ON AKI'S NONPARAMETRIC TEST FOR SYMMETRY

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Summary

Aki (1987, Ann. Inst. Statist. Math., 39, 457–472) develops a theory of extending the test for symmetry about zero of a continuous distribution function F. In this paper we discuss the same testing problem in the case where the probability F(0) of being negative is unknown, which is assumed to be known in Aki's paper.

1. Introduction

Aki [1] discusses an extension of the test for symmetry about zero of a continuous distribution function F. F is usually called symmetric about c, if it satisfies

\[ F(c-x) + F(c+x) = 1 \quad \text{for all} \quad x. \]

(1.1)

Let \( X_1, \ldots, X_n \) be independent random variables with a common continuous distribution function \( F \), not necessarily symmetric, let \( \alpha \) \((0 < \alpha < 1)\) be a given constant; and he considers testing for the hypothesis,

\( H_0: \) There exists a continuous distribution function \( G \) which is symmetric about zero and satisfies

\[ F(x) = \begin{cases} 2\alpha G(x), & \text{if} \quad x \leq 0, \\ \alpha + 2(1-\alpha)(G(x) - \frac{1}{2}), & \text{if} \quad x > 0. \end{cases} \]

(1.2)

If \( \alpha = 1/2 \), then \( H_0 \) reduces to the hypothesis of symmetry about zero in the sense (1.1).

In this paper we consider testing for the symmetry in the sense of \( H_0 \) assuming that \( \alpha \) is an unknown constant. The hypothesis we are going to test in this paper is thus,

\( H_0: \) There exist \( \alpha \) \((0 < \alpha < 1)\) and a continuous distribution function \( G \), symmetric about zero in the sense (1.1), which satisfy (1.2).

Key words and phrases: Brownian bridge, test for symmetry, weak convergence.
In the next section we shall give a test statistic and its limiting null distribution, and then prove that the test is consistent.

2. Extended test for symmetry

Let $H$ be a continuous and strictly increasing distribution function symmetric about zero in the sense (1.1). Considering $Y_i = H(X_i)$ ($i = 1, \cdots, n$) instead of $X_i$ ($i = 1, \cdots, n$) as in Aki [1], the testing problem of $H'_0$ is reduced to testing for the symmetry about 1/2 of the continuous distribution function $F$ defined on the unit interval $(0, 1)$.

$H'_0$: There exist $a$ ($0 < a < 1$) and a continuous distribution function $G^*$ defined on $(0, 1)$ which satisfy

$$F(t) = \begin{cases} 
\alpha G^*(2t), & \text{if } 0 < t \leq \frac{1}{2}, \\
\alpha + (1 - \alpha)(1 - G^*(2 - 2t)), & \text{if } \frac{1}{2} < t < 1.
\end{cases}$$

We give below a test statistic for $H'_0$ and two theorems concerning it.

Let $X_1, \cdots, X_n$ be independent random variables on the unit interval $(0, 1)$ with the same distribution function $F$. Define

$$(2.1) \quad \eta_i = I_{(0, 1/2)}(X_i) \quad (i = 1, \cdots, n),$$

$$(2.2) \quad m = \sum_{i=1}^{n} \eta_i \quad \text{and} \quad \hat{a} = \frac{m}{n},$$

and put in the case $0 < m < n$,

$$\xi_i = \sqrt{\frac{1 + \hat{a}}{\hat{a}}} = \sqrt{\frac{n - m}{m}} \quad \text{and} \quad Y_i = 2X_i, \quad \text{if } \eta_i = 1,$$

whereas

$$\xi_i = -\sqrt{\frac{\hat{a}}{1 - \hat{a}}} = -\sqrt{\frac{m}{n - m}} \quad \text{and} \quad Y_i = 2(1 - X_i), \quad \text{if } \eta_i = 0.$$ 

Define for $t \in [0, 1]$

$$u_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i I_{(0, t]}(Y_i), \quad \text{if } 0 < m < n,$$

and

$$u_n(t) = 0, \quad \text{if } m = 0 \text{ or } m = n.$$

We consider as a test statistic for $H'_0$ a functional of $u_n$,.
\[ T_n = \sup_{0 \leq t \leq 1} |u_n(t)|. \]

It is a natural substitute of the test statistic proposed by Aki [1] for the case where \( \alpha \) is a constant.

**Theorem 2.1.** If \( F \) satisfies \( H'_0 \), then \( u_n(t) \) converges weakly to \( W^\circ(G^*(t)) \) where \( W^\circ \) denotes Brownian bridge.

