_n^*(t,\theta)]\le 0$$

$$\text{for all } t>0 \quad \text{and all } \theta\in R^1\ ,$$

$$\overline{\lim_{n\to\infty}}\ \sqrt{n}[P_{\theta,n}\{\sqrt{n}(\hat{\theta}_n-\theta)\le t\}-F_n^*(t,\theta)]\ge 0$$

$$\text{for all } t<0 \quad \text{and all } \theta\in R^1\ ,$$

(Akahira and Takeuchi (1981)).

In the definition, it may not be necessary to assume that the bound $F_n^*(t, \theta)$ admits an expansion in powers of the order $n^{-1/2}$. The bound $F_n^*(t, \theta)$ is sharp in the sense that for each real number r there exists a second order AMU estimator $\hat{\theta}_n^r$ attaining the bound at $t=r$. Indeed, we consider a discretized likelihood estimator (DLE) which is defined to be a solution $\theta=\hat{\theta}_n^r$ of the discretized likelihood equation

$$\sum_{i=1}^n \log f(X_i, \theta-rn^{-1/2})-\sum_{i=1}^n \log f(X_i,\theta)=a_n(\theta, r)\ ,$$

where $a_n(\theta, r)$ is determined so that $\hat{\theta}_n^r$ is second order AMU. Then it is shown by Akahira and Takeuchi (1979, 1981) that the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n^r-\theta)$ attains the bound for the asymptotic distribution of all second order AMU estimators at r, up to the second order, i.e., the order $n^{-1/2}$.

3. The bound for the second order asymptotic distributions

In this section we shall obtain the bound for the second order asymptotic distribution of all second order AMU estimators using the log-likelihood ratio test statistic.

Without loss of generality we assume that the true parameter θ_0 is equal to zero.

In order to obtain the bound we consider the problem of testing hypothesis H^+: $\theta=\Delta(\Delta>0)$ against A: $\theta=0$. Then the log-likelihood ratio test statistic $\log L$ is given by

$$\begin{aligned}(3.1)\quad \log L &= \log\prod_{i=1}^n \frac{f(X_i)}{f(X_i-\Delta)}=\sum_{i=1}^n\{\log f(X_i)-\log f(X_i-\Delta)\}\\ &= \sum_{i=1}^n\Bigg[\frac{f'(X_i)}{f(X_i)}\Delta\chi_{\left(\bigcup_{j=1}^m I_j(\Delta)\right)^c}(X_i)\\ &\quad -\frac{f(X_i)f''(X_i)-f'(X_i)^2}{2f(X_i)^2}\Delta^2\chi_{\left(\bigcup_{j=1}^m I_j(\Delta)\right)^c}(X_i)\\ &\quad +\frac{\Delta^3}{6}l^{(3)}(X_i)\,\chi_{\left(\bigcup_{j=1}^m I_j(\Delta)\right)^c}(X_i)+\sum_{j=1}^m\Bigg\{\frac{f'(s_j+0)}{f(s_j)}(X_i-s_j)\end{aligned}$$

$$
\begin{aligned}
&+\frac{f'(s_j-0)}{f(s_j)}(s_j-(X_i-\Delta))\Big\}\chi_{I_j(\Delta)}(X_i)\\
&-\frac{1}{2}\sum_{j=1}^{m}\{(c_{j-}(X_i-\Delta-s_j)^2\\
&-c_{j+}(X_i-s_j)^2)\chi_{I_j(\Delta)}(X_i)\}+o_p(\Delta^3)\Big]\\
=&\sum_{i=1}^{n}\Big[\Delta\frac{f'(X_i)}{f(X_i)}\Big(1-\sum_{j=1}^{m}\chi_{I_j(\Delta)}(X_i)\Big)\\
&-\frac{\Delta^2}{2}\cdot\frac{f(X_i)f''(X_i)-f'(X_i)^2}{f(X_i)^2}\cdot\Big(1-\sum_{j=1}^{m}\chi_{I_j(\Delta)}(X_i)\Big)\\
&+\frac{\Delta^3}{6}l^{(3)}(X_i)\Big(1-\sum_{j=1}^{m}\chi_{I_j(\Delta)}(X_i)\Big)\\
&+\sum_{j=1}^{m}\Big\{\frac{b_{j+}-b_{j-}}{a_j}(X_i-s_j)+\Delta\frac{b_{j-}}{a_j}\Big\}\chi_{I_j(\Delta)}(X_i)\\
&-\frac{1}{2}\sum_{j=1}^{m}\{c_{j-}(X_i-s_j-\Delta)^2-c_{j+}(X_i-s_j)^2\}\chi_{I_j(\Delta)}(X_i)+o_p(\Delta^3)\Big]\\
=&\sum_{i=1}^{n}Z_i(\Delta)+o_p(n\Delta^3)\quad\text{(say)},
\end{aligned}
$$

where for each $j=1,\ldots,m$, $\chi_{I_j(\Delta)}(x)$ is the indicator of the interval $I_j(\Delta)=(s_j, s_j+\Delta)$, $I_j^c(\Delta)$ denotes the complement of $I_j(\Delta)$, $a_j=f(s_j)$, $b_{j\pm}=\lim_{x\to s_j\pm 0}f'(x)$ and $c_{j\pm}=\lim_{x\to s_j\pm 0}\{f(x)f''(x)-f'(x)\}^2/f(x)^2$ in which the signs $+$ and $-$ should be read consistently.

