Ann. Inst. Statist. Math.
Vol. 40, No. 2, 279-296 (1988)

ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTION OF QUADRATIC FORMS IN NORMAL VARIABLES

SADANORI KONISHI[1], NAOTO NIKI[1*] AND ARJUN K. GUPTA[2]

[1] *The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan*
[2] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, U.S.A.*

(Received June 25, 1986; revised October 16, 1986)

Abstract. Higher order asymptotic expansions for the distribution of quadratic forms in normal variables are obtained. The Cornish-Fisher inverse expansions for the percentiles of the distribution are also given. The resulting formula for a definite quadratic form guarantees accuracy almost up to fourth decimal place if the distribution is not very skew. The normalizing transformation investigated by Jensen and Solomon (1972, *J. Amer. Statist. Assoc.*, **67**, 898-902) is reconsidered based on the rate of convergence to the normal distribution.

Key words and phrases: Cornish-Fisher inverse expansion, distribution of quadratic forms, Edgeworth expansion, normalizing transformation.

1. Introduction

The distributions of quadratic forms arise in a variety of problems in statistics. A number of authors have studied the distribution problems. Work has been done on the derivation of both exact and approximate distributions, and tables of percentiles and probabilities have been prepared for selected values of the parameters. A comprehensive survey of the pre-1970 work in this area, including applications, is given by Johnson and Kotz ((1970), Chapter 29). For related work, see Gupta *et al.* (1975), Gupta and Chattopadhyay (1979) and references given therein.

It may be emphasized that the problem of actually tabulating percentiles and probabilities using earlier results still remains to be investigated. Exact distribution expressed as an infinite series is not convenient for computations, where numerical difficulties increase rapidly with the number of variables

---

This work was completed while the first author was visiting at Bowling Green State University, Department of Mathematics and Statistics.

*Now at Faculty of Science, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812, Japan.

involved. In order to avoid the numerical difficulties arising from the series representations, several approximate methods have been proposed (see, e.g., Jensen and Solomon (1972) and Solomon and Stephens (1977)). The common weakness of approximate distributions suggested previously appears to lie in accuracy. It is desirable to obtain more accurate approximations which yield percentage points and probabilities for various combinations of parameter values.

The purpose of this paper is to derive higher order asymptotic expansions for the distribution of quadratic forms in normal variables. In Section 2 an asymptotic expansion is obtained for the distribution of a transformed variate of a definite quadratic form (positive definite case) and expressed as power series in terms of the first moment. The resulting formula is accurate almost to fourth decimal place if the skewness of the distribution is not so large, as will be seen in Section 4. In order to obtain the desired percentiles, Cornish-Fisher inverse expansion is also given. Section 3 deals with the case of an indefinite quadratic form in normal variables. In the Appendix the normalizing transformation given by Jensen and Solomon (1972) is shown to be derivable from the viewpoint discussed in Konishi (1981).

The REDUCE-III system (Hearn (1983)) has been used to obtain many coefficients in this paper.

2. Definite quadratic form

2.1 *Asymptotic expansion*

Let $X_1, X_2, \ldots, X_k$ be independent standard normal variables, and let $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu=(\mu_1, \mu_2, \ldots, \mu_k)$ where λ_j and μ_j are constants. In this section we assume that $\lambda_j>0, j=1, 2, \ldots, k$. Then the distribution of a definite quadratic form in normal variables is the same as that of

$$Q_k = Q_k(\lambda, \mu) = \sum_{j=1}^{k} \lambda_j(X_j - \mu_j)^2 , \tag{2.1}$$

where λ has $p\ (\leq k)$ distinct elements $\lambda_1', \lambda_2', \ldots, \lambda_p'$ with multiplicities $\nu_1, \nu_2, \ldots, \nu_p$, respectively, so that $\nu_1+\nu_2+\cdots+\nu_p=k$.

It is known (e.g., Johnson and Kotz (1970), p. 153) that the r-th cumulant of Q_k is

$$\kappa_r^* = 2^{r-1}(r-1)!m_r , \quad r = 1, 2, \ldots , \tag{2.2}$$

where $m_r=\sum_{j=1}^{k} \lambda_j^r(1+r\mu_j^2)$. We assume that for the mean $\kappa_1^*=m_1$ of Q_k

$$w_j = m_j/m_1 = O(1) \quad \text{for} \quad j = 2, 3, \ldots , \tag{2.3}$$

as m_1 tends to infinity. Then the standardized quantity

$$(Q_k - m_1)/\sqrt{2m_2} = \sqrt{m_1}\{(Q_k/m_1) - 1\}/\sqrt{2m_2/m_1}\ ,$$

is asymptotically normally distributed with mean 0 and variance 1 as $m_1 \to +\infty$. This standardized form suggests to derive an asymptotic expansion for the distribution of Q_k/m_1 itself in terms of m_1. However, Niki and Konishi (1986) have pointed out that a higher order asymptotic expansion for the distribution of the statistic itself, e.g., a chi-square variate and a sample correlation coefficient in a normal sample, could be much less accurate in the tail for a small sample size. They also have shown that this weakness may be overcome by the use of normalizing transformations like Wilson and Hilferty's (1931) cube root transformation for a chi-square variate and Fisher's (1921) z-transformation for a sample correlation coefficient. A higher order asymptotic expansion for the distribution of a transformed variate gives, in general, extremely high accuracy over the whole domain of the variate. Niki and Konishi (1984) obtained an asymptotic expansion for the distribution of Fisher's z for a sample correlation coefficient, up to terms of order of the reciprocal of the fourth power of the sample size, and showed that the resulting formula guarantees accuracy to five decimal places even when the sample size is as small as 11. A general procedure for finding normalizing transformations was given by Konishi (1981).

A normalizing transformation of Q_k/m_1 is given in the form

$$T_k = \sqrt{m_1}\left\{\left(\frac{Q_k}{m_1}\right)^h - 1 - \frac{1}{m_1}h(h-1)w_2\right\}\Big/\sqrt{2h^2w_2}\ , \tag{2.4}$$

where $h=1-2m_1m_3/(3m_2^2)$ and $w_2=m_2/m_1$.

The derivation of the transformation, including the concept of normalization, is outlined in the Appendix. Jensen and Solomon (1972) obtained the quantity (2.4), by using the approach discussed in Wilson and Hilferty (1931), and suggested to approximate the distribution of T_k by a normal distribution with mean 0 and variance 1. To improve upon this approximation based on the asymptotic distribution of T_k, we obtain an asymptotic expansion for the distribution of T_k up to the terms of order m_1^{-3}.

The characteristic function of T_k can be expressed as

$$\exp\left(-\frac{1}{2}t^2\right)\exp\left\{\kappa_1(it) + \frac{(\kappa_2-1)(it)^2}{2} + \sum_{j=3}^{8}\frac{\kappa_j(it)^j}{j!}\right\} + O(m_1^{-7/2})\ , \tag{2.5}$$

where κ_j is the j-th cumulant of T_k and the orders of expansions of the cumulants are of the form

$$\kappa_1 = O(m_1^{-1/2})\ ,\quad \kappa_2 = 1 + O(m_1^{-1})\ ,\quad \kappa_j = O(m_1^{-j/2+1})\ ,\quad j \geq 3\ .$$

Expanding $(Q_k/m_1)^h$ in a Taylor's series around $Q_k/m_1=1$ and taking expectations term by term with the help of (2.2), we obtain the first eight cumulants of T_k each of which is expanded up to terms of order m_1^{-3}. Substituting the cumulants into the characteristic function (2.5) and inverting the result give an asymptotic expansion for the distribution of T_k. The result is summarized in the following theorem.