**Proof.** Assume that \( H'_0 \) is satisfied. Then, \( \eta_1, \ldots, \eta_n \) are independent with the same distribution

\[ P(\eta_i = 1) = \alpha, \quad P(\eta_i = 0) = 1 - \alpha, \]

and

\[ Y_1, \ldots, Y_n \]

are independent with the same distribution

\[ P(Y_i \leq t) = G^*(t), \]

furthermore \( (\eta_1, \ldots, \eta_n) \) and \( (Y_1, \ldots, Y_n) \) are independent. The distribution of \( m \) is binomial given by the expansion of \( (\alpha + (1 - \alpha))^n \).

We show first that \( \{u_n\} \) is tight. For the purpose it suffices to prove that there exists a constant \( C \) for which

\[
\begin{align}
E \left[ |u_n(t) - u_n(t_1)|^p | u_n(t_2) - u_n(t)|^q \right] \\
\leq C(G^*(t) - G^*(t_1))(G^*(t_2) - G^*(t))
\end{align}
\]  

holds for any \( n \) and for any \( t_i, t, t_j \) such that \( 0 \leq t_i < t < t_j \leq 1 \) (Billingsley [2]).

Put

\[ P(t_i < Y_i \leq t) = G^*(t) - G^*(t_i) = p \]

and

\[ P(Y_i < Y_i \leq t) = G^*(t) - G^*(t) = q. \]

and evaluate the expectation on the left hand side of (2.3), conditionally upon a given \( m \) such that \( 0 < m < n \). Then we have

\[
\begin{align}
E \left[ |u_n(t) - u_n(t_1)|^p | u_n(t_2) - u_n(t)|^q \right] \\
= \frac{1}{n^2} \sum_{i,j,k,l} E \left[ \xi_i \xi_j \xi_k \xi_l I(t_i, t_1)(Y_i)I(t_i, t_j)(Y_j)I(t_j, t_2)(Y_k)I(t_2, t)(Y_l) \right] \cdot m.
\end{align}
\]

The terms with \( i=k, i=l, j=k \) or \( j=l \) may be ignored in the above summation. Taking into account the fact that \( Y_1, \ldots, Y_n \) are independent with (2.4) and (2.5), and that \( (\xi_1, \ldots, \xi_n) \) and \((Y_1, \ldots, Y_n)\) are independent given \( m \), we have

\[
\begin{align}
(2.7) \quad (2.6) = \frac{1}{n^2} \left[ \sum_i \sum_j E[\xi_i \xi_j | m]pq + \sum_i \sum_j E[\xi_i \xi_k \xi_j | m]pq^2 \right] \\
+ \sum_i E[\xi_i \xi_j | m]pq^2 + \sum_i \sum_j E[\xi_i \xi_k \xi_j | m]pq^3
\end{align}
\]
\[
+ \sum' \mathbb{E} [\xi_i \xi_j \xi_k | m] p^3 q + \sum' \mathbb{E} [\xi_i \xi_j \xi_k \xi_l | m] p^4 q^2 \],
\]
where \( \sum' \) denotes the summation over all sets of different indices.

Now, for a given \( m \) such that \( 0 < m < n \), \( (\eta_1, \ldots, \eta_n) \) is distributed with equal probabilities over the set of \( \binom{n}{m} \) points in the \( n \)-dimensional space, where \( m \) of the coordinates are 1 while \( n - m \) other coordinates are 0. Hence, if \( i \neq k \), we have

\[
P(\eta_i = \eta_k = 1 | m) = \frac{m(m-1)}{n(n-1)},
\]

\[
P(\eta_i = 1, \eta_k = 0 | m) = P(\eta_i = 0, \eta_k = 1 | m) = \frac{m(n-m)}{n(n-1)},
\]

\[
P(\eta_i = \eta_k = 0 | m) = \frac{(n-m)(n-m-1)}{n(n-1)}.
\]

By using the value of \( \xi_i \xi_j \) in each case we obtain

\[
(2.8) \quad \mathbb{E} [\xi_i \xi_j \xi_k | m] = \left( \frac{n-m}{m} \right)^2 \frac{m(m-1)}{n(n-1)} + 2 \frac{m(n-m)}{n(n-1)}
\]

\[
+ \left( \frac{m}{n-m} \right)^2 \frac{(n-m)(n-m-1)}{n(n-1)}
\]

\[
= \frac{A-B}{n(n-1)m(n-m)},
\]

putting

\[
A = n^2 m(n-m) \quad \text{and} \quad B = m^3 + (n-m)^3.
\]