In the following lemma we obtain the asymptotic mean, variance and third order cumulant of $\sum_{i=1}^{n}Z_i(\Delta)$ up to the order Δ^3, under the distributions $P_{0,n}$ and $P_{\Delta,n}$.

LEMMA 3.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. Then the asymptotic mean, variance and third order cumulant of* $\sum_{i=1}^{n}Z_i(\Delta)$ *with* $\Delta=tn^{-1/2}$ $(t>0)$, *under the distributions* $P_{0,n}$ *and* $P_{\Delta,n}$, *are given as follows: Under the distribution* $P_{0,n}$,

$$
\begin{aligned}
(3.2)\qquad E_0\Big[\sum_{i=1}^{n}Z_i(\Delta)\Big]=\frac{t^2I}{2}+\frac{t^3}{6\sqrt{n}}\Big[L_3-\sum_{j=1}^{m}\Big\{\frac{1}{a_j}b_{j+}(b_{j+}-b_{j-})\\
+a_j(c_{j-}-c_{j+})\Big\}\Big]+o\Big(\frac{1}{\sqrt{n}}\Big);
\end{aligned}
$$

(3.3) $$V_0\left(\sum_{i=1}^{n} Z_i(\Delta)\right)=t^2 I-\frac{t^3}{\sqrt{n}}\left\{J+\sum_{j=1}^{m}\frac{1}{3a_j}(b_{j+}-b_{j-})(2b_{j+}+b_{j-})\right\}+o\left(\frac{1}{\sqrt{n}}\right);$$

(3.4) $$\kappa_0\left(\sum_{i=1}^{n} Z_i(\Delta)\right)=\frac{t^3K}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right),$$

respectively, and under the distribution $P_{\Delta,n}$,

(3.5) $$E_\Delta\left[\sum_{i=1}^{n} Z_i(\Delta)\right]=-\frac{t^2 I}{2}+\frac{t^3}{6\sqrt{n}}\left[L_3-\sum_{j=1}^{m}\left\{\frac{1}{a_j}b_{j-}(b_{j+}-b_{j-})-a_j(c_{j+}-c_{j-})\right\}\right]+o\left(\frac{1}{\sqrt{n}}\right);$$

(3.6) $$V_\Delta\left(\sum_{i=1}^{n} Z_i(\Delta)\right)=t^2 I+\frac{t^3}{\sqrt{n}}\left\{J+\sum_{j=1}^{m}\frac{1}{3a_j}(b_{j+}-b_{j-})(b_{j+}+2b_{j-})\right\}+o\left(\frac{1}{\sqrt{n}}\right);$$

(3.7) $$\kappa_\Delta\left(\sum_{i=1}^{n} Z_i(\Delta)\right)=\frac{t^3K}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right),$$

respectively.

PROOF. First we have from (3.1)

$$\begin{aligned}E_0[Z_i(\Delta)]=&\ \Delta\left(\int_{-\infty}^{\infty}-\sum_{j=1}^{m}\int_{s_j}^{s_j+\Delta}\right)f'(x)dx-\frac{\Delta^2}{2}\left(\int_{-\infty}^{\infty}-\sum_{j=1}^{m}\int_{s_j}^{s_j+\Delta}\right)\\&\cdot\frac{f(x)f''(x)-f'(x)^2}{f(x)}dx+\frac{\Delta^3}{6}L_3+\sum_{j=1}^{m}\left\{\frac{b_{j+}-b_{j-}}{a_j}\right.\\&\left.\cdot\int_{s_j}^{s_j+\Delta}(x-s_j)f(x)dx+\Delta\frac{b_{j-}}{a_j}\int_{s_j}^{s_j+\Delta}f(x)dx\right\}\\&-\frac{1}{2}\sum_{j=1}^{m}\int_{s_j}^{s_j+\Delta}\{c_{j-}(x-s_j-\Delta)^2-c_{j+}(x-s_j)^2\}f(x)dx\\&+o(\Delta^3).\end{aligned}$$

In order to obtain $E_0[Z_i(\Delta)]$ $(i=1,\dots,n)$ up to the order Δ^2, we get the following: For each $j=1,\dots, m$

$$(3.8)\quad \int_{s_j}^{s_j+\Delta} f'(x)dx = f(s_j+\Delta) - f(s_j) = \Delta b_{j+} + \frac{\Delta^2}{2} f''(s_j+0) + o(\Delta^2)\ ;$$

$$(3.9)\quad \int_{s_j}^{s_j+\Delta} (x-s_j)f(x)dx = \int_0^{\Delta} xf(x+s_j)dx = \frac{\Delta^2}{2} a_j + \frac{\Delta^3}{3} b_{j+} + o(\Delta^3)\ ;$$

$$(3.10)\quad \int_{s_j}^{s_j+\Delta} f(x)dx = \int_0^{\Delta} f(x+s_j)dx = \Delta a_j + \frac{\Delta^2}{2} b_{j+} + o(\Delta^2)\ ;$$

