ON THE ROBUSTNESS OF BALANCED FRACTIONAL
2\textsuperscript{m} FACTORIAL DESIGNS OF RESOLUTION 2\ell+1
IN THE PRESENCE OF OUTLIERS*

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Summary

By use of the algebraic structure, we obtain a simplified expression for the outlier-insensitivity factor for balanced fractional (2\textsuperscript{m}-BFF) designs of resolution 2\ell+1 derived from simple arrays (S-arrays), whose measure has been introduced by Ghosh and Kipngeno (1985, J. Statist. Plann. Inference, 11, 119–129). It is defined by use of the measure suggested by Box and Draper (1975, Biometrika, 62 (2), 347–352). As examples, we study the sensitivity of A-optimal 2\textsuperscript{m}-BFF designs of resolution VII (i.e., \ell=3) given by Shirakura (1976, Ann. Statist., 4, 515–531; 1977, Hiroshima Math. J., 7, 217–285). We observe that these designs are robust in the sense that they have low sensitivities.

1. Introduction

The concept of a balanced array (B-array) was introduced and first studied by Chakravartii [2]. A general connection between a B-array of strength 2\ell and a 2\textsuperscript{m}-BFF design of resolution 2\ell+1 was established by Yamamoto, Shirakura and Kuwada [15]. Furthermore, these authors ([16]) obtained an explicit expression for the characteristic polynomial of the information matrix of a 2\textsuperscript{m}-BFF design of resolution 2\ell+1 by utilizing the decomposition of the triangular multidimensional partially balanced (TMDPB) association algebra into its \ell+1 two-sided ideals. This polynomial includes the results obtained by Srivastava and Chopra [12] as a special case. A- and/or D-optimal 2\textsuperscript{m}-BFF designs of resolution V and VII have been obtained by several authors (e.g., Shirakura [9] and [10] and Srivastava and/or Chopra [3]–[7], [13] and [14]).

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As a measure of sensitivity in the sense that the design should be insensitive to wild observations, Box and Draper [1] introduced the sum of squares of diagonal elements of the projection matrix, when the number of observations and the number of unknown effects in the model assumed are both fixed. Recently, using the measure suggested by Box and Draper [1], Ghosh and Kippenbo [8] have defined a new measure of robustness of a design with respect to outliers, which is called the “outlier-insensitivity factor”., and they have found these values for A-optimal 2^m-BFF designs of resolution V given by Srivastava and/or Chopra [3]–[7], [13] and [14].

In this paper, we obtain a simplified expression for the outlier-insensitivity factor for a 2^m-BFF design of resolution 2^l+1 derived from an S-array, by use of the properties of the TMDP association scheme and its algebra. As examples, we find its value for A-optimal 2^m-BFF designs of resolution VII given by Shirakura [9] and [10], when 6 ≤ m ≤ 9.

2. Measures of sensitivity

Consider a fractional experiment with m factors each at two levels (0 and 1, say). Then the ordinary linear model associated with a fraction T with N assemblies is

\begin{equation}
\mathcal{E}[\mathbf{y}(T)] = E_T \mathbf{\theta}, \quad \text{Var}[\mathbf{y}(T)] = \sigma^2 I_N, \quad \text{Rank}(E_T) = \nu_l,
\end{equation}

where \( \mathcal{E}[\mathbf{y}] \) stands for the expected value of \( \mathbf{y} \), \( \mathbf{y}(T) \) is a vector of \( N \) observations, \( E_T \) is the \( N \times \nu_l \) design matrix, \( \mathbf{\theta} \) is a vector of unknown effects up to the \( l \)-factor interactions, \( \sigma^2 \) is a constant which may or may not be known, and \( \nu_l = 1 + \binom{m}{1} + \cdots + \binom{m}{l} \). Here \( l \leq [m/2] \), where \([x]\) denotes the largest integer not exceeding \( x \). The predicted value of \( \mathbf{y}(T) \) is \( \hat{\mathbf{y}}(T) = R \mathbf{y}(T) \), where \( R = E_T (E_T' E_T)^{-1} E_T' \) which is known as the projection matrix. Here \( A' \) denotes the transpose of a matrix \( A \). Suppose that the \( u \)-th observation in \( \mathbf{y}(T) \) is an outlier in the sense that an unknown aberration \( c \), a fixed constant, is added to it. And we denote the resulting observation vector as \( \mathbf{y}^*(T) \) and the corresponding predicted value as \( \hat{\mathbf{y}}^*(T) = R \mathbf{y}^*(T) \). Then the quantity \( d_u = (\hat{\mathbf{y}}^*(T) - \hat{\mathbf{y}}(T))' (\mathbf{y}^*(T) - \hat{\mathbf{y}}(T)) \) is a measure of overall discrepancy caused by the effect of \( c \) on the \( u \)-th observation, and it is equal to \( \sigma^2 r_{uu} \), where \( r_{uu} \) is the \( u \)-th diagonal element of \( R \). Clearly