In the same way we have for different sets of indices that

\[
(2.9) \quad \mathbb{E} [\xi_i \xi_j \xi_k | m] = \frac{-A + 2B}{n(n-1)(n-2)m(n-m)},
\]

\[
(2.10) \quad \mathbb{E} [\xi_i \xi_j \xi_k | m] = \frac{-A + 2B}{n(n-1)(n-2)m(n-m)},
\]

and

\[
(2.11) \quad \mathbb{E} [\xi_i \xi_j \xi_k \xi_l | m] = \frac{3A - 6B}{n(n-1)(n-2)(n-3)m(n-m)}.
\]

In (2.7) terms of the types (2.8), (2.9), (2.10) and (2.11) appear \( n(n-1) \), \( n(n-1)(n-2) \), \( n(n-1)(n-2) \) and \( n(n-1)(n-2)(n-3) \) times, respectively, hence we have
\[(2.12) \quad (2.7) = \frac{A-B}{n^2m(n-m)} pq + \frac{-A+2B}{n^2m(n-m)} (pq^3 + p^3q) + \frac{3A-6B}{n^2m(n-m)} p^3q^3\]
\[= pq \left[ \frac{A}{n^2m(n-m)} (1 - p - q + 3pq) \right. \]
\[\left. + \frac{B}{n^2m(n-m)} (-1 + 2p + 2q - 6pq) \right].\]

Since
\[\frac{A}{n^2m(n-m)} = 1 \quad \text{and} \quad \frac{B}{n^2m(n-m)} = \frac{m^4}{n^4(n-m)} + \frac{(n-m)^4}{n^4m} \leq 1\]
for \(1 \leq m \leq n-1\), we have
\[(2.12) \leq pq(|1 - p - q + 3pq| + |-1 + 2p + 2q - 6pq|) .\]

As the expression in the parentheses on the right hand side is bounded for
\[(2.13) \quad p \geq 0, \quad q \geq 0 \quad \text{and} \quad p+q \leq 1,\]
so denoting by \(C\) its upper bound over the region (2.13) we have
\[(2.14) \quad E \left[ \|u_n(t) - u_n(t_i)\|^2 | u_n(t) - u_n(t_i) \|^2 \mid m \right] \]
\[\leq Cpq = C(G^*(t) - G^*(t_i))(G^*(t) - G^*(t)) .\]

In the case \(m=0\) or \(m=n\) we defined \(u_n(t)\) to be \(0\), hence (2.14) holds trivially. Thus (2.3) is proved, establishing the tightness of \(\{u_n\}\).

Next we shall find the limiting distribution of the finite dimensional random variable
\[(2.15) \quad (u_n(t), \cdots, u_n(t_n))\]
for \(0 < t_1 < \cdots < t_k < 1\). Note that \(u_n(0) = u_n(1) = 0\). First we deal with the case \(k=2\) and evaluate the limiting characteristic function of \((u_n(t), u_n(t_2) - u_n(t_1))\).

For a given \(m\) such that \(0 < m < n\), \(m\) of \(\eta_i\)'s are 1 and \(n-m\) of \(\eta_i\)'s are 0, and \((\eta_1, \cdots, \eta_n)\) and \((Y_1, \cdots, Y_n)\) are independent; therefore the conditional distribution of \((u_n(t), u_n(t_2) - u_n(t_1))\) given \(m\) is the same as the conditional distribution given
\[(2.16) \quad \eta_1 = \cdots = \eta_m = 1 \quad \text{and} \quad \eta_{m+1} = \cdots = \eta_n = 0 .\]

Assuming (2.16) we have
\[u_n(t_1) = \frac{1}{\sqrt{n}} \left[ \sqrt{\frac{n-m}{m}} \sum_{j=1}^{m} I(\alpha, t_1) Y_j - \sqrt{\frac{m}{n-m}} \sum_{j=m+1}^{n} I(\alpha, t_1) Y_j \right] \]
and
\[ u_n(t_2) - u_n(t_1) = \frac{1}{\sqrt{n}} \left[ \sqrt{\frac{n-m}{m}} \sum_{j=1}^{m} I_{i(t_1,t_2)}(Y_j) - \sqrt{\frac{m}{n-m}} \sum_{j=m+1}^{n} I_{i(t_1,t_2)}(Y_j) \right]. \]

