ON A CHARACTERIZATION OF MONOTONE LIKELIHOOD RATIO EXPERIMENTS

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Summary

Pfanzagl (1962, Zeit. Wahrscheinlichkeitsthe., 1, 109–115) showed that a dominated family of probability measures has monotone likelihood ratios with respect to some real valued statistic if there exists a set of tests which has certain nice properties. A similar characterization was given by Dettweiler (1978, Metrika, 25, 247–254), who did not assume domination. However, Pfanzagl’s result is not a special case of the one proved by Dettweiler. We present a theorem which comprises the results of both authors. Our proof shows that not all conditions introduced by them are needed. Furthermore, we investigate the question concerning the generality we get if we do not assume domination.

1. Introduction

Let \((\Omega, \mathcal{A})\) be a measurable space and let \(\mathcal{A} = \sigma_{\mathcal{A}}(\mathcal{A})\) be the set of all probability measures on \(\mathcal{A}\). We use \(E_P\) to denote the expectation with respect to \(P \in \sigma_{\mathcal{A}}(\mathcal{A})\). We write \([P]\) instead of “\(P\)-almost everywhere” and \([\mathcal{D}]\) instead of “\([P]\) for all \(P \in \mathcal{D}\)” whenever \(\mathcal{D} \subset \sigma_{\mathcal{A}}(\mathcal{A})\). A test \(\varphi\) is a real valued measurable function defined on \(\Omega\) such that \(0 \leq \varphi \leq 1\). For \(P, Q \in \sigma_{\mathcal{A}}(\mathcal{A})\) and a test \(\varphi\) we write \(P_\varphi Q\) if \(\varphi\) is most powerful for testing \(P\) against \(Q\) at level \(E_P\varphi\) and \(1 - \varphi\) is most powerful for testing \(Q\) against \(P\) at level \(E_Q(1 - \varphi)\). If \(\varphi\) is fixed, this definition provides a reflexive and transitive binary relation (preorder) on \(\sigma_{\mathcal{A}}(\mathcal{A})\). In Pfanzagl’s ([10]) notation \(P_\varphi Q\) means “\(\varphi\) trennscharf \(P:Q\)”. If \(\varphi, \psi\) are tests and if \(\varphi\) is most powerful for testing \(P\) against \(Q\) at level \(E_P\varphi\), then the following assertions hold.

\[
\begin{align*}
(1.1) \quad & E_P\varphi \leq E_Q\varphi. \\
(1.2) \quad & \text{If } E_P\varphi = 0, \text{ then } \varphi_{1_{\{\varphi > 0\}}} = 1_{\{\varphi > 0\}}[Q]. \\
(1.3) \quad & \text{If } P[0 < \varphi < 1] = 0, \text{ then } Q[0 < \varphi < 1] = 0.
\end{align*}
\]

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(1.4) If $\varphi \leq \varphi [P]$, then $\varphi \leq \varphi [Q]$.

We say that $P, Q \in \mathcal{A}$ have monotone likelihood ratio with respect to an extended real valued statistic $T$ and write $P \leq_T Q$ if there is a non-decreasing function $h$ from $\bar{R}$ to $\bar{R}$ such that
\[
\frac{dQ}{d\mu} \frac{dP}{d\mu} = h \circ T[\mu],
\]
where $\mu = P + Q$ and $a/0 = \infty$ for $a > 0$. Each pair $P, Q \in \mathcal{A}$ has monotone likelihood ratio with respect to a statistic $T$ which is equal to $(dQ/d\mu)/(dP/d\mu)$ on $\{dP/d\mu > 0\} \cup \{dQ/d\mu > 0\}$. If $T$ is fixed, $\leq_T$ defines a partial order on $\mathcal{A}$. In the literature, $T$ is usually assumed to be real valued. The set $\mathcal{A}$ of all tests $\varphi$ of the type $\varphi = 1_{\{T > c\}} + \gamma 1_{\{T = c\}}$, $c \in \bar{R}$, $0 \leq \gamma \leq 1$, has some nice and well-known properties of optimality for each pair $P, Q$ with $P \leq_T Q$ (see for example, Karlin and Rubin [6], Lehmann ([7], p. 68), Pfanzagl ([10], p. 112), Heyer ([5], p. 84) and the next section of this paper). Furthermore, $\mathcal{A}$ is obviously totally ordered.