\begin{equation}
\sum_{u=0}^{N} d_u = \sigma^2 \sum_{u=0}^{N} r_{uu} = \sigma^2 \nu_l,
\end{equation}

because of the idempotency of \( R \). If it is equally likely that \( c \) occurs
with any of $N$ observations, giving rise to $d_1, \ldots, d_N$, the average discrepancy is $\bar{d} = c^d \frac{\sum_{u=1}^{N} r_{uu}}{N} = c^d \nu_1/N$. If the $d_u (u=1, \ldots, N)$ are as small as possible, then a design is said to be insensitive with respect to outliers. Because $\sum d_u$ is fixed as in (2.2), this means that the $d_u$ are as uniformly as possible for a design insensitive to outliers. As one convenient measure of uniformity, Box and Draper [1] introduced the following:

$$V(d) = \sum_{u=1}^{N} (d_u - \bar{d})^2/N = c^d \frac{r - (\nu_1)^2/N}{N},$$

where

$$r = \sum_{u=1}^{N} (r_{uu})^2.$$

Thus to ensure insensitivity to outliers, $V(d)$ should be made small. Minimization of $V(d)$ is equivalent to minimization of $r$, when $N$ and $\nu_1$ are both fixed. Recently, Ghosh and Kipngenov [8] have defined the outlier-insensitivity factor, $E$, say, by

$$E = 100 \times (\nu_1)^2/(Nr),$$

because $r \leq (\nu_1)^2/N$.

Note that under the model (2.1), for an orthogonal design or $N = \nu_1$, i.e., saturated design, we have $E=100$.

3. Outlier-insensitivity factors

Under the model (2.1), the expected value of an observation associated with an assembly $(\epsilon_1, \ldots, \epsilon_m)$ with $\epsilon_x = 0$ or 1 is given by

$$E[y(\epsilon_1, \ldots, \epsilon_m)] = \sum_{\eta_1, \ldots, \eta_m} d_1(\eta_1) \cdots d_m(\eta_m) \theta(\eta_1, \ldots, \eta_m),$$

where the summation is over all binary numbers $(\eta_1, \ldots, \eta_m)$ with $\eta_x = 0$ or 1 such that $0 \leq \eta_1 + \cdots + \eta_m \leq l$, and

$$d_0(0) = d_1(0) = d_1(1) = 1 \quad \text{and} \quad d_0(1) = -1.$$

Note that when $\eta_1 + \cdots + \eta_m = j$ ($j=0, 1, \ldots, l$), $\theta(\eta_j)$ is called the $j$-factor interaction, where $\eta_j = (\eta_1, \ldots, \eta_m)$.

When $T$ is a $B$-array of strength $m$, size $N$, $m$ constraints, 2 levels and index set $\{\lambda_0, \lambda_1, \ldots, \lambda_m\}$, $T$ is called an $S$-array, written $SA(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$ for brevity (see [9]). For $T$ being an $SA(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$, $T$ can be expressed as $T = \|J_i \otimes T_i\|$ if $\lambda_i \geq 1$, where $T_i$ are the $(0, 1)$ matrices of size $\binom{m}{i} \times m$ whose rows denote all distinct
vectors with weight \( i \). Here \( \mathbf{j}_p \) and \( A \otimes B \) denote, respectively, the \( p \times 1 \) vector with all unity and the Kronecker product of two matrices \( A \) and \( B \), and the weight of a \((0,1)\) vector means the number of ones in the vector.