Put

\[ P(0 < Y_j \leq t_i) = p, \quad P(t_i < Y_j \leq t_i) = q \quad \text{and} \quad P(t_i < Y_j < 1) = r, \]

then we have clearly

\[ p \geq 0, \quad q \geq 0, \quad r \geq 0 \quad \text{and} \quad p + q + r = 1, \]

and the random variables defined by

\[ (U_1, U_2, U_3) = \left( \sum_{j=1}^{m} I_{0,t_2}^{(1)}(Y_j), \sum_{j=1}^{m} I_{0,t_2}^{(2)}(Y_j), \sum_{j=1}^{m} I_{0,t_2}^{(3)}(Y_j) \right) \]

and

\[ (V_1, V_2, V_3) = \left( \sum_{j=m+1}^{n} I_{t_1,t_2}^{(1)}(Y_j), \sum_{j=m+1}^{n} I_{t_1,t_2}^{(2)}(Y_j), \sum_{j=m+1}^{n} I_{t_1,t_2}^{(3)}(Y_j) \right) \]

are independent with the multinomial distribution given by the expansion of \((p+q+r)^n\) and \((p+q+r)^{n-m}\), respectively.

Furthermore we have

\[ u_n(t_i) = \frac{1}{\sqrt{n}} \left( \sqrt{\frac{n-m}{m}} U_1 - \sqrt{\frac{m}{n-m}} V_i \right) \]

and

\[ u_n(t_2) - u_n(t_1) = \frac{1}{\sqrt{n}} \left( \sqrt{\frac{n-m}{m}} U_1 - \sqrt{\frac{m}{n-m}} V_1 \right), \]

hence we obtain the conditional characteristic function of \((u_n(t_i), u_n(t_2) - u_n(t_1))\) given \(m\) as in the following,

\[ \phi_n(\theta_1, \theta_2 | m) = E \{ \exp \{ i \theta_1 u_n(t_i) + i \theta_2 (u_n(t_2) - u_n(t_1)) \} \mid m \} \]

\[ = \frac{1}{\sqrt{n}} \left( \sqrt{\frac{n-m}{m}} (\theta_1 U_1 + \theta_2 U_2) - \frac{i}{\sqrt{n}} \sqrt{\frac{m}{n-m}} (\theta_1 V_1 + \theta_2 V_2) \right) \mid m \]

\[ = \left[ p \exp \left( \frac{i}{\sqrt{n}} \sqrt{\frac{n-m}{m}} \theta_1 \right) + q \exp \left( \frac{i}{\sqrt{n}} \sqrt{\frac{n-m}{m}} \theta_2 \right) + r \right]^m \]

\[ \times \left[ p \exp \left( -\frac{i}{\sqrt{n}} \sqrt{\frac{m}{n-m}} \theta_1 \right) + q \exp \left( -\frac{i}{\sqrt{n}} \sqrt{\frac{m}{n-m}} \theta_2 \right) + r \right]^{n-m} \]

\[ = \phi_n^{(1)}(\theta_1, \theta_2 | m) \phi_n^{(2)}(\theta_1, \theta_2 | m), \quad \text{say.} \]
Let an arbitrary $\varepsilon > 0$ be given, then by the central limit theorem there exist $\gamma > 0$ and a natural number $n_0$ such that for any $n$ satisfying $n \geq n_0$ we have

$$(2.18) \quad P(n\alpha - \sqrt{n} \gamma < m < n\alpha + \sqrt{n} \gamma) > 1 - \varepsilon.$$ 

We shall show below that $\phi_n(\theta_1, \theta_2 | m)$ tends, as $n \to \infty$, to a limiting function $\phi(\theta_1, \theta_2)$ uniformly in $m$ for which

$$(2.19) \quad n\alpha - \sqrt{n} \gamma < m < n\alpha + \sqrt{n} \gamma$$ 

holds.