THEOREM 2.1. *Under the assumption that* $w_j=m_j/m_1=O(1)$ *for* $j=2, 3,\dots$, *an asymptotic expansion for the distribution of* $Q_k=Q_k(\lambda,\mu)$ *defined by* (2.1), *as* $m_1\to\infty$, *is given by*

$$\begin{aligned}(2.6)\qquad &\Pr\left[\sqrt{m_1}\left\{\left(\frac{Q_k}{m_1}\right)^h-1-\frac{1}{m_1}h(h-1)w_2\right\}\Big/\sqrt{2h^2w_2}<x\right]\\ &\quad=\Phi(x)-\varphi(x)\left(\sum_{j=1}^{6}m_1^{-j/2}a_j\right)+O(m_1^{-7/2})\,,\end{aligned}$$

where m_r *and* h *are given in* (2.2) *and* (2.4), *respectively*, $\Phi(x)$ *and* $\varphi(x)$ *are the standard normal distribution function and its derivative and the coefficients* a_j *are given below.*

$$a_1=0\,,$$

$$a_2=w_2^{-3}\left\{H_3\left(\frac{1}{2}w_4w_2-\frac{20}{27}w_3^2+\frac{2}{9}w_3w_2^2\right)+H_1w_3\left(-\frac{2}{3}w_3+\frac{2}{3}w_2^2\right)\right\},$$

$$\begin{aligned}a_3=\sqrt{2}w_2^{-9/2}&\left\{H_4\left(\frac{2}{5}w_5w_2^2-\frac{4}{3}w_4w_3w_2+\frac{76}{81}w_3^3+\frac{1}{9}w_3^2w_2^2-\frac{1}{9}w_3w_2^4\right)\right.\\ &+H_2w_3\left(-2w_4w_2+\frac{184}{81}w_3^2+\frac{4}{9}w_3w_2^2-\frac{2}{3}w_2^4\right)\\ &\left.+w_3\left(\frac{2}{9}w_3^2+\frac{1}{9}w_3w_2^2-\frac{1}{3}w_2^4\right)\right\},\end{aligned}$$

$$\begin{aligned}a_4=w_2^{-6}&\left\{H_7\left(\frac{1}{8}w_4^2w_2^2-\frac{10}{27}w_4w_3^2w_2+\frac{1}{9}w_4w_3w_2^3\right.\right.\\ &\left.+\frac{200}{729}w_3^4-\frac{40}{243}w_3^3w_2^2+\frac{2}{81}w_3^2w_2^4\right)\\ &+H_5\left(\frac{2}{3}w_6w_2^3-\frac{8}{3}w_5w_3w_2^2+\frac{7}{3}w_4w_3^2w_2+\frac{5}{3}w_4w_3w_2^3+\frac{56}{405}w_3^4\right.\\ &\left.-\frac{1016}{405}w_3^3w_2^2+\frac{32}{135}w_3^2w_2^4+\frac{2}{15}w_3w_2^6\right)\\ &+H_3w_3\left(-\frac{16}{3}w_5w_2^2+\frac{104}{9}w_4w_3w_2+\frac{16}{3}w_4w_2^3-\frac{1106}{243}w_3^3\right.\end{aligned}$$

$$- \frac{76}{9} w_3^2 w_2^2 + \frac{2}{27} w_3 w_2^4 + \frac{4}{3} w_2^6 \bigg)$$
$$+ H_1 w_3 \bigg(\frac{28}{9} w_4 w_3 w_2 + \frac{8}{3} w_4 w_2^3 - \frac{560}{243} w_3^3 - \frac{392}{81} w_3^2 w_2^2$$
$$- \frac{20}{27} w_3 w_2^4 + 2 w_2^6 \bigg) \bigg\} ,$$

$$a_5 = \sqrt{2} w_2^{-15/2} \bigg\{ H_8 \bigg(\frac{1}{5} w_5 w_4 w_2^3 - \frac{8}{27} w_5 w_3^2 w_2^2 + \frac{4}{45} w_5 w_3 w_2^4 - \frac{2}{3} w_4^2 w_3 w_2^2$$
$$+ \frac{118}{81} w_4 w_3^3 w_2 - \frac{13}{54} w_4 w_3^2 w_2^3 - \frac{1}{18} w_4 w_3 w_2^5 - \frac{1520}{2187} w_3^5$$
$$+ \frac{92}{729} w_3^4 w_2^2 + \frac{26}{243} w_3^3 w_2^4 - \frac{2}{81} w_3^2 w_2^6 \bigg)$$
$$+ H_6 \bigg(\frac{4}{7} w_7 w_2^4 - \frac{8}{3} w_6 w_3 w_2^3 + \frac{148}{45} w_5 w_3^2 w_2^2 + \frac{8}{5} w_5 w_3 w_2^4 - \frac{7}{3} w_4^2 w_3 w_2^2$$
$$+ \frac{508}{81} w_4 w_3^3 w_2 - \frac{14}{3} w_4 w_3^2 w_2^3 - \frac{11}{9} w_4 w_3 w_2^5 - \frac{19016}{3645} w_3^5$$
$$+ \frac{70}{27} w_3^4 w_2^2 + \frac{556}{243} w_3^3 w_2^4 - \frac{34}{81} w_3^2 w_2^6 - \frac{4}{45} w_3 w_2^8 \bigg)$$
$$+ H_4 w_3 \bigg(- \frac{20}{3} w_6 w_2^3 + \frac{56}{3} w_5 w_3 w_2^2 + \frac{20}{3} w_5 w_2^4 - \frac{67}{9} w_4 w_3^2 w_2$$
$$- \frac{391}{18} w_4 w_3 w_2^3 - \frac{41}{6} w_4 w_2^5 - \frac{2168}{405} w_3^4 + \frac{3124}{243} w_3^3 w_2^2 + \frac{1012}{81} w_3^2 w_2^4$$
$$- \frac{4}{3} w_3 w_2^6 - \frac{4}{3} w_2^8 \bigg)$$
$$+ H_2 w_3 \bigg(\frac{80}{9} w_5 w_3 w_2^2 + \frac{16}{3} w_5 w_2^4 - \frac{328}{27} w_4 w_3^2 w_2 - \frac{196}{9} w_4 w_3 w_2^3$$
$$- \frac{28}{3} w_4 w_2^5 + \frac{1316}{729} w_3^4 + \frac{3394}{243} w_3^3 w_2^2 + \frac{1360}{81} w_3^2 w_2^4 + \frac{14}{27} w_3 w_2^6 - 4 w_2^8 \bigg)$$
$$+ w_3 \bigg(- \frac{8}{27} w_4 w_3^2 w_2 - \frac{4}{3} w_4 w_3 w_2^3 - \frac{4}{3} w_4 w_2^5 + \frac{88}{729} w_3^4 + \frac{248}{243} w_3^3 w_2^2$$
$$+ \frac{194}{81} w_3^2 w_2^4 + \frac{22}{27} w_3 w_2^6 - \frac{4}{3} w_2^8 \bigg) \bigg\} ,$$