Let \( E_T = \| \mathbf{j}_i \otimes E_i(j) \| \) if \( \lambda_i \geq 1 \) \((i=0,1,\ldots,m; j=0,1,\ldots,l)\), where \( E_i(j) \) denote the submatrices of \( E_T \) corresponding, respectively, to \( T_i \) and \( \theta(\eta_i) \). Then from (3.1), (3.2) and Appendix, we have

\[
E_i(j) = \sum_{a=0}^{\min(i, j, m-i)} (-1)^{j-\min(i, j)+a} A_a^{(i, j)}
\]

for \( i=0,1,\ldots,m \) and \( j=0,1,\ldots,l \),

where \( \min(a_i, \ldots, a_n) \) denotes the minimum value of integers \( a_i, \ldots, a_n \), and \( A_a^{(i, j)} \) are the local association matrices which are given in Appendix. Hence, it follows from Appendix that

\[
E_i(j) = \begin{cases} 
\sum_{a=0}^{\min(i, j)} (-1)^{j-\min(i, j)+a} A_a^{(i, j)} & \text{for } 0 \leq i \leq \lceil m/2 \rceil, \\
\sum_{a=0}^{\min(m-i, j)} (-1)^{j-\min(m-i, j)-a} A_a^{(m-i, j)} & \text{for } \lfloor m/2 \rfloor < i \leq m.
\end{cases}
\]

Thus, from (3.3) and Appendix, the following is immediate.

**Lemma 3.1.** The submatrices \( E_i(j) \) of \( E_T \) can be expressed as

\[
E_i(j) = \begin{cases} 
\sum_{\beta=0}^{\min(i, j)} h_{\beta}^{(i, j)} A_{\beta}^{(i, j)} & \text{for } 0 \leq i \leq \lceil m/2 \rceil \text{ and } 0 \leq j \leq l, \\
\sum_{\beta=0}^{\min(m-i, j)} h_{\beta}^{*(m-i, j)} A_{\beta}^{(m-i, j)} & \text{for } \lfloor m/2 \rfloor < i \leq m \text{ and } 0 \leq j \leq l,
\end{cases}
\]

where

\[
h_{\beta}^{(i, j)} = \begin{cases} 
(-1)^{i} (-2)^i \left(\begin{array}{c} m-i-\beta \\ j-i \end{array}\right) \left(\begin{array}{c} j-\beta \\ j-i \end{array}\right)^{1/2} \\
\times \left\{ \sum_{\delta=0}^{\lceil \frac{i-\beta}{2} \rceil} (-1)^{\delta} \left(\begin{array}{c} j-\beta \\ i-\beta-\delta \end{array}\right) \left(\begin{array}{c} m-i-\beta+\delta \\ b \end{array}\right) \right\} & \text{if } 0 \leq i \leq j \leq \lceil m/2 \rceil, \\
2^j \left(\begin{array}{c} m-j-\beta \\ i-j \end{array}\right) \left(\begin{array}{c} i-\beta \\ i-j \end{array}\right)^{1/2} \\
\times \left\{ \sum_{\delta=0}^{\lceil \frac{i-\beta}{2} \rceil} (-1)^{\delta} \left(\begin{array}{c} i-\beta \\ j-\beta-\delta \end{array}\right) \left(\begin{array}{c} m-j-\beta+\delta \\ b \end{array}\right) \right\} & \text{if } 0 \leq j \leq i \leq \lfloor m/2 \rfloor,
\end{cases}
\]

and \( h_{\beta}^{*(m-i, j)} \) are given by replacing \( i \) in \( h_{\beta}^{(i, j)} \) by \( m-i \).
For $T$ being an SA $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$, the submatrices $M_{u,v}$ of $E_r^t E_T$ corresponding, respectively, to $\{\theta(\eta_u)\}$ and $\{\theta(\eta_v)\}$ can be expressed as

$$M_{u,v} = \sum_{a=0}^{\min(u,v)} \gamma_{u-\alpha+2a} A^{(u,v)}_{a}$$

for $0 \leq u, v \leq l$,

where a connection between $\gamma$'s and $\lambda$'s is given by

$$\gamma_i = \sum_{j=0}^{m} \sum_{p=0}^{j} (-1)^p \binom{m-i}{p} \binom{m-i}{j-i+p} \lambda_j$$