Note that $(n-m)/m$ and $m/(n-m)$ are bounded for $n$ satisfying $n \geq n_0$ and for $m$ satisfying (2.19), and apply Taylor expansion to obtain

$$\log \phi_n^{(i)}(\theta_1, \theta_2 | m)$$

$$= m \log \left[ 1 + p \left\{ \exp \left( \frac{i}{\sqrt{n}} \sqrt{\frac{n-m}{m}} \theta_1 \right) - 1 \right\} 
+ q \left\{ \exp \left( \frac{i}{\sqrt{n}} \sqrt{\frac{n-m}{m}} \theta_1 \right) - 1 \right\} \right]$$

$$= m \log \left[ 1 + \frac{i}{\sqrt{n}} \sqrt{\frac{n-m}{m}} (p\theta_1 + q\theta_2) - \frac{n-m}{2nm} (p\theta_1^2 + q\theta_2^2 + O(n^{-1/2})) \right]$$

$$= i \sqrt{\frac{m(n-m)}{n}} (p\theta_1 + q\theta_2) - \frac{n-m}{2n} (p\theta_1^2 + q\theta_2^2 - (p\theta_1 + q\theta_2)^2) + O(n^{-1/2}),$$

and similarly,

$$\log \phi_n^{(ii)}(\theta_1, \theta_2 | m)$$

$$= -i \sqrt{\frac{m(n-m)}{n}} (p\theta_1 + q\theta_2) - \frac{n-m}{2n} (p\theta_1^2 + q\theta_2^2 - (p\theta_1 + q\theta_2)^2) + O(n^{-1/2}),$$

uniformly in $m$ satisfying (2.19).

Adding both equations side by side we get

$$\log \phi_n(\theta_1, \theta_2 | m) = - \frac{1}{2} (p\theta_1^2 + q\theta_2^2 - (p\theta_1 + q\theta_2)^2) + O(n^{-1/2}),$$

from which it follows that

$$(2.20) \quad \phi_n(\theta_1, \theta_2 | m) = \phi(\theta_1, \theta_2) + O(n^{-1/2}),$$

uniformly in $m$ satisfying (2.19), where we put

$$\phi(\theta_1, \theta_2) = \exp \left[ - \frac{1}{2} (p\theta_1^2 + q\theta_2^2 - (p\theta_1 + q\theta_2)^2) \right].$$

Therefore, if we take some integer $n_1$ such that $n_1 \geq n_0$, then the
$O(n^{-1/2})$ term in (2.20) is less than $\varepsilon$ in absolute value for any $n$ such that $n \geq n_1$ and any $m$ satisfying (2.19), that is,

$$|\phi_n(\theta_1, \theta_2|m) - \phi(\theta_1, \theta_2)| < \varepsilon.$$  

(2.21)

Let

$$\phi_n(\theta_1, \theta_2) = \sum_{m=0}^{n} \binom{n}{m} \alpha^{m}(1-\alpha)^{n-m} \phi_n(\theta_1, \theta_2|m) = \Sigma' + \Sigma'' ,$$

where $\Sigma'$ denotes the summation over all values of $m$ satisfying (2.19), while $\Sigma''$ denotes the summation over all other values of $m$. Taking into account of (2.21) for $\Sigma'$ and of (2.18) we have

$$|\Sigma' - \phi(\theta_1, \theta_2)| < P \left( |m - na| \geq \sqrt{n\gamma} \right) + \varepsilon < 2\varepsilon$$

and

$$|\Sigma''| < P \left( |m - na| \geq \sqrt{n\gamma} \right) < \varepsilon ,$$

therefore we get

$$|\phi_n(\theta_1, \theta_2) - \phi(\theta_1, \theta_2)| < 3\varepsilon$$

for all $n$ such that $n \geq n_1$.

Thus we have proved

$$\lim_{n \to \infty} \phi_n(\theta_1, \theta_2) = \phi(\theta_1, \theta_2)$$

for all $\theta_1$ and $\theta_2$. $\phi(\theta_1, \theta_2)$ is clearly the characteristic function of $(W^o(p), W^o(q)-W^o(p))$, where $W^o$ denotes Brownian bridge. In particular it has been shown that the variances and the covariance of the limiting distribution of $(u_n(t_1), u_n(t_2))$ coincide with those of $(W^o(p), W^o(q)) = (W^o(G*(t_1)), W^o(G*(t_2)))$.

By using a similar argument we can find the limit of the characteristic function of

$$\phi_n(\theta_1, \theta_2|m) = \phi_n(\theta_1, \theta_2|m), \quad \phi_n(\theta_1, \theta_2|m) - \phi_n(\theta_1, \theta_2|m), \ldots, \phi_n(\theta_1, \theta_2|m) - \phi_n(\theta_1, \theta_2|m)$$

for any $k$ and any $t_1, \ldots, t_k$ such that $0 < t_1 < \ldots < t_k < 1$, and we can see that it is of the form

$$\exp \left[ -\frac{1}{2} \text{ (quadratic form in } \theta \text{'s}) \right] ,$$

which shows that the limiting distribution of (2.22) is $k$-variate normal with zero means. Since the variances and covariances of the limiting distribution of (2.15) have been already shown to coincide with those of

$$W^o(G*(t_1)), \ldots, W^o(G*(t_k)) ,$$

(2.23)
the limiting distribution of (2.15) is proved to be that of (2.23).