$$a_6 = w_2^{-9} \bigg\{ H_{11} \bigg(\frac{1}{48} w_4^3 w_2^3 - \frac{5}{54} w_4^2 w_3^2 w_2^2 + \frac{1}{36} w_4^2 w_3 w_2^4 + \frac{100}{729} w_4 w_3^4 w_2$$
$$- \frac{20}{243} w_4 w_3^3 w_2^3 + \frac{1}{81} w_4 w_3^2 w_2^5 - \frac{4000}{59049} w_3^6$$
$$+ \frac{400}{6561} w_3^5 w_2^2 - \frac{40}{2187} w_3^4 w_2^4 + \frac{4}{2187} w_3^3 w_2^6 \bigg)$$
$$+ H_9 \bigg(\frac{1}{3} w_6 w_4 w_2^4 - \frac{40}{81} w_6 w_3^2 w_2^3 + \frac{4}{27} w_6 w_3 w_2^5 + \frac{4}{25} w_5^2 w_2^4 - \frac{12}{5} w_5 w_4 w_3 w_2^3$$

$$
\begin{aligned}
&+\frac{368}{135}w_5w_3^3w_2^2-\frac{68}{135}w_5w_3^2w_2^4-\frac{4}{45}w_5w_3w_2^6+\frac{109}{36}w_4^2w_3^2w_2^2\\
&+\frac{3}{4}w_4^2w_3w_2^4-\frac{5356}{1215}w_4w_3^4w_2-\frac{788}{405}w_4w_3^3w_2^3+\frac{32}{45}w_4w_3^2w_2^5+\frac{1}{15}w_4w_3w_2^7\\
&+\frac{6304}{6561}w_3^6+\frac{19736}{10935}w_3^5w_2^2-\frac{2927}{3645}w_3^4w_2^4-\frac{106}{1215}w_3^3w_2^6+\frac{17}{405}w_3^2w_2^8\Bigg)\\
&+H_7\Bigg(w_8w_2^5-\frac{16}{3}w_7w_3w_2^4+\frac{76}{9}w_6w_3^2w_2^3+\frac{28}{9}w_6w_3w_2^5-\frac{48}{5}w_5w_4w_3w_2^3\\
&+\frac{4336}{405}w_5w_3^3w_2^2-\frac{1552}{135}w_5w_3^2w_2^4-\frac{104}{45}w_5w_3w_2^6+\frac{76}{3}w_4^2w_3^2w_2^2\\
&+\frac{16}{3}w_4^2w_3w_2^4-\frac{12233}{243}w_4w_3^4w_2-\frac{682}{81}w_4w_3^3w_2^3+\frac{325}{27}w_4w_3^2w_2^5+2w_4w_3w_2^7\\
&+\frac{4275352}{229635}w_3^6+\frac{1120492}{76545}w_3^5w_2^2-\frac{285704}{25515}w_3^4w_2^4-\frac{31132}{8505}w_3^3w_2^6\\
&+\frac{2872}{2835}w_3^2w_2^8+\frac{8}{63}w_3w_2^{10}\Bigg)\\
&+H_5w_3\Bigg(-16w_7w_2^4+\frac{496}{9}w_6w_3w_2^3+16w_6w_2^5-\frac{5416}{135}w_5w_3^2w_2^2\\
&-\frac{1012}{15}w_5w_3w_2^4-\frac{244}{15}w_5w_2^6+\frac{266}{9}w_4^2w_3w_2^2+\frac{28}{3}w_4^2w_2^4-\frac{7000}{81}w_4w_3^3w_2\\
&+\frac{220}{9}w_4w_3^2w_2^3+\frac{214}{3}w_4w_3w_2^5+17w_4w_2^7+\frac{1732396}{32805}w_3^5+\frac{23480}{729}w_3^4w_2^2\\
&-\frac{13954}{243}w_3^3w_2^4-\frac{7900}{243}w_3^2w_2^6+\frac{778}{135}w_3w_2^8+\frac{8}{3}w_2^{10}\Bigg)\\
&+H_3w_3\Bigg(\frac{1040}{27}w_6w_3w_2^3+\frac{160}{9}w_6w_2^5-\frac{1984}{27}w_5w_3^2w_2^2-\frac{896}{9}w_5w_3w_2^4\\
&-32w_5w_2^6+\frac{272}{81}w_4w_3^3w_2+\frac{2668}{27}w_4w_3^2w_2^3+132w_4w_3w_2^5+44w_4w_2^7\\
&+\frac{55616}{2187}w_3^5+\frac{1040}{243}w_3^4w_2^2-\frac{2552}{27}w_3^3w_2^4-\frac{2216}{27}w_3^2w_2^6\\
&+\frac{112}{27}w_3w_2^8+\frac{40}{3}w_2^{10}\Bigg)\\
&+H_1w_3\Bigg(-\frac{64}{9}w_5w_3^2w_2^2-\frac{160}{9}w_5w_3w_2^4-\frac{32}{3}w_5w_2^6+\frac{160}{27}w_4w_3^3w_2\\
&+\frac{784}{27}w_4w_3^2w_2^3+\frac{416}{9}w_4w_3w_2^5+24w_4w_2^7+\frac{196}{729}w_3^5-\frac{3932}{729}w_3^4w_2^2\\
&-\frac{7081}{243}w_3^3w_2^4-\frac{3358}{81}w_3^2w_2^6-\frac{197}{27}w_3w_2^8+\frac{40}{3}w_2^{10}\Bigg)\Bigg\}.
\end{aligned}
$$

Here H_j is the Hermite polynomial of degree j. For $j=1, 2, \ldots, 10$, these are given in Kendall and Stuart ((1977), p. 167) *and for $j=11, \ldots, 15$, in Niki and Konishi* (1984).

It is interesting to note that the term of order $1/\sqrt{m_1}$ in the asymptotic expansion (2.6) reduces to zero, which will be discussed in the Appendix.

In the case when $\mu_1=\mu_2=\cdots=\mu_k=0$ in $Q_k=Q_k(\lambda, \mu)$, the formula (2.6) yields an asymptotic expansion for the distribution of linear combination of independent chi-square variables. A number of papers have been published on the distribution of $Q_k(\lambda, 0)$. Among them, an approach based on linear differential equation by Davis (1977) appears to be useful for computation.

If $\lambda=(1, 1,\ldots, 1)=e$, say, in $Q_k(\lambda, \mu)$, then $Q_k(e, \mu)$ has the noncentral chi-square distribution with k degrees of freedom and noncentrality parameter $\omega^2=\sum_{j=1}^{k} \mu_j^2$. An asymptotic expansion for the distribution of $Q_k(e, \mu)$ is given by (2.6) with

$$
\begin{aligned}
&m_r = k + r\omega^2\,, \quad w_j = (k + j\omega^2)/(k + \omega^2)\,, \\
&h = \frac{1}{3} + 2\omega^4/\{3(k + 2\omega^2)^2\}\,.
\end{aligned} \tag{2.7}
$$

In the special case when $\lambda=e$ and $\mu=0$, $Q_k(\lambda, \mu)$ has a chi-square distribution with k degrees of freedom, for which (2.7) further reduces to $m_r=k$, $w_j=1$ and $h=1/3$. In this case the formula (2.6) gives an asymptotic expansion for the distribution of the cube root transformation of the chi-square variate $Q_k(e, 0)$. A multivariate extension of the quadratic forms has been discussed by Khatri (1966) and Hayakawa (1966).

2.2 *Cornish-Fisher expansion*

The asymptotic expansion (2.6) can be used to calculate the probability $\Pr[Q_k<q_0]$ for an assigned value q_0. To obtain desired percentiles of the distribution of $Q_k(\lambda,\mu)$, the Cornish-Fisher inverse expansion is very convenient. The method suggested by Hill and Davis (1968) is useful for deriving the expansion of this type.