(see [15]). It follows from Appendix that $M_{u,v}$ can be expressed as

$$M_{u,v} = \sum_{\beta=0}^{\min(u,v)} \kappa^{u,v}_{\beta} A^{(u,v)}_{\beta}$$

for $0 \leq u, v \leq l$,

where

$$\kappa^{u,v}_{\beta} = \sum_{\alpha=0}^{\min(u,v)} \gamma_{u-\alpha+2\beta} \kappa^{(\alpha,v)}_{\beta}$$

for $0 \leq u \leq v \leq l - \beta$

and $\kappa^{u,v}_{\beta} = \kappa^{v,u}_{\beta}$ for $u \leq v$ (see [16]). If $E_r^t E_T$ is non-singular, then the submatrices $M^{*}_{u,v}$ of $(E_r^t E_T)^{-1}$ corresponding, respectively, to $\{\theta(\eta_u)\}$ and $\{\theta(\eta_v)\}$ is

$$M^{*}_{u,v} = \sum_{\beta=0}^{\min(u,v)} \kappa^{u,v}_{\beta} A^{(u,v)}_{\beta}$$

for $0 \leq u, v \leq l$,

where $\|\kappa^{u,v}_{\beta}\| = \|\kappa^{v,u}_{\beta}\|^{-1}$. Note that the order of $\|\kappa^{u,v}_{\beta}\|$ is $l+1-\beta$ ($\beta=0, 1, \ldots, l$) and does not depend on $m$, while the order of $E_r^t E_T$ is $\nu_i$. Thus from Lemma 3.1, (3.4) and Appendix, the following is immediate.

**LEMMA 3.2.** The diagonal submatrices $R_{ui}$ of $R$ corresponding to $T_i$ can be expressed as

$$R_{ui} = \begin{cases} \sum_{j=0}^{1} \sum_{k=0}^{1} \left( \sum_{\beta=0}^{\min(m-i, j, k)} \kappa^{(j,i,k)}_{\beta} h^{(j,k)}_{\beta} A^{(i,k)}_{\beta} \right) & \text{if } 0 \leq i \leq [m/2], \\ \sum_{j=0}^{1} \sum_{k=0}^{1} \left( \sum_{\beta=0}^{\min(m-i, j, k)} \kappa^{(m-i,k)}_{\beta} h^{(m-i,k)}_{\beta} A^{(m-i,m-i)}_{\beta} \right) & \text{if } [m/2] < i \leq m. \end{cases}$$

Since $A^{(i,i)}_{0} = I_{\binom{m}{i}}$ for $i = 0, 1, \ldots, m$, it follows from Appendix that the coefficients of $A^{(i,i)}_{0}$ in $A^{(i,i)}_{\beta}$ are given by

$$z^{(i,i)}_{\beta} = \phi_{\beta} z^{(i,i)}_{\beta} \left[ \binom{m}{i} \binom{m-i}{0} \right] = \phi_{\beta} \binom{m}{i},$$

(3.5)

where $z^{(u,v)}_{\beta}, z^{(v,u)}_{\beta}$, and $\phi_{\beta}$ are given in Appendix. Hence, from Lemma 3.2 and (3.5), the diagonal elements $r_{ui}$ of $R_{ui}$ are given by
Thus the following is immediate.

**Lemma 3.3.** For $T$ being an SA $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$, under the model (2.1), $r$ is given by

$$r = \sum_{t=0}^{m} \lambda_t \binom{m}{i} (r_{it})^2,$$

where $r_{it}$ are given by (3.6).

From Lemma 3.3, we have the main results of this paper as follows.

**Theorem 3.1.** Let $T$ be an SA $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$. Then under the model (2.1), the outlier-insensitivity factor $E$ is given by

$$E = 100 \times (v_i)^3/(Nr),$$

where $r$ is given in Lemma 3.3.