The proof of Theorem 2.1 is thus completed by Theorem 15.6 of Billingsley [2].

By Theorem 2.1 the limiting distribution of $T_n$ is given by

$$\lim_{n \to \infty} P(T_n \leq z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{-t^2} dt.$$ 

See Billingsley [2], p. 85.

We prove next the consistency of the test given by the critical region $T_n \geq c$ for any constant $c$. We give a proof of a slightly different character from that of Theorem 3.2 in Aki [1].

**Theorem 2.2.** If a continuous distribution function $F$ defined on $(0, 1)$ does not satisfy $H'$, then $P(T_n \geq c)$ converges to 1 for any constant $c$, provided that $0 < F(1/2) < 1$.

**Proof.** Put $\alpha = F(1/2)$ ($0 < \alpha < 1$) and define for any $t$ such that $0 \leq t \leq 1$,

$$G^*_t(t) = \frac{1}{\alpha} F\left(\frac{t}{2}\right) \quad \text{and} \quad G^*_t(t) = \frac{1}{1-\alpha} \left(1 - F\left(1 - \frac{t}{2}\right)\right),$$

then we have

$$G^*_t(t) = P(Y_i \leq t| \eta_i = 1) \quad \text{and} \quad G^*_t(t) = P(Y_i \leq t| \eta_i = 0).$$

By the assumption of Theorem 2.2 there exists $t_0$ such that

$$0 < t_0 < 1 \quad \text{and} \quad G^*_t(t_0) = G^*_t(t_0).$$

Put $G^*_t(t_0) = p_1$ and $G^*_t(t_0) = p_2$.

As (2.17) in the proof of Theorem 2.1 we have

$$u_n(t_0) = \frac{1}{\sqrt{n}} \left(\sqrt{\frac{n-m}{m}} U_0 - \sqrt{\frac{m}{n-m}} V_0\right),$$

where $m$ is defined by (2.2) and is assumed to be $0 < m < n$. For a given $m$, $U_0$ and $V_0$ are independent random variables with the binomial distribution given by the expansion of $(p_1 + (1-p_1))^m$ and $(p_2 + (1-p_2))^{n-m}$, respectively. Hence we have, if $0 < m < n$,

$$E[u_n(t_0)| m] = \sqrt{\frac{m(n-m)}{n}} (p_1 - p_2)$$

and

$$V[u_n(t_0)| m] = \frac{n-m}{n} p_1(1-p_1) + \frac{m}{n} p_2(1-p_2).$$
If $m=0$ or $m=n$, we have clearly
\[ E[u_n(t_0)|m] = V[u_n(t_0)|m] = 0. \]

Since $\sqrt{n(n-m)/n^2}$ is the sample standard deviation calculated from \(\eta_1, \cdots, \eta_n\), its mean and variance are derived from the formulae given in Cramér [3], p. 353. Using the results we have
\[ E[u_n(t_0)] = E \sum_m E[u_n(t_0)|m] = \sqrt{n\alpha(1-\alpha)(p_1-p_2)(1+O(n^{-1}))} \]
and
\[ V[u_n(t_0)] = V \sum_m E[u_n(t_0)|m] + E \sum_m V[u_n(t_0)|m] \]
\[ = \left( \frac{1}{4} - \alpha(1-\alpha) \right)(p_1-p_2)^2 + (1-\alpha)p_1(1-p_1) \]
\[ + \alpha p_2(1-p_2) + O(n^{-1}) \]
\[ = O(1). \]

By $p_1 - p_2 \neq 0$, $|E[u_n(t_0)]|$ tends to infinity whereas $V[u_n(t_0)]$ is bounded as $n \to \infty$; therefore by using Tchebychev's inequality we have $P(|u_n(t_0)| \geq c) \to 1$ for any $c$ so that
\[ \lim_{n \to \infty} P(T_n \geq c) = \lim_{n \to \infty} P(\sup_{0 \leq t \leq 1} |u_n(t)| \geq c) = 1, \]
completing the proof.

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