$$(3.11)\quad \int_{s_j}^{s_j+\Delta} \frac{\{f'(x)\}^2}{f(x)} dx = \int_0^{\Delta} \frac{\{f'(x+s_j)\}^2}{f(x+s_j)} dx = \Delta \frac{b_{j+}^2}{a_j} + o(\Delta)\ ;$$

$$(3.12)\quad \int_{s_j}^{s_j+\Delta} (x-s_j)^2 f(x)dx = \int_0^{\Delta} x^2 f(x+s_j)dx = \frac{\Delta^3}{3} a_j + o(\Delta^3)\ ;$$

$$(3.13)\quad \int_{s_j}^{s_j+\Delta} (x-s_j-\Delta)^2 f(x)dx = \int_0^{\Delta} (x-\Delta)^2 f(x+s_j)dx = \frac{\Delta^3}{3} a_j + o(\Delta^3)$$

From (3.8) to (3.13) we have

$$(3.14)\quad \begin{aligned} E_0[Z_i(\Delta)] &= \Delta \sum_{j=1}^m \left\{ -\Delta b_{j+} - \frac{\Delta^2}{2} f''(s_j+0) \right\} - \frac{\Delta^2}{2} \left\{ (\alpha - I) \right. \\ &\quad \left. - \Delta \sum_{j=1}^m f''(s_j+0) + \Delta \sum_{j=1}^m \frac{b_{j+}^2}{a_j} \right\} + \frac{\Delta^3}{6} L_3 \\ &\quad + \sum_{j=1}^m \frac{b_{j+} - b_{j-}}{a_j} \left(\frac{\Delta^2}{2} a_j + \frac{\Delta^3}{3} b_{j+} \right) + \Delta \sum_{j=1}^m \frac{b_{j-}}{a_j} \\ &\quad \cdot \left(\Delta a_j + \frac{\Delta^2}{2} b_{j+} \right) - \frac{\Delta^3}{6} \sum_{j=1}^m a_j (c_{j-} - c_{j+}) + o(\Delta^3) \\ &= -\frac{\Delta^2}{2} \left\{ (\alpha - I) + \sum_{j=1}^m (b_{j+} - b_{j-}) \right\} + \frac{\Delta^3}{6} \left[L_3 \right. \\ &\quad \left. - \sum_{j=1}^m \left\{ \frac{b_{j+}(b_{j+} - b_{j-})}{a_j} + a_j (c_{j-} - c_{j+}) \right\} \right] + o(\Delta^3)\ , \end{aligned}$$

where $\alpha = \int_{-\infty}^{\infty} f''(x)dx$ and $I = \int_{-\infty}^{\infty} [\{f'(x)\}^2 / f(x)]dx$. Since

$$(3.15)\qquad \alpha = \int_{-\infty}^{\infty} f''(x)dx = \sum_{j=1}^m (b_{j-} - b_{j+})\ ,$$

and $\Delta = tn^{-1/2}$, it follows from (3.14) that (3.2) holds. Second we obtain from (3.1)

$$(3.16)\quad E_0[Z_i^2(\Delta)]$$
$$= \Delta^2 E_0\left[\left\{\frac{f'(X)}{f(X)}\right\}^2\left(1 - \sum_{j=1}^{m}\chi_{I_j(\Delta)}(X)\right)\right]$$
$$+ 2\Delta \sum_{k=1}^{m} E_0\left[\frac{f'(X)}{f(X)}\left\{\frac{b_{k+} - b_{k-}}{a_k}(X - s_k) + \Delta\frac{b_{k-}}{a_k}\right\}\chi_{I_k(\Delta)}(X)\right]$$
$$- 2\Delta \sum_{j=1}^{m}\sum_{k=1}^{m} E_0\left[\frac{f'(X)}{f(X)}\left\{\frac{b_{k+} - b_{k-}}{a_k}(X - s_k) + \Delta\frac{b_{k-}}{a_k}\right\}\chi_{I_j(\Delta)}(X)\right.$$
$$\left.\cdot\chi_{I_k(\Delta)}(X)\right] - \Delta^3 E_0\left[\frac{f'(X)(f(X)f''(X) - f'(X)^2)}{f(X)^3}\right.$$
$$\left.\cdot\left\{1 - \sum_{j=1}^{m}\chi_{I_j(\Delta)}(X)\right\}\right] + \sum_{j=1}^{m} E_0\left[\left\{\frac{b_{j+} - b_{j-}}{a_j}(X - s_j)\right.\right.$$
$$\left.\left. + \Delta\frac{b_{j-}}{a_j}\right\}^2\chi_{I_j(\Delta)}(X)\right] + o(\Delta^3)\,.$$

From (3.11) we have for each $j = 1, \ldots, m$

$$(3.17)\qquad E_0\left[\left\{\frac{f'(X)}{f(X)}\right\}^2\chi_{I_j(\Delta)}(X)\right] = \Delta\frac{b_{j+}^2}{a_j} + o(\Delta)\,.$$