The aim of the papers of Pfanzagl [8], [10] and Dettweiler [2] is to show that under suitable additional conditions families of probability measures which have monotone likelihood ratios with respect to a statistic $T$ can be characterized by the existence of a set of tests which has some of the properties of the set $\mathcal{A}$. We will prove a more general result. It will be shown that some of the conditions introduced by Pfanzagl [10] or Dettweiler [2] are not needed. Furthermore, we will see that families of probability measures which are totally ordered with respect to $\leq_T$ for some statistic $T$ and which are not necessarily dominated are majorized in the sense of Siebert [12].

2. Some properties of statistical experiments with monotone likelihood ratios

Let $T$ be an extended real valued statistic on $\Omega$ and let $\mathcal{A}$ have the same meaning as in Section 1. For each $P \in \mathcal{A}$ we define $H_P = \inf \{t \in \bar{R}: P[T > t] = 0\}$ and $h_P = \sup \{t \in \bar{R}: P[T \geq t] = 1\}$ (we make the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$). Put $I_P = [-\infty, H_P]$ if $P[T = H_P] > 0$ and otherwise $I_P = [-\infty, H_P]$. Furthermore, set $J_p = [h_P, \infty]$ if $P[T = h_P] > 0$ and otherwise $J_P = [h_P, \infty]$. Define $Z_P = I_P \cap J_P$. The following lemmas are essentially known (see for example, Karlin and Rubin [6], Lehmann ([7], p. 68), Pfanzagl ([10], p. 112), or Heyer ([5], p. 84)).

**Lemma 2.1.** a) If $P \in \mathcal{A}$, then $P[T \in Z_P] = 1$.

b) If $P, Q \in \mathcal{A}$ and $P \leq_T Q$, then $Q(\cdot \cap \{T \in I_P\}) \ll P$ and $P(\cdot \cap \{T \in I_P\})$. 


Suppose $P, Q \in \varphi(\mathcal{A})$ and $P \leq_T Q$. Then $Z_P = Z_Q$ if and only if $P$ and $Q$ are equivalent. If $Z_P = Z_Q$ and $Z_P$ is degenerated, then $P = Q$.

**Lemma 2.2.**

a) If $P \in \varphi(\mathcal{A})$ and $\alpha \in [0, 1]$, then there is a $\varphi \in \mathcal{A}$ such that $E_{\varphi} = \alpha$.

b) If $P, Q \in \varphi(\mathcal{A})$, $P \leq_T Q$, $\varphi \in \mathcal{A}$, $0 < E_{\varphi} \varphi$ and $E_{\varphi} \varphi < 1$, then $P, Q$.

c) If $P, Q, R \in \varphi(\mathcal{A})$ such that $R$ is equivalent to $P$ and $Q \geq_T R$ and if $\varphi_P = 1_{[T \in I]}$, then $\varphi, \varphi \in \mathcal{A}$, $E_{\varphi} \varphi_P = 0$ and $R_{\varphi_P} Q$.

d) Suppose $\mathcal{O} \neq \mathcal{M} \subseteq \varphi(\mathcal{A})$ and $k = \sup\{H_P : P \in \mathcal{M}\}$. Put $D = [-\infty, k]$ if there is a $P \in \mathcal{M}$ with $P\{T = k\} > 0$, and otherwise put $D = [-\infty, k]$. If $\varphi \in \mathcal{A}$ such that $\varphi = 1_{[T \in I]}$ or $E_{\varphi} \varphi > 0$ for some $P \in \mathcal{M}$, then $\varphi 1_{[T \in D]} = 1_{[T \in D]}$.

e) If $P \in \varphi(\mathcal{A})$, $\varphi \in \mathcal{A}$, $P\{0 < \varphi < 1\} = 0$ and $0 < E_{\varphi} \varphi < 1$, then $Q\{0 < \varphi < 1\} = 0$ for all $Q \in \varphi(\mathcal{A})$ with the property $Q \geq_T P$ or $Q \leq_T P$.

f) Suppose $\varphi = 1_{[T > 1]} + \gamma 1_{[T = 1]}$. Then $0 < E_{\varphi} \varphi$ implies $s \in I_P$ and $E_{\varphi} \varphi < 1$ implies $s \in J_Q$.

A subset $\mathcal{M} \subseteq \varphi(\mathcal{A})$ is called majorized if there is a measure $\mu$ on $\mathcal{A}$ such that every $P \in \mathcal{M}$ has a density with respect to $\mu$ (see Siebert [12]).

**Proposition 2.3.** A subset $\mathcal{M}$ of $\varphi(\mathcal{A})$ which is totally ordered with respect to $\leq_T$ is majorized.