4. Calculation of $E$ for Shirakura’s designs

In this section, we study the sensitivity of $A$-optimal $2^m$-BFF designs of resolution VII (i.e., $l=3$) given by Shirakura [9] and [10]. It follows from Lemma 3.1 that

$$h_{ij}^{(0,0)} = \binom{m}{i}^{1/2}$$ for $0 \leq i \leq [m/2],$$

$$h_{ij}^{(0,j)} = (-1)^j \binom{m}{j}^{1/2}$$ for $j=1, 2, 3,$

$$h_{ij}^{(1,1)} = -(m-2i) \binom{m-1}{i-1}^{1/2}$$ for $1 \leq i \leq [m/2],$$

$$h_{ij}^{(2,1)} = 2 \binom{m-2}{i-1}^{1/2}$$ for $1 \leq i \leq [m/2],$$

$$h_{ij}^{(1,j)} = (-1)^{j-1} h_{ij}^{(1,1)}$$ for $j=2, 3$ and $\beta=0, 1,$

$$h_{ij}^{(2,2)} = 4 \binom{i}{2} - 2(m-1)i + \binom{m}{2} \left[ \binom{m-2}{i-2}/2 \right]^{1/2}$$ for $2 \leq i \leq [m/2],$
\[ h_{i}^{(i,s)} = -2(m-2i) \left\{ \binom{m-3}{i-2}/(i-1) \right\}^{1/2} \quad \text{for} \quad 2 \leq i \leq [m/2] , \]

\[ h_{i}^{(s,2)} = 4 \left\{ \binom{m-4}{i-2} \right\}^{1/2} \quad \text{for} \quad 2 \leq i \leq [m/2] , \]

\[ h_{\beta}^{(2,3)} = -h_{\beta}^{(3,2)} \quad \text{for} \quad \beta = 0, 1, 2 , \]

\[ h_{0}^{(i,s)} = \left\{ 8 \binom{i}{3} - 4(m-2) \binom{i}{2} + 2 \binom{m-1}{2} i - \binom{m}{3} \right\} \left\{ \binom{m-3}{i}/(i-3) \right\}^{1/2} \quad \text{for} \quad 3 \leq i \leq [m/2] , \]

\[ h_{i}^{(i,s)} = \left\{ 8 \binom{i-1}{2} - 4(m-3)(i-1) + 2 \binom{m-2}{2} \right\} \left\{ \binom{m-4}{i-3}/(i-1) \right\}^{1/2} \quad \text{for} \quad 3 \leq i \leq [m/2] , \]

\[ h_{i}^{(5,s)} = -4(m-2i) \left\{ \binom{m-5}{i-3}/(i-2) \right\}^{1/2} \quad \text{for} \quad 3 \leq i \leq [m/2] , \]

\[ h_{i}^{(4,s)} = 8 \left\{ \binom{m-6}{i-3} \right\}^{1/2} \quad \text{for} \quad 3 \leq i \leq [m/2] , \]

and \( h_{\beta}^{(m-i,s)} \) are given by replacing \( i \) in \( h_{\beta}^{(i,s)} \) by \( m-i \). Thus for \( T \) being an SA \((m; \lambda_{0}, \lambda_{1}, \ldots, \lambda_{m})\), we have

\[ E = 100 \times (\nu_{s})^{2}/(N \times r) , \]

where

\[ r = \sum_{i=0}^{m} \lambda_{i} \binom{m}{i} (r_{ui})^{2} \]

and

\[ r_{ui} = \begin{cases} \left[ \sum_{f=0}^{3} \sum_{k=0}^{3} \left\{ \frac{\min_{\beta=0}^{(i,f,k)} \phi_{\beta} h_{\beta}^{(i,f)} h_{\beta}^{(i,k)} \kappa_{\beta}^{i-f,k-f}}{\binom{m}{i}} \right\} \right]^{1/2} & \text{if} \quad 0 \leq i \leq [m/2] , \\ \left[ \sum_{f=0}^{3} \sum_{k=0}^{3} \left\{ \frac{\min_{\beta=0}^{(m-i,f,k)} \phi_{\beta} h_{\beta}^{(m-i,f)} h_{\beta}^{(m-i,k)} \kappa_{\beta}^{i-f,k-f}}{\binom{m}{i}} \right\} \right]^{1/2} & \text{if} \quad [m/2] < i \leq m . \end{cases} \]