We also have

$$(3.18)\qquad E_0\left[\frac{f'(x)\{f(x)f''(x) - f'(x)^2\}}{f(x)^3}\left(1 - \sum_{j=1}^{m}\chi_{I_j(\Delta)}(X)\right)\right]$$
$$= E_0[l^{(1)}(X)\, l^{(2)}(X)] + o(1) = J + o(1)\,.$$

Since for each $j = 1, \ldots, m$

$$E_0\left[\left\{\frac{b_{j+} - b_{j-}}{a_j}(X - s_j) + \Delta\frac{b_{j-}}{a_j}\right\}^2\chi_{I_j(\Delta)}(X)\right]$$
$$= \frac{\Delta^3}{3a_j}\{(b_{j+} - b_{j-})^2 + 3b_{j+}b_{j-}\} + o(\Delta^3)\,,$$

and $\Delta = tn^{-1/2}$, it follows from (3.16) to (3.18) that (3.3) holds. Third we have from (3.1)

$$(3.19)\quad E_0[Z_i^3(\Delta)] = \Delta^3 E_0\left[\left\{\sum_{j=1}^m \frac{f'(X)}{f(X)}\left(1-\sum_{j=1}^m \chi_{I_j(\Delta)}(X)\right)\right\}^3\right] + o(\Delta^3)$$

$$= \Delta^3 E_0\left[\left\{\frac{f'(X)}{f(X)}\right\}^3\left(1-\sum_{j=1}^m \chi_{I_j(\Delta)}(X)\right)\right] + o(\Delta^3)\ .$$

Since for each $j=1,\dots,m$

$$(3.20)\quad \int_{s_j}^{s_j+\Delta} \frac{f'(x)^3}{f(x)^2}\,dx = \Delta\,\frac{b_{j+}^3}{a_j^2} + o(\Delta)\ ,$$

it follows that

$$(3.21)\quad E_0[Z_i^3(\Delta)] = \Delta^3 E_0\left[\left\{\frac{f'(X)}{f(X)}\right\}^3\left\{1-\sum_{j=1}^m \chi_{I_j(\Delta)}(X)\right\}\right] + o(\Delta^3)$$

$$= \Delta^3 K - \Delta^4 \sum_{j=1}^m \frac{b_{j+}^3}{a_j^2} + o(\Delta^4)\ ,$$

which implies (3.4) since $\Delta = tn^{-1/2}$.

In a similar way to the above, we obtain the asymptotic mean (3.5), variance (3.6) and third order cumulant (3.7) of $\sum_{i=1}^n Z_i(\Delta)$, under the distribution $P_{\Delta,n}$. Thus we complete the proof.

In order to obtain the bound for the second order asymptotic distribution of all second order AMU estimators, we need the following.

LEMMA 3.2. *Assume that the asymptotic mean, variance and third order cumulant of* $\sum_{i=1}^n Z_i(\Delta)$ *with* $\Delta = tn^{-1/2}$, *under the distributions* $P_{\theta,n}(\theta=0,\Delta)$, *are given by the following form.*

$$E_\theta\left[\sum_{i=1}^n Z_i(\Delta)\right] = \mu(t,\theta) + \frac{1}{\sqrt{n}}c_1(t,\theta) + o\left(\frac{1}{\sqrt{n}}\right);$$

$$V_\theta\left(\sum_{i=1}^n Z_i(\Delta)\right) = v^2(t,\theta) + \frac{1}{\sqrt{n}}c_2(t,\theta) + o\left(\frac{1}{\sqrt{n}}\right);$$

$$\kappa_\theta\left(\sum_{i=1}^n Z_i(\Delta)\right) = \frac{1}{\sqrt{n}}c_3(t,\theta) + o\left(\frac{1}{\sqrt{n}}\right).$$

Then

$$a=\mu(t,\theta)+\frac{1}{\sqrt{n}}c_1(t,\theta)-\frac{1}{6\sqrt{n}}\frac{c_3(t,\theta)}{v^2(t,\theta)}+o\left(\frac{1}{\sqrt{n}}\right) \tag{3.22}$$

if and only if

$$P_{\theta,n}\left\{\sum_{i=1}^{n} Z_i(\Delta)\leq a\right\}=\frac{1}{2}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.23}$$

The proof is essentially given in Akahira and Takeuchi (1981, pp. 132, 133).

In the following theorem we obtain the bound for asymptotic distribution of all second order AMU estimators.

THEOREM 3.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. Then the bound for the second order asymptotic distribution of all second order AMU estimators* $\hat{\theta}_n$ *is given as follows.*

$$\overline{\lim_{n\to\infty}}\sqrt{n}\left[P_{\theta,n}\{\sqrt{In}\,(\hat{\theta}_n-\theta)\leq t\}-\Phi(t)-\frac{t^2\phi(t)}{6I^{3/2}\sqrt{n}}\cdot\left\{3J+K+\sum_{j=1}^{m}\frac{(b_{j+}-b_{j-})(b_{j+}+2b_{j-})}{a_j}\right\}\right]\leq 0, \tag{3.24}$$

for all $t>0$;

$$\underline{\lim_{n\to\infty}}\sqrt{n}\left[P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n-\theta)\leq t\}-\Phi(t)-\frac{t^2\phi(t)}{6I^{3/2}\sqrt{n}}\cdot\left\{3J+K+\sum_{j=1}^{m}\frac{(b_{j+}-b_{j-})(2b_{j+}+b_{j-})}{a_j}\right\}\right]\geq 0, \tag{3.25}$$

for all $t<0$, *where* $\Phi(t)$ *and* $\phi(t)$ *denote the standard normal distribution function and its density function, respectively.*