**Proof.** We define an equivalence relation on $\bigcup_{P \in \mathcal{M}} Z_P$. Two points $x, y$ of this set are called equivalent if there is a non-empty finite subset $\mathcal{F} \subseteq \mathcal{M}$ such that $\bigcup_{P \in \mathcal{F}} Z_P$ is an interval and $x, y \in \bigcup_{P \in \mathcal{F}} Z_P$. Let $\{A_i, i \in I\}$ be the family of all equivalence classes where $i \in A_i \subseteq \bar{R}$ for all $i \in I$.

Suppose $A_i$ is not degenerated. It easily follows from the above definition that there is a countable subset $\mathcal{W}_i \subseteq \mathcal{M}$ with $A_i = \bigcup_{Q \in \mathcal{W}_i} Z_Q$. If $P \in \mathcal{M}$ and $Z_P \subseteq A_i$, then $P(A) = P\left(\bigcup_{P \in \mathcal{F}} (A \cap \{T \leq Z_P\})\right) \leq \sum_{Q \in \mathcal{W}_i} P(A \cap \{T \leq Z_P\})$. Hence Lemma 2.1.b) implies that $P$ is absolutely continuous with respect to $\mathcal{W}_i$ since $\mathcal{M}$ is totally ordered. We conclude that for each $i \in I$ there is a $\mu_i \in \varphi(\mathcal{A})$ such that $P \ll \mu_i$ for all $P \in \mathcal{M}$ with $Z_P \subseteq A_i$ (if $A_i$ is degenerated, use Lemma 2.1.c)). $\mathcal{M}$ is majorized by the measure $\mu = \sum_{i \in I} \mu_i$.

**Remark 2.4.**

a) If $\Omega = R$, $\mathcal{A} = \mathcal{B}_1$ and $T = id_\mathcal{B}$, then $Z_P$ is identified with Pfanzagl's ([11], p. 1219) convex support of $P$. The set of
all \( i \in I \) such that \( A_i \) is not degenerated is countable. \( \mathcal{M} \) is dominated if it contains at most countably many Dirac measures (then the set of all \( i \in I \) such that \( A_i \) is degenerated is at most countable). This result was proved by Pfanzagl ([11], Theorem 3) by means of a topological argument.

b) Let \( \hat{\mathcal{A}} \) consist of all sets \( A \subset \Omega \) such that \( A \cap \{ T \in Z_p \} \in \mathcal{A} \) for every \( P \in \mathcal{M} \). From the definition of \( \mu \) (proof of Proposition 2.3) it is clear that it can be extended to a measure \( \hat{\mu} \) on \( \hat{\mathcal{A}} \) and that \( (\Omega, \hat{\mathcal{A}}, \hat{\mu}) \) is strictly localizable (Fremlin [3], p. 172, a direct sum of finite measure spaces). Experiments majorized by a localizable measure retain several properties of experiments dominated by a finite measure (see for example, Ghosh et al. [4] and the references given there).

Let \( \Lambda^* \) be the set of all \( \varphi \in \Lambda \) such that for some \( P \in \mathcal{M} \), we have \( \varphi = 1_{\{ T \in I \}} \) or \( 0 < E_P \varphi < 1 \) and \( P[0 < \varphi < 1] = 0 \). The following properties Propositions 2.5.a) and b) are quite obvious for a dominated set \( \mathcal{M} \) which is totally ordered with respect to \( \leq_T \) if the map \( S \) is defined by \( S(\varphi) = E_P(1 - \varphi) \) for some equivalent finite dominating measure \( \mu \).

**Proposition 2.5.** Let \( \mathcal{M} \) be a non-empty subset of \( \alpha _1 (\Lambda) \) which is totally ordered with respect to \( \leq_T \). Then there is a map \( S \) from \( \Lambda^* \) to \( \bar{R} \) with the following properties.

a) If \( \varphi, \psi \in \Lambda^* \) and \( \varphi \leq \psi[\mathcal{M}] \), then \( S(\varphi) \geq S(\psi) \); moreover, if \( \varphi \leq \psi[\mathcal{M}] \) and \( P[\varphi < \psi] > 0 \) for some \( P \in \mathcal{M} \), then \( S(\varphi) > S(\psi) \).

b) If \( \varphi \in \Lambda^* \) and \( I \) is a non-empty, countable subset of \( \Lambda^* \) such that \( \varphi \leq \psi[\mathcal{M}] \) for all \( \psi \in I \) and \( \inf \{ S(\psi) : \psi \in I \} = S(\varphi) \), then \( \sup \{ E_Q \psi : \psi \in I \} = E_Q \varphi \) for all \( Q \in \mathcal{M} \).