Suppose that an asymptotic expansion for the distribution of a certain variate X_n has the form

$$\Pr[X_n < x] = \Phi(x) - \varphi(x) \sum_{j=1}^{\infty} A_j(x)n^{-j/2}$$

We take u_α so that, for an assigned probability $(1-\alpha)$, $1-\alpha=\Pr[X_n<x_\alpha]=\Phi(u_\alpha)$. Then the Cornish-Fisher inverse expansion for x_α is given by

$$x_\alpha = u_\alpha + \sum_{r=1}^{\infty} D_{(r)}\left\{ - \sum_{j=1}^{\infty} A_j(u)\right\}^r \Big/ r! \,\Bigg|_{u=u_\alpha}\,,$$

where $D_{(1)}$ denotes the identity operator and

$$D_{(r)} = (u - D_u)(2u - D_u) \cdots \{(r-1)u - D_u\} \quad \text{for} \quad r = 2, 3, \ldots ,$$

with $D_u = d/du$, the differential operator.

Applying this general formula to our problem, we have the following theorem.

THEOREM 2.2. *The Cornish-Fisher inverse expansion for the percentile* q_α *of the distribution of* $Q_k(\lambda, \mu)$ *defined by* (2.1) *is given by*

$$q_\alpha = m_1\{(2h^2 w_2/m_1)^{1/2} x_\alpha + 1 + h(h-1)w_2/m_1\}^{1/h}$$

and

$$x_\alpha = u_\alpha + \left(\sum_{j=1}^{6} m_1^{-j/2} b_j\right) + O(m_1^{-7/2}) , \tag{2.8}$$

where m_r *and* h *are, respectively, defined in* (2.2) *and* (2.4), u_α *is the percentile point of the standard normal distribution and the coefficients* b_j, *using the notation* $w_j = m_j/m_1$ *for* $j = 2, 3, \ldots$, *are given below.*

$$b_1 = 0 ,$$

$$b_2 = w_2^{-3}\left\{u_\alpha^3\left(\frac{1}{2} w_4 w_2 - \frac{20}{27} w_3^2 + \frac{2}{9} w_3 w_2^2\right) + u_\alpha\left(-\frac{3}{2} w_4 w_2 + \frac{14}{9} w_3^2\right)\right\} ,$$

$$\begin{aligned} b_3 = \sqrt{2} w_2^{-9/2}\Big\{& u_\alpha^4\left(\frac{2}{5} w_5 w_2^2 - \frac{4}{3} w_4 w_3 w_2 + \frac{76}{81} w_3^3 + \frac{1}{9} w_3^2 w_2^2 - \frac{1}{9} w_3 w_2^4\right) \\ &+ u_\alpha^2\left(-\frac{12}{5} w_5 w_2^2 + 6 w_4 w_3 w_2 - \frac{272}{81} w_3^3 - \frac{2}{9} w_3^2 w_2^2\right) \\ &+ \frac{6}{5} w_5 w_2^2 - 2 w_4 w_3 w_2 + \frac{62}{81} w_3^3\Big\} , \end{aligned}$$

$$\begin{aligned} b_4 = w_2^{-6}\Big\{& u_\alpha^5\Big(\frac{2}{3} w_6 w_2^3 - \frac{8}{3} w_5 w_3 w_2^2 - \frac{9}{8} w_4^2 w_2^2 + 6 w_4 w_3^2 w_2 + \frac{1}{3} w_4 w_3 w_2^3 \\ &- \frac{1144}{405} w_3^4 - \frac{52}{135} w_3^3 w_2^2 - \frac{2}{15} w_3^2 w_2^4 + \frac{2}{15} w_3 w_2^6\Big) \\ &+ u_\alpha^3\Big(-\frac{20}{3} w_6 w_2^3 + \frac{64}{3} w_5 w_3 w_2^2 + 9 w_4^2 w_2^2 - \frac{367}{9} w_4 w_3^2 w_2 - w_4 w_3 w_2^3 \\ &+ \frac{4144}{243} w_3^4 + \frac{20}{27} w_3^3 w_2^2 + \frac{8}{27} w_3^2 w_2^4\Big) \\ &+ u_\alpha\Big(10 w_6 w_2^3 - 24 w_5 w_3 w_2^2 - \frac{87}{8} w_4^2 w_2^2 + \frac{113}{3} w_4 w_3^2 w_2 \\ &- \frac{350}{27} w_3^4 + \frac{4}{27} w_3^3 w_2^2\Big)\Big\} , \end{aligned}$$

$$
\begin{aligned}
b_5 = \sqrt{2}\, w_2^{-15/2} \Big\{ & u_\alpha^6 \Big(\frac{4}{7} w_7 w_2^4 - \frac{8}{3} w_6 w_3 w_2^3 - \frac{12}{5} w_5 w_4 w_2^3 + \frac{64}{9} w_5 w_3^2 w_2^2 \\
& + \frac{4}{15} w_5 w_3 w_2^4 + \frac{20}{3} w_4^2 w_3 w_2^2 - \frac{1192}{81} w_4 w_3^3 w_2 - \frac{2}{3} w_4 w_3^2 w_2^3 - \frac{2}{9} w_4 w_3 w_2^5 \\
& + \frac{59392}{10935} w_3^5 + \frac{256}{729} w_3^4 w_2^2 + \frac{64}{243} w_3^3 w_2^4 + \frac{8}{81} w_3^2 w_2^6 - \frac{4}{45} w_3 w_2^8 \Big) \\
& + u_\alpha^4 \Big(- \frac{60}{7} w_7 w_2^4 + \frac{100}{3} w_6 w_3 w_2^3 + \frac{144}{5} w_5 w_4 w_2^3 - \frac{1144}{15} w_5 w_3^2 w_2^2 \\
& - \frac{8}{5} w_5 w_3 w_2^4 - 70 w_4^2 w_3 w_2^2 + \frac{3784}{27} w_4 w_3^3 w_2 + \frac{28}{9} w_4 w_3^2 w_2^3 + \frac{2}{3} w_4 w_3 w_2^5 \\
& - \frac{6416}{135} w_3^5 - \frac{340}{243} w_3^4 w_2^2 - \frac{38}{81} w_3^3 w_2^4 - \frac{2}{9} w_3^2 w_2^6 \Big) \\
& + u_\alpha^2 \Big(\frac{180}{7} w_7 w_2^4 - 80 w_6 w_3 w_2^3 - \frac{348}{5} w_5 w_4 w_2^3 + \frac{768}{5} w_5 w_3^2 w_2^2 + \frac{4}{5} w_5 w_3 w_2^4 \\
& + 142 w_4^2 w_3 w_2^2 - \frac{6592}{27} w_4 w_3^3 w_2 - \frac{2}{3} w_4 w_3^2 w_2^3 + \frac{264272}{3645} w_3^5 - \frac{8}{243} w_3^4 w_2^2 \\
& - \frac{4}{27} w_3^3 w_2^4 \Big) - \frac{60}{7} w_7 w_2^4 + 20 w_6 w_3 w_2^3 + \frac{96}{5} w_5 w_4 w_2^3 - \frac{472}{15} w_5 w_3^2 w_2^2 \\
& - 32 w_4^2 w_3 w_2^2 + \frac{1192}{27} w_4 w_3^3 w_2 - \frac{41096}{3645} w_3^5 - \frac{4}{243} w_3^4 w_2^2 \Big\},
\end{aligned}
$$