Remark 3.1. In the third terms $\{\cdots\}$ of (3.24) and (3.25), the first term $3J+K$ and the remainder correspond to the regular part and the non-regular one, i.e., the cusps of the density $f(x)$, respectively. If all cusps of $f(x)$ vanish, that is, $b_{j+}=b_{j-}$ $(j=1,\ldots,m)$, then the terms $\{\cdots\}$ consist of only $3J+K$, which coincides with the fact by Akahira and Takeuchi (1981).

Remark 3.2. A result on the validity of formal Edgeworth expansions for the sum of i.i.d. random variables was given by Bhattacharya and Ghosh (1978). Here it is not applicable, since the log-likelihood ratio is approximated by a sum $\sum_{i=1}^{n} Z_i(\Delta)$, with $\Delta=tn^{-1/2}$, of i.i.d. functions depending on n. In this case, we denote by $\varphi_n(u)$ the characteristic function of $Z_i(\Delta)$. If for any $\xi>0$, there exists a constant d_ξ such that

$$\sup_{|u|\geq\xi} |\varphi_n(u)| \leq d_\xi < 1 ,$$

then the Edgeworth expansion for sums of i.i.d. random variables is valid.

PROOF OF THEOREM 3.1. We consider the case when $t>0$. In order to choose a such that

$$P_{\Delta,n}\left\{\sum_{i=1}^{n} Z_i(\Delta) \leq a\right\} = \frac{1}{2} + o\left(\frac{1}{\sqrt{n}}\right), \tag{3.26}$$

we have by Lemmas 3.1 and 3.2

$$a = -\frac{t^2 I}{2} + \frac{t^3}{6\sqrt{n}}\left[L_3 - \sum_{j=1}^{m}\left\{\frac{b_{j-}(b_{j+} - b_{j-})}{a_j} - a_j(c_{j+} - c_{j-})\right\}\right]$$
$$- \frac{tK}{6I\sqrt{n}} ,$$

where $\Delta = tn^{-1/2}$. Since

$$P_{0,n}\left\{\sum_{i=1}^{n} Z_i(\Delta) \geq a\right\} = P_{0,n}\left\{-\left\{\sum_{i=1}^{n} Z_i(\Delta) - t^2 I - a\right\} \leq t^2 I\right\}, \tag{3.27}$$

putting

$$W_n^+ = -\left\{\sum_{i=1}^{n} Z_i(\Delta) - t^2 I - a\right\},$$

we have from Lemma 3.1

$$E_0(W_n^+) = \frac{t^3}{6\sqrt{n}}\sum_{j=1}^{m}\frac{(b_{j+} - b_{j-})^2}{a_j} - \frac{tK}{6I\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right); \tag{3.28}$$

$$V_0(W_n^+) = t^2 I - \frac{t^3}{\sqrt{n}}\left\{J + \sum_{j=1}^{m}\frac{(b_{j+} - b_{j-})(2b_{j+} + b_{j-})}{3a_j}\right\} + o\left(\frac{1}{\sqrt{n}}\right); \tag{3.29}$$

$$\kappa_0(W_n^+) = -\frac{t^3 K}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right). \tag{3.30}$$

We obtain by (3.27) to (3.30) and the Edgeworth expansion

$$(3.31)\quad P_{0,n}\left\{\sum_{i=1}^{n} Z_i(\Delta) \geq a\right\} = P_{0,n}\{W_n^+ \leq It^2\}$$
$$= \Phi(\sqrt{I}\,t) + \frac{t^2}{6\sqrt{In}}\,\phi(\sqrt{I}\,t)\left\{3J + K + \sum_{j=1}^{m} \frac{1}{a_j}(b_{j+} - b_{j-})(b_{j+} + 2b_{j-})\right\} + o\left(\frac{1}{\sqrt{n}}\right).$$

Here, from (3.1), (3.26) and the fundamental lemma of Neyman-Pearson it is noted that a test with the rejection region $\left\{\sum_{i=1}^{n} Z_i(\Delta)\geq a\right\}$ is the most powerful test of level $1/2+o(1/\sqrt{n})$.

Let $\hat{\theta}_n$ be any second order AMU estimator. Putting $A_{\hat{\theta}_n}=\{\sqrt{n}\,\hat{\theta}_n\leq t\}$, we have

$$P_{\Delta,n}(A_{\hat{\theta}_n}) = P_{\Delta,n}\{\hat{\theta}_n \leq tn^{-1/2}\} = \frac{1}{2} + o\left(\frac{1}{\sqrt{n}}\right).$$

Then it is seen that $\chi_{A_{\hat{\theta}_n}}$ of indicators of $A_{\hat{\theta}_n}$ is a test of level $1/2+o(1/\sqrt{n})$. From (3.31) we obtain for any second order AMU estimator $\hat{\theta}_n$

$$P_{0,n}\{\sqrt{n}\hat{\theta}_n \leq t\} \leq P_{0,n}\{W_n^+ \leq It^2\}$$
$$= \Phi(\sqrt{I}\,t) + \frac{t^2}{6\sqrt{In}}\,\phi(\sqrt{I}\,t)\left\{3J + K + \sum_{j=1}^{m} \frac{1}{a_j}(b_{j+} - b_{j-}) \cdot (b_{j+} + 2b_{j-})\right\} + o\left(\frac{1}{\sqrt{n}}\right),$$

for all $t>0$.