**Proof.** Let \( \{ A_i, i \in I \} \) be defined as in the proof of Proposition 2.3. We assume that \( i \) is a real interior point of \( A_i \) if \( A_i \) is not degenerated. We define a partition of \( \Lambda^* \). Put \( A_i = \{ 1_{\{ T > s \}} \} \) if \( i \in I \) and \( A_i \) is degenerated. If \( A_i \) is not degenerated, then let \( A_i \) denote the set of all \( \varphi \in \Lambda^* \) of the type \( \varphi = 1_{\{ T > s \}} + \gamma 1_{\{ T = s \}} \) for some \( s \in A_i \). Since \( \mathcal{M} \) is totally ordered with respect to \( \leq_T \), we get from Lemma 2.2.e) that \( P[0 < \varphi < 1] = 0 \) for all \( P \in \mathcal{M} \) and \( \varphi \in \Lambda^* \). We have \( \Lambda^* = \bigcup_{i \in I} A_i \). If \( \varphi = 1_{\{ T > s \}} + \gamma 1_{\{ T = s \}} \in \Lambda^* \) and \( 0 < E_P \varphi < 1 \), then \( s \in Z_P \) and \( Z_P \cap A_i \) for some \( i \in I \) (see Lemma 2.2.f)).

This \( A_i \) is not degenerated. Thus \( \varphi \in A_i \). If \( \varphi \in \Lambda^* \) and \( E_Q \varphi \in [0, 1] \) for all \( Q \in \mathcal{M} \), then \( \varphi = 1_{\{ T \notin I \}} \) for some \( P \in \mathcal{M} \). Hence \( \varphi = 1_{\{ T > i \}} \) or \( \varphi = 1_{\{ T \geq i \}} \) for some \( i \in I \). If the corresponding \( A_i \) is degenerated, we have \( \varphi = 1_{\{ T > i \}} \) indeed \( \varphi = 1_{\{ T \geq i \}} \) would imply that \( Z_P \cup \{ i \} \) is a non-degenerated interval which is contained in \( A_i \).

Obviously, for each \( i \in I \) such that \( A_i \) is not degenerated, there is an \( a_i > 0 \) with the property that \( i + a_i \) is an interior point of \( A_i \). We
define $S(\varphi) = c_i E_{\varphi} (1 - \varphi) + i$ if $\varphi \in \Lambda_i$ and $A_i$ is not degenerated and $S(\varphi) = S(1_{\varphi = i}) = i$ if $\varphi \in \Lambda_i$ and $A_i$ is degenerated ($\mu_i$ as in the proof of Proposition 2.3). Straightforward calculations show that $S$ has the above properties.

3. Conditions for the existence of monotone likelihood ratios

In order to characterize (among other things) subsets $\mathcal{M} \subset \alpha_i(\mathcal{A})$ which are totally ordered with respect to $\leq_T$ for some suitable statistic $T$, we introduce the following set-up. Let $\mathcal{M}$ be a non-empty subset of $\alpha_i(\mathcal{A})$. Suppose $\mathcal{M} \subset \mathcal{D} \subset \alpha_i(\mathcal{A})$ in such a way that for each $P \in \mathcal{D}$ there is a $Q \in \mathcal{M}$ which is equivalent to $P$. For each $P \in \mathcal{D}$ let $\mathcal{W}_P$ be a subset of $\alpha_i(\mathcal{A})$ with $P \in \mathcal{W}_P$. Assume that for $P, Q \in \mathcal{M}$ we have $P \in \mathcal{W}_Q$ or $Q \in \mathcal{W}_P$. Put $Q = \bigcup_{P \in \mathcal{D}} \mathcal{W}_P$.

**Remark 3.1.** It is not generally true that an order $\leq$ is defined by "$V \leq W$ iff $W \in \mathcal{W}_V$". Indeed, put $\mathcal{M} = \mathcal{D} = \{\varepsilon_0, \varepsilon_1, \varepsilon_i\}$ and $\mathcal{W}_i = \{\varepsilon_{(i+1)\mod 3}\}$, where $\varepsilon_i$ denotes the Dirac measure at $i$. But if $\leq$ is an arbitrary partial order on a subset $\mathcal{D}' \subset \alpha_i(\mathcal{A})$ such that $\mathcal{M} \subset \mathcal{D}'$ and $\mathcal{M}$ is totally ordered with respect to $\leq$, then the above conditions on $\mathcal{W}_P$ are fulfilled with $\mathcal{M} \subset \mathcal{D} \subset \mathcal{D}'$ and $\mathcal{W}_P = \{Q \in \mathcal{D}' : Q \geq P\}$, $P \in \mathcal{D}$.