$$
\begin{aligned}
b_6 = w_2^{-9} \Big\{ & u_\alpha^7 \Big(w_8 w_2^5 - \frac{16}{3} w_7 w_3 w_2^4 - 5 w_6 w_4 w_2^4 + \frac{440}{27} w_6 w_3^2 w_2^3 + \frac{4}{9} w_6 w_3 w_2^5 \\
& - \frac{64}{25} w_5^2 w_2^4 + \frac{476}{15} w_5 w_4 w_3 w_2^3 - \frac{15584}{405} w_5 w_3^3 w_2^2 - \frac{64}{45} w_5 w_3^2 w_2^4 \\
& - \frac{16}{45} w_5 w_3 w_2^6 + \frac{81}{16} w_4^3 w_2^3 - \frac{1021}{18} w_4^2 w_3^2 w_2^2 - \frac{7}{12} w_4^2 w_3 w_2^4 + \frac{18892}{243} w_4 w_3^4 w_2 \\
& + \frac{70}{27} w_4 w_3^3 w_2^3 + \frac{23}{27} w_4 w_3^2 w_2^5 + \frac{1}{3} w_4 w_3 w_2^7 - \frac{1090784}{45927} w_3^6 - \frac{5248}{5103} w_3^5 w_2^2 \\
& - \frac{2048}{5103} w_3^4 w_2^4 - \frac{220}{567} w_3^3 w_2^6 - \frac{92}{567} w_3^2 w_2^8 + \frac{8}{63} w_3 w_2^{10} \Big) \\
& + u_\alpha^5 \Big(- 21 w_8 w_2^5 + 96 w_7 w_3 w_2^4 + 85 w_6 w_4 w_2^4 - \frac{6892}{27} w_6 w_3^2 w_2^3 - \frac{40}{9} w_6 w_3 w_2^5 \\
& + \frac{1056}{25} w_5^2 w_2^4 - \frac{2364}{5} w_5 w_4 w_3 w_2^3 + \frac{215728}{405} w_5 w_3^3 w_2^2 + \frac{176}{15} w_5 w_3^2 w_2^4 \\
& + \frac{32}{15} w_5 w_3 w_2^6 - \frac{1179}{16} w_4^3 w_2^3 + \frac{3049}{4} w_4^2 w_3^2 w_2^2 + 5 w_4^2 w_3 w_2^4 \\
& - \frac{236348}{243} w_4 w_3^4 w_2 - \frac{524}{27} w_4 w_3^3 w_2^3 - \frac{35}{9} w_4 w_3^2 w_2^5 - w_4 w_3 w_2^7 + \frac{9095056}{32805} w_3^6 \\
& + \frac{5320}{729} w_3^5 w_2^2 + \frac{1132}{729} w_3^4 w_2^4 + \frac{52}{81} w_3^3 w_2^6 + \frac{16}{45} w_3^2 w_2^8 \Big) \\
& + u_\alpha^3 \Big(105 w_8 w_2^5 - 400 w_7 w_3 w_2^4 - 345 w_6 w_4 w_2^4 + \frac{24560}{27} w_6 w_3^2 w_2^3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{20}{3} w_6 w_3 w_2^5 - \frac{4224}{25} w_5^2 w_2^4 + \frac{8228}{5} w_5 w_4 w_3 w_2^3 - \frac{669856}{405} w_5 w_3^3 w_2^2 \\
&- \frac{64}{5} w_5 w_3^2 w_2^4 - \frac{16}{15} w_5 w_3 w_2^6 + \frac{4059}{16} w_4^3 w_2^3 - \frac{4663}{2} w_4^2 w_3^2 w_2^2 - \frac{23}{4} w_4^2 w_3 w_2^4 \\
&+ \frac{649937}{243} w_4 w_3^4 w_2 + \frac{440}{27} w_4 w_3^3 w_2^3 + \frac{4}{9} w_4 w_3^2 w_2^5 - \frac{167432}{243} w_3^6 \\
&- \frac{3364}{729} w_3^5 w_2^2 + \frac{296}{729} w_3^4 w_2^4 + \frac{8}{27} w_3^3 w_2^6 \Big) \\
&+ u_\alpha \Big(- 105 w_8 w_2^5 + 320 w_7 w_3 w_2^4 + 285 w_6 w_4 w_2^4 - \frac{5500}{9} w_6 w_3^2 w_2^3 \\
&+ \frac{3456}{25} w_5^2 w_2^4 - \frac{5676}{5} w_5 w_4 w_3 w_2^3 + \frac{130064}{135} w_5 w_3^3 w_2^2 - \frac{16}{15} w_5 w_3^2 w_2^4 \\
&- \frac{2889}{16} w_4^3 w_2^3 + \frac{5591}{4} w_4^2 w_3^2 w_2^2 - \frac{112673}{81} w_4 w_3^4 w_2 + \frac{14}{9} w_4 w_3^3 w_2^3 \\
&+ \frac{2089036}{6561} w_3^6 - \frac{376}{729} w_3^5 w_2^2 + \frac{16}{243} w_3^4 w_2^4 \Big) \Big\} .
\end{aligned}
$$

We note that the standardized quantity can also be rewritten as $(Q_k - m_1)/\sqrt{2m_2} = \sqrt{2m_2}\{Q_k/(2m_2) - m_1/(2m_2)\}$ and that $(Q_k/2m_2)^h = a(Q_k/m_1)^h$ with $a = (m_1/2m_2)^h$. Hence, under the assumption that $m_r/m_2 = O(1)$ for $r = 1, 2, \ldots$, as $m_2 \to +\infty$, similar results can be obtained for the transformed variate $(Q_k/2m_2)^h$.

3. Indefinite quadratic form

This section contains results concerning an indefinite quadratic form in normal variables.

Suppose that the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_k$ in $Q_k = Q_k(\lambda, \mu)$ defined by (2.1) are ordered and that $\lambda_1 \geq \cdots \geq \lambda_l > 0 > \lambda_{l+1} \geq \cdots \geq \lambda_k$. Under this assumption a power transformation is not valid, since Q_k may have negative real numbers for the domain.

It is easily seen that the characteristic function of the standardized quantity can be expressed as

$$
\begin{aligned}
&E[\exp\{(it)(Q_k - m_1)/\sqrt{2m_2}\}] \\
&= \exp\left(-\frac{1}{2}t^2\right) \exp\left\{\sum_{r=3}^{\infty} (it)^r \frac{1}{(\sqrt{2m_2})^{r-2}} \frac{2^{r-2}}{r} \left(\frac{m_r}{m_2}\right)\right\} \\
&= \varphi_1(t)\varphi_2(t) .
\end{aligned}
$$

This implies that if

$$
v_j = m_j/m_2 = O(1) \quad \text{for} \quad j = 1, 2, 3, \ldots ,
$$

as m_2 tends to infinity, then the standardized quantity is asymptotically normally distributed with mean 0 and variance 1. The distribution of the indefinite quadratic form Q_k can be expressed as power series in terms of $(2m_2)^{-1/2}$. Expanding $\varphi_2(t)$ in a Taylor's series and inverting the result, we have the following theorem.