For \( 6 \leq m \leq 9 \), all \( A \)-optimal \( 2^{m} \)-BFF designs of resolution VII given by Shirakura [9] and [10] except for the designs corresponding to \( m = 8 \) and \( N = 127b, 128b \) are \( S \)-arrays. In Tables 1, 2, 3 and 4, the values of the outlier-insensitivity factor \( E \) of \( A \)-optimal \( 2^{m} \)-BFF designs of resolution VII are presented for \( m = 6 \) and \( 42 \leq N \leq 64 \), \( m = 7 \) and \( 64 \leq N \leq 90 \), \( m = 8 \) and \( 93 \leq N \leq 128 \), and \( m = 9 \) and \( 130 \leq N \leq 150 \), respectively.
These values are greater than 93, 87, 92 and 91 for \( m=6, 7, 8 \) and 9, respectively. Therefore we conclude that the sensitivities of the Shirakura’s designs to outliers are low.

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5. Concluding remarks

Let \( T^* \) be an SA \((7; 3, 0, 0, 1, 1, 0, 1, 3)\), which is also a B-array of strength 6 and size 83. Then we have \( \det (\| \mathbf{x}^* \|^\ast) = 1294666368 \), \( \det (\| \mathbf{x}^{(c)} \|^\ast) = 78728 \), \( \det (\| \mathbf{x}^c \|^\ast) = 3072 \) and \( \det (\| \mathbf{x}^t \|^\ast) = 128 \), where \( \det (A) \) denotes the determinant of a matrix \( A \). Hence, \( T^* \) is a 2\(^{9}\)-BFF design of resolution VII. It follows from Section 4 that \( E = 96.91502 \) for \( T^* \), which is the most insensitive design to outliers in the class of balanced designs derived from S-arrays, while, from Table 2, we have \( E = 89.85773 \).
for an $A$-optimal $2^r$-BFF design $T$ of resolution VII. On the other hand, it follows from Theorem 2.1 of Shirakura [9] that

$$\text{tr} \{(E'_T'E_T)^{-1}\} = 1.50710 \quad \text{for} \quad T^*$$

and

$$\text{tr} \{(E'_T'E_T)^{-1}\} = 0.88119 \quad \text{for} \quad T.$$

This implies that in the restricted class mentioned above, the most insensitive design to outliers is not always good design in some sense. It, however, is worth to calculate the values of $E$ for some optimal designs with respect to the popular criteria (e.g., $A$-, $D$- and $E$-optimal).

Acknowledgements

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REFERENCES

designs of resolution V, \( m \leq 6 \), *Technometrics*, **13**, 257–269.


Appendix

Let \((e_1, \ldots, e_m) (=e_i, \text{ say})\) be a \((0, 1)\) vector with weight \(i\). Further let \(S_i = \{e_i\} (i = 0, 1, \ldots, m)\). Then \(|S| = \binom{m}{i} (=n_i, \text{ say})\), where \(|S|\) denotes the cardinality of a set \(S\). Suppose a relation of association defined among the sets \(S_i\) in such a way that \(e_i \in S_i\) and \(e_j \in S_j\) are the \(\alpha\)-th associates if

\[
(A.1) \quad e_i e_j = \min (i, j) - \alpha,
\]

where \(\alpha = 0, 1, \ldots, \min (i, m - i, j, m - j)\). Let \(\tilde{e}_i = j - e_i\). Then if \(e_i\) is the \(\alpha\)-th associate of \(e_j\), then (A.1) shows that \(e_i\) is the \(\alpha^*\)-th associate of \(\tilde{e}_j\), \(\tilde{e}_i\) is the \(\alpha^{**}\)-th associate of \(e_j\), and \(\tilde{e}_i\) is the \(\alpha^{***}\)-th associate of \(\tilde{e}_j\), where \(\alpha^* = \min (i, m - j) - i + \min (i, j) - \alpha\), \(\alpha^{**} = \min (m - i, j) - j + \min (i, j) - \alpha\), and \(\alpha^{***} = \min (m - i, m - j) - m + i + j - \min (i, j) + \alpha\).