Hence we see that the bound for the second order asymptotic distribution of all second order AMU estimators for all $t>0$ is given by (3.24).

In a similar way to the case $t>0$, we obtain the bound (3.25) for all $t<0$. Thus we complete the proof.

COROLLARY 3.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. If*

$$(3.32)\qquad \sum_{j=1}^{m} \frac{b_{j+}^2}{a_j} = \sum_{j=1}^{m} \frac{b_{j-}^2}{a_j},$$

then the bound for the second order asymptotic distribution of all second order AMU estimators $\hat{\theta}_n$ is given by

$$\Phi(t) + \frac{t^2\phi(t)}{6I^{3/2}\sqrt{n}}\left\{3J + K + \sum_{j=1}^{m}\frac{1}{a_j}(b_{j+} - b_{j-})(b_{j+} + 2b_{j-})\operatorname{sgn} t\right\} + o\left(\frac{1}{\sqrt{n}}\right).$$

The proof is omitted since it is straightforward.

In some case of the double exponential distribution, the condition (3.32) is satisfied, as is shown later.

4. The second order asymptotic distribution of a bias-adjusted maximum likelihood estimator

In this section we shall obtain the second order asymptotic distribution of a bias-adjusted maximum likelihood estimator and compare it with the bound obtained in the previous section.

We denote by θ_0 and $\hat{\theta}_{\mathrm{ML}}$ the true parameter and the maximum likelihood estimator (MLE), respectively. It is seen that for each real t, $\hat{\theta}_{\mathrm{ML}} < \theta_0 + tn^{-1/2}$ if and only if $(\partial/\partial\theta)\sum_{i=1}^{n}\log f(X_i - \theta_0 - tn^{-1/2}) < 0$. Without loss of generality we assume that $\theta_0 = 0$. Hence we see that for each real t

$$\hat{\theta}_{\mathrm{ML}} < tn^{-1/2} \quad \text{if and only if} \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} l^{(1)}(X_i - tn^{-1/2}) > 0 \tag{4.1}$$

with probability larger than $1 - o(n^{-1})$. By the Taylor's expansion we have for each $t > 0$

$$\begin{aligned}\sum_{i=1}^{n} l^{(1)}\left(X_i - \frac{t}{\sqrt{n}}\right) = \sum_{i=1}^{n}\Bigg[\Bigg\{&\frac{f'(X_i)}{f(X_i)} - \frac{t}{\sqrt{n}}\cdot\frac{f''(X_i)f(X_i) - f'(X_i)^2}{f(X_i)^2} \\ &+ \frac{t^2}{2n} l^{(3)}(X_i)\Bigg\}\left\{1 - \sum_{j=1}^{m}\chi_{I_j}(X_i)\right\} \\ &+ \sum_{j=1}^{m}\left\{\frac{b_{j-}}{a_j} + \frac{a_j f''(s_j - 0) - b_{j-}^2}{a_j^2}\left(X_i - s_j\right.\right. \\ &\left.\left. - \frac{t}{\sqrt{n}}\right)\right\}\chi_{I_j}(X_i)\Bigg] + o_p(1) \\ = \sum_{i=1}^{n} Y_i + o_p(1) \quad &\text{(say)},\end{aligned}$$

where $I_j = (s_j, s_j + tn^{-1/2})$ $(j = 1, \ldots, m)$. Then we have the following:

LEMMA 4.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. Then the asymptotic mean, variance and third order cumulant of* $\sum_{i=1}^{n} Y_i/\sqrt{n}$ *under the distribution* $P_{0,n}$ *up to the order* $n^{-1/2}$ *are given as follows: For each* $t > 0$

$$E_0\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i\right] = tI + \frac{t^2}{2\sqrt{n}}\left[L_3 - \sum_{j=1}^m \frac{1}{a_j}(b_{j+} - b_{j-})(2b_{j+} + b_{j-}) + \sum_{j=1}^m \{f''(s_j+0) - f''(s_j-0)\}\right] + o\left(\frac{1}{\sqrt{n}}\right);$$

$$V_0\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i\right) = I - \frac{t}{\sqrt{n}}\left\{2J + \sum_{j=1}^m \frac{1}{a_j}(b_{j+}^2 - b_{j-}^2)\right\} + o\left(\frac{1}{\sqrt{n}}\right);$$

$$\kappa_0\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i\right) = \frac{K}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

The proof is omitted since it is similar to that of Lemma 3.1.
From Lemma 4.1 we have the following:

THEOREM 4.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. Let* $\hat{\theta}^*_{\mathrm{ML}}$ *be a bias-adjusted MLE defined by*

$$\hat{\theta}^*_{\mathrm{ML}} = \hat{\theta}_{\mathrm{ML}} - \frac{K}{6nI^2}.$$

Then the second order asymptotic distribution of $\hat{\theta}^*_{\mathrm{ML}}$ *is given by*

$$(4.2)\quad P_{\theta,n}\{\sqrt{In}(\hat{\theta}^*_{\mathrm{ML}} - \theta) \le t\}$$
$$= \begin{cases} \Phi(t) + \dfrac{t^2\phi(t)}{6I^{3/2}\sqrt{n}}\left[6J + K + 3L_3 - 3\sum_{j=1}^m\left\{\dfrac{1}{a_j}b_{j+}(b_{j+} - b_{j-}) - f''(s_j+0) + f''(s_j-0)\right\}\right] + o\left(\dfrac{1}{\sqrt{n}}\right) & \text{for all } t > 0; \\ \Phi(t) + \dfrac{t^2\phi(t)}{6I^{3/2}\sqrt{n}}\left[6J + K + 3L_3 - 3\sum_{j=1}^m\left\{\dfrac{1}{a_j}b_{j-}(b_{j+} - b_{j-}) - f''(s_j+0) + f''(s_j-0)\right\}\right] + o\left(\dfrac{1}{\sqrt{n}}\right) & \text{for all } t < 0. \end{cases}$$

Remark 4.1. In the terms $[\cdots]$ of the right-hand side of (4.2), the first term $6J+K+3L_3$ and the remainder correspond to the regular part and the non-regular one, i.e., the cusps of the density $f(x)$, respectively. If all cusps of $f(x)$ vanish, that is, for each $j=1,\dots,m$, $b_{j+}=b_{j-}$ and $f''(s_j+0)=f''(s_j-0)$, then the term $[\cdots]$ consists of only $6J+K+3L_3$. Since $L_3=E_0[l^{(3)}(X)]=-3J-K$ and $K=-2J$ in the regular and location parameter case, it follows that $6J+K+3L_3=J$ which coincides with the fact by Akahira and Takeuchi (1981).

PROOF OF THEOREM 4.1. From (4.1) and Lemma 4.1 it follows that the Edgeworth expansion of the distribution of the MLE $\hat{\theta}_{\mathrm{ML}}$ is given as follows: For each $t>0$

$$\begin{aligned}
&P_{0,n}\{\sqrt{n}\hat{\theta}_{\mathrm{ML}} \leq t\} \\
&= 1 - P_{0,n}\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i < 0 \right\} \\
&= 1 - \Phi(-\sqrt{I}\, t) + \phi(\sqrt{I}\, t)\Bigg[- \frac{K}{6I^{3/2}\sqrt{n}} \\
&\quad + \frac{t^2}{2\sqrt{In}} \left\{ 2J + \frac{K}{3} + L_3 + \sum_{j=1}^{m} (f''(s_j+0) - f''(s_j-0)) \right\} \\
&\quad + \frac{t^2}{2\sqrt{In}} \sum_{j=1}^{m} \Bigg\{ - \frac{1}{a_j} (b_{j+} - b_{j-})(2b_{j+} + b_{j-}) \\
&\quad + \frac{1}{a_j} (b_{j+}^2 - b_{j-}^2) \Bigg\} \Bigg] + o\left(\frac{1}{\sqrt{n}} \right) \\
&= \Phi(\sqrt{I}\, t) + \phi(\sqrt{I}\, t)\Bigg[- \frac{K}{6I^{3/2}\sqrt{n}} + \frac{t^2}{6\sqrt{In}} (6J + K + 3L_3) \\
&\quad - \frac{t^2}{2\sqrt{In}} \sum_{j=1}^{m} \left\{ \frac{1}{a_j} b_{j+}(b_{j+} - b_{j-}) - f''(s_j+0) + f''(s_j-0) \right\} \Bigg] \\
&\quad + o\left(\frac{1}{\sqrt{n}} \right).
\end{aligned}$$

Hence we have for each $t>0$

$$\begin{aligned}
&P_{0,n}\{\sqrt{In}\hat{\theta}^*_{\mathrm{ML}} \leq t\} \\
&= P_{0,n}\left\{ \sqrt{In}\hat{\theta}^*_{\mathrm{ML}} \leq t + \frac{K}{6I^{3/2}\sqrt{n}} \right\} \\
&= \Phi(t) + \frac{t^2\phi(t)}{6I^{3/2}\sqrt{n}} \Bigg[6J + K + 3L_3 - 3 \sum_{j=1}^{m} \Bigg\{ \frac{1}{a_j} b_{j+}(b_{j+} - b_{j-}) \\
&\quad - f''(s_j+0) + f''(s_j-0) \Bigg\} \Bigg] + o\left(\frac{1}{\sqrt{n}} \right).
\end{aligned}$$

In a similar way to the case $t>0$, we obtain (4.2) for all $t<0$. Thus we complete the proof.