THEOREM 3.1. *Under the assumption that $v_j=m_j/m_2=O(1)$ for $j=1, 2,\dots$, an asymptotic expansion for the distribution of the indefinite quadratic form Q_k is, as $m_2\to\infty$, given by*

$$\Pr\left[\sqrt{2m_2}\left(\frac{Q_k}{2m_2}-\frac{m_1}{2m_2}\right)<x\right]$$
$$=\Phi(x)-\varphi(x)\left(\sum_{j=1}^{6}(2m_2)^{-j/2}a_j\right)+O((2m_2)^{-7/2}),$$

where m_r is given in (2.2) *and the coefficients a_j* are given below.

$$a_1=\frac{2}{3}H_2v_3, \quad a_2=\frac{2}{9}H_5v_3^2+H_3v_4, \quad a_3=\frac{4}{81}H_8v_3^3+\frac{2}{3}H_6v_4v_3+\frac{8}{5}H_4v_5,$$

$$a_4=\frac{2}{243}H_{11}v_3^4+\frac{2}{9}H_9v_4v_3^2+H_7\left(\frac{16}{15}v_5v_3+\frac{1}{2}v_4^2\right)+\frac{8}{3}H_5v_6,$$

$$a_5=\frac{4}{3645}H_{14}v_3^5+\frac{4}{81}H_{12}v_4v_3^3+H_{10}v_3\left(\frac{16}{45}v_5v_3+\frac{1}{3}v_4^2\right)$$
$$+H_8\left(\frac{16}{9}v_6v_3+\frac{8}{5}v_5v_4\right)+\frac{32}{7}H_6v_7,$$

$$a_6=\frac{4}{32805}H_{17}v_3^6+\frac{2}{243}H_{15}v_4v_3^4+H_{13}v_3^2\left(\frac{32}{405}v_5v_3+\frac{1}{9}v_4^2\right)$$
$$+H_{11}\left(\frac{16}{27}v_6v_3^2+\frac{16}{15}v_5v_4v_3+\frac{1}{6}v_4^3\right)$$
$$+H_9\left(\frac{64}{21}v_7v_3+\frac{8}{3}v_6v_4+\frac{32}{25}v_5^2\right)+8H_7v_8.$$

By an argument similar to that discussed in Subsection 2.2, the Cornish-Fisher inverse expansion is given in the following theorem.

THEOREM 3.2. *The Cornish-Fisher inverse expansion for the percentile q_α of the distribution of the indefinite quadratic form Q_k is given by*

$$q_\alpha=\sqrt{2m_2}\,x_\alpha+m_1,$$

and

$$x_\alpha = u_\alpha + \sum_{j=1}^{6} (2m_2)^{-j/2} b_j + O((2m_2)^{-7/2}) ,$$

where m_r are defined in (2.2), u_α is the percentile point of the standard normal distribution and the coefficients b_j, using the notation $v_j = m_j/m_2$, for $j=1, 2, 3, \ldots$, are given below.

$$b_1 = \frac{2}{3} u_\alpha^2 v_3 - \frac{2}{3} v_3 , \quad b_2 = u_\alpha^3\left(v_4 - \frac{8}{9} v_3^2\right) + u_\alpha\left(-3v_4 + \frac{20}{9} v_3^2\right) ,$$

$$b_3 = u_\alpha^4\left(\frac{8}{5} v_5 - 4v_4v_3 + \frac{64}{27} v_3^3\right) + u_\alpha^2\left(-\frac{48}{5} v_5 + 20v_4v_3 - \frac{848}{81} v_3^3\right) + \frac{24}{5} v_5 - 8v_4v_3 + \frac{272}{81} v_3^3 ,$$

$$\begin{aligned} b_4 = {} & u_\alpha^5\left(\frac{8}{3} v_6 - \frac{128}{15} v_5v_3 - \frac{9}{2} v_4^2 + \frac{56}{3} v_4v_3^2 - \frac{224}{27} v_3^4\right) \\ & + u_\alpha^3\left(-\frac{80}{3} v_6 + \frac{1088}{15} v_5v_3 + 36v_4^2 - \frac{412}{3} v_4v_3^2 + \frac{13504}{243} v_3^4\right) \\ & + u_\alpha\left(40v_6 - \frac{448}{5} v_5v_3 - \frac{87}{2} v_4^2 + \frac{428}{3} v_4v_3^2 - \frac{12088}{243} v_3^4\right) , \end{aligned}$$

$$\begin{aligned} b_5 = {} & u_\alpha^6\left(\frac{32}{7} v_7 - \frac{160}{9} v_6v_3 - \frac{96}{5} v_5v_4 + \frac{2048}{45} v_5v_3^2 + 48v_4^2v_3 \right. \\ & \left. - \frac{2560}{27} v_4v_3^3 + \frac{8192}{243} v_3^5\right) \\ & + u_\alpha^4\left(-\frac{480}{7} v_7 + \frac{2080}{9} v_6v_3 + \frac{1152}{5} v_5v_4 - \frac{23168}{45} v_5v_3^2 - 516v_4^2v_3 \right. \\ & \left. + \frac{25696}{27} v_4v_3^3 - \frac{42368}{135} v_3^5\right) \\ & + u_\alpha^2\left(\frac{1440}{7} v_7 - \frac{1760}{3} v_6v_3 - \frac{2784}{5} v_5v_4 + \frac{16768}{15} v_5v_3^2 \right. \\ & \left. + 1084v_4^2v_3 - \frac{48416}{27} v_4v_3^3 + \frac{1927936}{3645} v_3^5\right) \\ & - \frac{480}{7} v_7 + 160v_6v_3 + \frac{768}{5} v_5v_4 - 256v_5v_3^2 - 256v_4^2v_3 + \frac{9728}{27} v_4v_3^3 \\ & - \frac{339328}{3645} v_3^5 , \end{aligned}$$

$$\begin{aligned} b_6 = {} & u_\alpha^7\left(8v_8 - \frac{256}{7} v_7v_3 - 40v_6v_4 + \frac{320}{3} v_6v_3^2 - \frac{512}{25} v_5^2 + \frac{1152}{5} v_5v_4v_3 \right. \\ & \left. - \frac{11264}{45} v_5v_3^3 + \frac{81}{2} v_4^3 - 396v_4^2v_3^2 + \frac{4576}{9} v_4v_3^4 - \frac{36608}{243} v_3^6\right) \\ & + u_\alpha^5\left(-168v_8 + \frac{4736}{7} v_7v_3 + 680v_6v_4 - \frac{46880}{27} v_6v_3^2 + \frac{8448}{25} v_5^2 \right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{17472}{5} v_5 v_4 v_3 + \frac{1461248}{405} v_5 v_3^3 - \frac{1179}{2} v_4^3 + 5462 v_4^2 v_3^2 \\
&- \frac{536032}{81} v_4 v_3^4 + \frac{20135296}{10935} v_3^6 \Big) \\
&+ u_\alpha^3 \Big(840 v_8 - \frac{20480}{7} v_7 v_3 - 2760 v_6 v_4 + \frac{176000}{27} v_6 v_3^2 - \frac{33792}{25} v_5^2 \\
&+ \frac{62336}{5} v_5 v_4 v_3 - \frac{960256}{81} v_5 v_3^3 + \frac{4059}{2} v_4^3 - \frac{52012}{3} v_4^2 v_3^2 \\
&+ \frac{1562072}{81} v_4 v_3^4 - \frac{161078848}{32805} v_3^6 \Big) \\
&+ u_\alpha \Big(- 840 v_8 + \frac{17280}{7} v_7 v_3 + 2280 v_6 v_4 - \frac{42400}{9} v_6 v_3^2 + \frac{27648}{25} v_5^2 \\
&- 8896 v_5 v_4 v_3 + \frac{1007872}{135} v_5 v_3^3 - \frac{2889}{2} v_4^3 + \frac{32734}{3} v_4^2 v_3^2 \\
&- \frac{876424}{81} v_4 v_3^4 + \frac{81364384}{32805} v_3^6 \Big) .
\end{aligned}
$$

The exact distribution of indefinite quadratic forms has been studied by Imhof (1961), Press (1966) and so on. Algorithm to calculate probabilities of Q_k was proposed by Davies (1980), based on the method of Davies (1973) involving the numerical inversion of the characteristic function.