Let \(A_a^{(u, v)} (=A_a^{(u, v)})\) be the \(n_x n_x\) local association matrix of the TMDPB association scheme, where \(0 \leq u \leq v \leq [m/2]\) and \(\alpha = 0, 1, \ldots, l\) (e.g., [15]). Further let \(A_a^{(u, v)} (=A_a^{(u, v)})\) \((0 \leq u \leq v \leq [m/2] ; \beta = 0, 1, \ldots, l)\) be the \(n_x n_x\) matrices which are linearly linked with \(A_a^{(u, v)}\) as follows (e.g., [11] and [16]).

\[
(A.2) \quad A_a^{(u, v)} = \sum_{\beta=0}^{u} z_{\beta}^{(u, v)} A_{\beta}^{(u, v)} \quad \text{for} \quad 0 \leq \alpha \leq u \leq v \leq [m/2]
\]

and

\[
(A.3) \quad A_a^{(u, v)} = \sum_{\alpha=0}^{u} z_{\alpha}^{(u, v)} A_{a}^{(u, v)} \quad \text{for} \quad 0 \leq \beta \leq u \leq v \leq [m/2],
\]

where

\[
z_{\beta}^{(u, v)} = \sum_{b=0}^{u} \binom{u - \beta}{b} \binom{u - b}{u - \alpha} \binom{m - u - \beta + b}{b} \binom{m - u - \beta}{v - u} \binom{v - \beta}{v - u} \frac{1}{b} \frac{b}{v - u + b}
\]

and

\[
z_{\alpha}^{(u, v)} = \phi_{\beta} z_{\beta}^{(u, v)} \left\{ \frac{(m - u)}{u} \left( \frac{m - u}{v - u + \alpha} \right) \right\} \quad \text{for} \quad 0 \leq u \leq v \leq [m/2].
\]
Here

\[ \phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1} \quad \text{for} \quad \beta = 0, 1, \ldots, \min (u, v). \]

The matrices \( A^{(u,v)}_\beta \) have the following properties,

\[ \sum_{\beta=0}^{u} A^{(u,v)}_\beta = I_u, \]

\[ A^{(u,v)}_\beta A^{(v,u)}_\gamma = \delta_{\beta,\gamma} A^{(u,v)}_\beta, \]

\[ \text{Rank} \ (A^{(u,v)}_\beta) = \phi_\beta, \]

where \( \delta_{a,b} \) denotes Kronecker's delta, i.e., \( \delta_{a,b} = 1 \) or 0 according as \( a = b \) or not. As mentioned above, we have

\[
A^{(i,j)}_\alpha = \begin{cases} 
A^{(i,j)}_\alpha & \text{if} \quad 0 \leq i, j \leq \lfloor m/2 \rfloor, \\
A^{(i, m-j)}_\alpha & \text{if} \quad 0 \leq i \leq \lfloor m/2 \rfloor < j \leq m, \\
A^{(m-i,j)}_\alpha & \text{if} \quad 0 \leq j \leq \lfloor m/2 \rfloor < i \leq m, \\
A^{(m-i, m-j)}_\alpha & \text{if} \quad \lfloor m/2 \rfloor < i, j \leq m.
\end{cases}
\]

Thus \( A^{(i,j)}_\alpha \) \((0 \leq i, j \leq m; 0 \leq \alpha \leq \min (i, m-i, j, m-j))\) can be expressed as the linear combinations of \( A^{(*,*)}_\beta \) as in (A.2) and (A.3).

It is to be noted that the importance of the TMDPB association algebra \( \mathcal{A} \) generated by the ordered association matrices \( D^{(u,v)}_\alpha \) and also generated by \( D^{(u,v)}_\beta \) has been discussed in the works of Yamamoto, Shirakura and Kuwada [16], and others. A few references are given above; for further information the readers are requested to see the references therein.