COROLLARY 4.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. If*

$$\sum_{j=1}^{m}\left\{\frac{b_{j+}^2}{a_j}-2f''(s_j+0)\right\}=\sum_{j=1}^{m}\left\{\frac{b_{j-}^2}{a_j}-2f''(s_j-0)\right\},$$

then the second order asymptotic distribution of the bias-adjusted MLE $\hat{\theta}_{\mathrm{ML}}^*$ is given by

$$P_{\theta,n}\{\sqrt{In}(\hat{\theta}_{\mathrm{ML}}^*-\theta)\le t\}$$
$$=\Phi(t)+\frac{t^2\phi(t)}{6I^{3/2}\sqrt{n}}\left[6J+K+3L_3-3\sum_{j=1}^{m}\left\{\frac{1}{a_j}b_{j+}(b_{j+}-b_{j-})\right.\right.$$
$$\left.\left.-f''(s_j+0)+f''(s_j-0)\right\}\operatorname{sgn} t\right]+o\left(\frac{1}{\sqrt{n}}\right). \tag{4.3}$$

The proof is omitted since it is straightforward.

Remark 4.2. Comparing the second order asymptotic distribution (4.2) or (4.3) of $\hat{\theta}_{\mathrm{ML}}^*$ with the bound given in Theorem 3.1 or Corollary 3.1, we see that $\hat{\theta}_{\mathrm{ML}}^*$ is not second order asymptotically efficient since its second order asymptotic distribution does not uniformly attain the bound.

5. Examples

In this section we shall give some examples on the previous sections.

Example 5.1. (Symmetric double exponential case) Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables with a density $f(x-\theta)=e^{-|x-\theta|}/2$ $(-\infty<x<\infty)$. Since $f(x)$ has a cusp at $x=0$, this corresponds to the case when $m=1$ and $s_1=0$ in the condition (A.1). Since $I=1$, $J=K=L_3=0$, $a_1=b_{1-}=-b_{1+}=1/2$ and $f''(0-0)=f''(0+0)=1/2$, it follows from Corollaries 3.1 and 4.1 that the bound for the second order asymptotic distribution of all second order AMU estimators is given by

$$\Phi(t)-\frac{t^2\phi(t)}{6\sqrt{n}}\operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right), \tag{5.1}$$

and also the second order asymptotic distribution of the MLE $\hat{\theta}_{\mathrm{ML}}$ is given by

$$\Phi(t)-\frac{t^2\phi(t)}{2\sqrt{n}}\operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right). \tag{5.2}$$

From (5.1) and (5.2) it follows that $\hat{\theta}_{\mathrm{ML}}$ is not second order asymptotically efficient since its second order asymptotic distribution does not uniformly attain the bound. These facts coincide with the result by Akahira and Takeuchi ((1981), p. 97).

Example 5.2. (Asymmetric density) Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables with a density

$$f(x-\theta)=\begin{cases} \dfrac{1}{2}e^{x-\theta} & \text{for} \quad x<\theta\,; \\[2ex] \dfrac{1}{2} & \text{for} \quad \theta \le x < \theta+\dfrac{1}{2}\,; \\[2ex] \dfrac{1}{2}e^{-2(x-\theta-1/2)} & \text{for} \quad \theta+\dfrac{1}{2}\le x\,. \end{cases}$$

Since $f(x)$ has cusps at $x=0, 1/2$, this corresponds to the case when $m=2, s_1=0$ and $s_2=1/2$ in the condition (A.1). Since $I=3/2$, $J=0$, $K=-3/2$, $L_3=0$, $a_1=a_2=1/2$, $b_{1-}=1/2$, $b_{1+}=b_{2-}=0$, $b_{2+}=-1$, $f''(0-0)=1/2$, $f''(0+0)=0$, $f''(1/2-0)=0$ and $f''(1/2+0)=2$, it follows from Theorems 3.1 and 4.1 that the bound is given by

$$\Phi(t)-\frac{t^2\phi(t)}{12\left(\dfrac{3}{2}\right)^{3/2}\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right) \quad \text{for} \quad t>0\,;$$

$$\Phi(t)+\frac{t^2\phi(t)}{3\left(\dfrac{3}{2}\right)^{3/2}\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right) \quad \text{for} \quad t<0\,,$$

and also the second order asymptotic distribution of the bias-adjusted MLE $\hat{\theta}^*_{\text{ML}}=\hat{\theta}_{\text{ML}}+(1/9n)$ is given by

$$\Phi(t)-\frac{t^2\phi(t)}{2\left(\dfrac{3}{2}\right)^{3/2}\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right) \quad \text{for} \quad t>0\,;$$

$$\Phi(t)+\frac{3t^2\phi(t)}{4\left(\dfrac{3}{2}\right)^{3/2}\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right) \quad \text{for} \quad t<0\,.$$

Hence it is seen that $\hat{\theta}^*_{\text{ML}}$ is not second order asymptotically efficient.

6. Discussion

As is seen in the previous sections, the density with (A.1) to (A.3) has both of the regular and the non-regular sides, which does not affect the first order asymptotic distribution but the second order one. Further, the corresponding parts to both sides appear in the second order asymptotic distribution. Since the bound for the second order asymptotic distribution of all second order AMU estimators are given in Theorem 3.1 and Corollary 3.1, its second order asymptotic difference with the second order asymptotic distribution of any second order AMU estimator could be discussed.

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