4. Accuracy of approximations

Several approximations have been suggested for the distribution of $Q_k(\lambda, \mu)$ with $\lambda_j>0$ for $j=1,\ldots, k$, including the central and noncentral chi-square distributions. It is known that, for the central chi-square distribution, Wilson and Hilferty's (1931) approximation gives high accuracy even for small values of degrees of freedom. Jensen and Solomon (1972) adapted the Wilson-Hilferty method to develop a normal approximation to the distribution of $Q_k(\lambda, \mu)$ and obtained the normalizing transformation T_k given by (2.4). They also gave extensive numerical comparisons and references, in which their approximation compares favorably with the previous approximations. However, the Jensen-Solomon approximation is based on the leading term $\Phi(x)$ in our asymptotic expansion (2.6). So it is sufficient to check the accuracy of the asymptotic expansion (2.6) itself. The Cornish-Fisher inverse expansion (2.8) gives the same order of accuracy as the asymptotic expansion.

The formula (2.6) expanded up to terms of $O(m_1^{-j/2})$ is referred to

$$
\Pr[T_k < x] \approx F_j = \Phi(x) - \varphi(x)(a_1 m_1^{-1/2} + a_2 m_1^{-1} + \cdots + a_j m_1^{-j/2}) ,
$$

for $j=1, 2,\ldots, 6$, in which F_1 stands for the Jensen-Solomon approximation.

Tables 1 and 2 contain an overall comparison of the four approximations

Table 1. Maximum errors in approximating the values of $\Pr[\chi_k^2(\omega^2)<x$

k	ω^2	β_1	β_2	Maximum error $\times 10^5$			
				F_1	F_2	F_4	F_6
4	1	1.347	2.667	516	333	110	27
	4	1.089	1.667	733	335	97	40
	7	0.926	1.185	629	240	39	20
	10	0.818	0.917	519	178	15	8
	12	0.764	0.796	460	149	13	4
5	1	1.222	2.204	375	228	56	10
	3	1.085	1.686	549	273	69	17
	6	0.928	1.204	544	216	38	14
	9	0.821	0.930	473	164	17	7
	12	0.743	0.756	406	129	8	3
6	1	1.125	1.875	291	167	32	4
	3	1.021	1.500	430	210	43	8
	5	0.928	1.219	458	192	34	9
	7	0.854	1.020	440	165	22	7
	9	0.794	0.875	409	141	14	5
	11	0.745	0.765	377	121	9	3
7	2	1.008	1.488	302	156	26	3
	4	0.925	1.227	371	166	27	5
	6	0.854	1.030	382	150	21	5
	8	0.795	0.885	369	131	14	4
	10	0.746	0.774	347	114	10	3
	12	0.705	0.687	324	100	6	2

Table 2. Maximum errors in approximating the values of $\Pr[\Sigma\lambda_i\chi_{\nu_i}^2<x]$.

	β_1	β_2	Maximum error $\times 10^5$			
			F_1	F_2	F_4	F_6
$L_1 = .6\chi_4^2 + .3\chi_4^2 + .1\chi_4^2$	1.106	1.954	360	162	33	17
$L_2 = .5\chi_4^2 + .3\chi_4^2 + .2\chi_4^2$	0.966	1.500	206	131	21	11
$L_3 = .6\chi_6^2 + .3\chi_6^2 + .1\chi_6^2$	0.903	1.302	226	89	12	4
$L_4 = .6\chi_8^2 + .3\chi_8^2 + .1\chi_8^2$	0.782	0.977	163	58	6	2
$L_5 = .6\chi_4^2 + .3\chi_4^2 + .1\chi_6^2$	1.090	1.913	423	186	36	17
$L_6 = .6\chi_6^2 + .3\chi_4^2 + .1\chi_4^2$	0.972	1.484	216	97	15	4
$L_7 = .6\chi_6^2 + .3\chi_4^2 + .1\chi_2^2$	0.982	1.507	170	78	14	4
$L_8 = L_1 + L_2$	0.742	0.893	152	70	5	1
$L_9 = 2L_1 + L_7$	0.837	1.179	327	185	24	13
$L_{10} = L_1 + 2L_7$	0.832	1.126	239	100	13	5
$L_{11} = L_2 + L_5$	0.736	0.883	164	72	5	1
$L_{12} = L_2 + L_6$	0.703	0.792	116	44	3	1
$L_{13} = L_2 + L_7$	0.707	0.800	105	42	3	1

F_1, F_2, F_4 and F_6 to the distributions of the noncentral chi-square variate $Q_k(e, \mu)$ and the linear combination of independent central chi-square variates $Q_k(\lambda, 0)$ in terms of the maximum error

$$\max_{x \in S} | \Pr[Q_k < x] - F_j | \times 10^5 ,$$

where $\Pr[Q_k < x]$ is the exact distribution. Exact values of the probabilities of $Q_k(e, \mu)$ and $Q_k(\lambda, 0)$ were calculated to sixteen decimal places at intervals of $0.02 \times (\kappa_2^*)^{1/2}$ between 0 and 99.9 percentile points by using the program in Yamauti (1972) and the formula (2.1) given by Imhof (1961), respectively.

Table 3 presents the exact and approximate probabilities for the distribution of linear combination of noncentral chi-square variates with positive coefficients, in which the exact values are due to Imhof (1961). The values of the skewness $\beta_1 = \kappa_3^* / (\kappa_2^*)^{3/2}$ and the kurtosis $\beta_2 = \kappa_4^* / (\kappa_2^*)^2$ where κ_r^* are defined by (2.2) are included in each table. The notation $\chi_k^2(\omega^2)$ refers to a noncentral chi-square variate with k degrees of freedom and noncentrality parameter ω^2.

It may be seen from these tables that the asymptotic expansion (2.6) gives high accuracy over the whole domain of $Q_k(\lambda, \mu)$ for various types of distributions, provided that the value of the skewness is not so large. For $\beta_1 \leq 1.0$, the approximation F_6 guarantees accuracy to about fourth decimal place and may be regarded as an expression which generates exact probabilities of $Q_k(\lambda, \mu)$. For $\beta_1 \geq 2$, the approximation is not so accurate and must be developed. Tables also show the efficacy of higher order terms in the asymptotic expansion.

The approximation F_6 appears to be lengthy. It should, however, be noticed that the asymptotic expansion is most convenient for computations in

Table 3. Comparison of exact and approximate values of $\Pr[\Sigma \lambda_i \chi_{\nu_i}^2(\omega_i^2) > x]$.

	β_1	β_2	x	Approximation F_1	F_4	F_6	Exact
$L_{14} = .7\chi_6^2(6) + .3\chi_2^2(2)$	.834	.998	2.0	.9934	.9938	.9939	.9939
			10.0	.4089	.4087	.4087	.4087
			20.0	.0217	.0220	.0221	.0221
$L_{15} = .7\chi_1^2(6) + .3\chi_1^2(2)$	1.065	1.567	1.0	.9592	.9550	.9548	.9549
			6.0	.4086	.4078	.4076	.4076
			15.0	.0211	.0224	.0223	.0223
$L_{16} = \frac{1}{2}(L_{14} + L_{15})$	.659	.612	3.5	.9570	.9563	.9563	.9563
			8.0	.4153	.4153	.4152	.4152
			13.0	.0456	.0462	.0462	.0462
$L_{17} = \frac{1}{4}(L_4 + L_{14} + L_{15})$	.547	.440	3.0	.9838	.9842	.9842	.9842
			6.0	.4264	.4264	.4264	.4264
			10.0	.0118	.0117	.0117	.0117

which no numerical difficulty is involved, unlike the case of exact distributions. The approximation F_4 simplified by neglecting the terms higher than $O(m_1^{-5/2})$ yields values which agree almost up to three significant figures with the exact values and differ only in the fourth decimal place, if $\beta_1 \leq 1.0$. It seems to be satisfactorily accurate for practical applications.

Appendix

Normalizing transformation of $Q_k(\lambda, \mu)$

Concerning the approach used by Wilson and Hilferty (1931) and Jensen and Solomon (1972) to obtain the normalizing transformations, obscurities remain in the following: (i) Why was the form of transformation restricted to a class of power transforms? and (ii) what does the normalization mean?

In order to make these points clear, we consider the normalizing transformation of $Q_k(\lambda, \mu)$ with positive coefficients, based on the viewpoint discussed in Konishi (1981, 1985).

As shown in Section 2, the variate Q_k/m_1 has asymptotic normality as m_1 tends to infinity and its expected value is one. Let $f(\alpha Q_k/m_1)$ be a strictly monotone and twice continuously differentiable function in a neighborhood of $\alpha Q_k/m_1=\alpha$, where α is a constant. Using the approach discussed in Section 2, an asymptotic expansion for the distribution of $f(\alpha Q_k/m_1)$ is given by

$$\Pr\left[\sqrt{m_1}\left\{f\left(\frac{\alpha Q_k}{m_1}\right)-f(\alpha)-\frac{c}{m_1}\right\}\Big/\tau < x\right]$$

$$\text{(A.1)}\qquad = \Phi(x) - \frac{1}{\sqrt{m_1}}\left[-\frac{\sqrt{2}}{3}w_3w_2^{-3/2}-\frac{c}{\tau}\right.$$
$$\left.+\frac{\sqrt{2}}{6}\left\{2w_3w_2^{-3/2}+3w_2^{1/2}\alpha f''(\alpha)f'(\alpha)^{-1}\right\}x^2\right]\varphi(x)+O(m_1^{-1}),$$

where $w_j=m_j/m_1$, $\tau=\{2w_2\alpha^2f'(\alpha)^2\}^{1/2}$ and c is the asymptotic bias of the transformed variate $f(\alpha Q_k/m_1)$.

From above it follows that the transformed variate

$$\frac{\sqrt{m_1}}{\tau}\left\{f\left(\frac{\alpha Q_k}{m_1}\right)-f(\alpha)-\frac{c}{m_1}\right\},$$

neglecting the term of order $O(m_1^{-1/2})$, is approximated by a standard normal variate. To get an accurate approximation, we search for a function which makes the term of order $m_1^{-1/2}$ vanish for all values of x. This requirement is achieved by solving the differential equation

$$\text{(A.2)}\qquad 2w_3w_2^{-3/2}+3w_2^{1/2}\alpha f''(\alpha)f'(\alpha)^{-1}=0,$$

from (A.1) and, for the solution of (A.2), choosing c as

$$c = -\frac{\sqrt{2}}{3} w_3 w_2^{-3/2} \tau .$$

Solution of this differential equation is found to be

$$(Q_k/m_1)^{1-2w_3/(3w_2^2)} = (Q_k/m_1)^{1-2m_1m_3/(3m_2^2)} ,$$

and then $c=h(h-1)w_2$, where h is given in (2.4). Substituting these results into (A.1), we have the normalizing transformation T_k given by (2.4). It is of interest to note that

$$\Pr[T_k < x] = \Phi(x) + O(m_1^{-1}) ,$$

while

$$\Pr[\sqrt{m_1}\{(Q_k/m_1) - 1\}/(2w_2)^{1/2} < x] = \Phi(x) + O(m_1^{-1/2}) .$$

This implies that by making a suitable transformation with an appropriate bias correction c, the term of order $1/\sqrt{m_1}$ in the asymptotic expansion can be made to vanish, so the error involved is of order m_1^{-1}. Then it can be said that the transformation T_k achieves normality.

Recently, Hayakawa (1987) has shown that Fisher's z-transformation is effective for a sample correlation, canonical correlation and multiple correlation coefficients under an elliptical population.

References

Davies, R. B. (1973). Numerical inversion of a characteristic function, *Biometrika*, **60**, 415–417.

Davies, R. B. (1980). The distribution of a linear combination of χ^2 random variables, *Appl. Statist.*, **29**, 323–333.

Davis, A. W. (1977). A differential equation approach to linear combinations of independent chi-squares, *J. Amer. Statist. Assoc.*, **72**, 212–214.

Fisher, R. A. (1921). On the 'probable error' of a coefficient of correlation deduced from a small sample, *Metron*, **1**, 1–32.

Gupta, A. K. and Chattopadhyay, A. K. (1979). Gammaization and Wishartness of dependent quadratic forms, *Comm. Statist. A—Theory Methods*, **8**, 945–951.

Gupta, A. K., Chattopadhyay, A. K. and Krishnaiah, P. R. (1975). Asymptotic distribution of the determinants of some random matrices, *Comm. Statist.*, **4**, 33–47.

Hayakawa, T. (1966). On the distribution of a quadratic form in a multivariate normal sample, *Ann. Inst. Statist. Math.*, **18**, 191–201.

Hayakawa, T. (1987). Normalizing and variance stabilizing transformations of multivariate statistics under an elliptical population, *Ann. Inst. Statist. Math.*, **39**, 299–306.

Hearn, A. C. (1983). *REDUCE User's Manual*, The Rand Corporation, Santa Monica.

Hill, G. W. and Davis, A. W. (1968). Generalized asymptotic expansions of Cornish-Fisher type, *Ann. Math. Statist.*, **39**, 1264–1273.

Imhof, J. P. (1961). Computing the distribution of quadratic forms in normal variables, *Biometrika*, **48**, 419–426.

Jensen, D. R. and Solomon, H. (1972). A Gaussian approximation to the distribution of a definite quadratic form, *J. Amer. Statist. Assoc.*, **67**, 898–902.

Johnson, N. L. and Kotz, S. (1970). *Distributions in Statistics: Continuous Univariate Distributions-2*, Houghton Mifflin, Boston.

Kendall, M. G. and Stuart, A. (1977). *The Advanced Theory of Statistics*, Vol. I, *Distribution Theory*, 4th ed., Charles Griffin, London.

Khatri, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors, *Ann. Math. Statist.*, **37**, 468–479.

Konishi, S. (1981). Normalizing transformations of some statistics in multivariate analysis, *Biometrika*, **68**, 647–651.

Konishi, S. (1985). Normalizing and variance stabilizing transformations for intraclass correlations, *Ann. Inst. Statist. Math.*, **37**, 87–94.

Niki, N. and Konishi, S. (1984). Higher order asymptotic expansions for the distribution of the sample correlation coefficient, *Comm. Statist. B—Simulation Comput.*, **13**, 169–182.

Niki, N. and Konishi, S. (1986). Effects of transformations in higher order asymptotic expansions, *Ann. Inst. Statist. Math.*, **38**, 371–383.

Patnaik, P. B. (1949). The non-central χ^2- and F-distributions and their applications, *Biometrika*, **36**, 202–232.

Press, S. J. (1966). Linear combinations of non-central chi-square variates, *Ann. Math. Statist.*, **37**, 480–487.

Solomon, H. and Stephens, M. A. (1977). Distribution of a sum of weighted chi-square variables, *J. Amer. Statist. Assoc.*, **72**, 881–885.

Wilson, E. B. and Hilferty, M. M. (1931). The distribution of chi-square, *Proc. National Academy of Sciences*, **17**, 684–688.

Yamauti, Z. (ed.) (1972). *Statistical Tables and Formulas with Computer Applications*, Japanese Standards Association (in Japanese).