Let $\Phi' \subset \Phi$ be sets of tests on $(\Omega, \mathcal{A})$. Our aim it to give conditions on $\Phi, \Phi', \mathcal{M}, \mathcal{D}$ and $\{\mathcal{W}_P, P \in \mathcal{D}\}$ which imply that there is a statistic $T$ such that each pair $P, Q$ with $P \in \mathcal{D}$ and $Q \in \mathcal{W}_P$ has monotone likelihood ratio with respect to $T$.

**CONDITION 3.2.** (On $\Phi, \Phi', \mathcal{M}, \mathcal{D}$ and $\{\mathcal{W}_P, P \in \mathcal{D}\}$) a) For every $\alpha \in ]0, 1[$ and $P \in \mathcal{D}$ there is a $\varphi \in \Phi$ such that $E_{P, \varphi} = \alpha$.

b) If $P \in \mathcal{D}$, $Q \in \mathcal{W}_P$, $\varphi \in \Phi$, $0 < E_{P, \varphi}$ and $E_{Q, \varphi} < 1$, then $P \leq Q$.

c) For every $P \in \mathcal{D}$ there is an $M_P \in \mathcal{M}$ which is equivalent to $P$ and a $\xi_P \in \Phi$ such that $E_{P, \xi_P} = 0$ and $P_{\xi_P} Q$ for every $Q \in \mathcal{W}_P$. Moreover, $(M_P)_{\xi_P} V$ for every $V \in \mathcal{W}_P \cap \mathcal{M}$.

d) If $\varphi, \psi \in \Phi$ and $\varphi \leq \psi [\mathcal{M}]$, then $\varphi \leq \psi [Q]$. $\Phi'$ is the set of all $\varphi \in \Phi$ such that $\varphi = \xi_P$ for some $P \in \mathcal{D}$ or $\varphi = 1_{[0, \mu_{P + Q}) \cup \mu_{P} + \mu_{P + Q})} [P]$ and $0 < E_{P, \varphi} < 1$ for some $P \in \mathcal{D}$, $Q \in \mathcal{W}_P$ and $0 \leq c < \infty$. There is a map $S : \Phi' \to \mathbb{R}$ with the following properties.

e) If $\varphi, \psi \in \Phi'$ and $\varphi \leq \psi [\mathcal{D}]$, then $S(\varphi) \geq S(\psi)$; moreover, if $\varphi \leq \psi [\mathcal{D}]$ and $P[\varphi < \psi] > 0$ for some $P \in \mathcal{D}$, then $S(\varphi) > S(\psi)$.

f) If $\varphi \in \Phi'$ and if $\Gamma$ is a non-empty, countable subset of $\Phi'$ such that $\varphi \leq \psi [\mathcal{D}]$ for all $\psi \in \Gamma$ and $\inf \{S(\psi) : \psi \in \Gamma\} = S(\varphi)$, then $\sup \{E_{Q, \psi} : \psi \in \Gamma\} = E_{Q, \varphi}$ for every $Q \in \mathcal{Q}$.

**Example 3.3.** a) Suppose $\mathcal{M}$ is totally ordered by $\leq_T$ for some
statistic $T$, $\mathcal{P} = \mathcal{M}$, $W_P = \{ Q \in \alpha_1(\mathcal{A}) : Q \geq_T P, Q \leq \mathcal{M} \}$, $\Phi = \Lambda$ and $\xi_P = 1_{\{ T \leq P \} }$. If $0 < \alpha = P(dQ/d\mu) > cP(d\mu) > 1$ where $0 \leq c < \infty$, $P, Q \in \alpha_1(\mathcal{A})$, $\mu = P + Q$ and $P \leq c, Q$, then there is a $\varphi \in \Lambda$ (see Section 1) with $\varphi = 1_{\{ dQ/d\mu > cP(d\mu) \} } [P]$. Indeed, by Lemma 2.2a) there is a $\varphi \in \Lambda$ with $E_{\varphi} = \alpha$. The rest follows from the Neyman-Pearson-Lemma. Thus we have $\Phi' \subset \Lambda^*$. Using Lemma 2.2 and Proposition 2.5, we see that Condition 3.2 is fulfilled.

b) If $\mathcal{M}$ is given as in a), $\mathcal{P} \supset \mathcal{M}$, and for each $P \in \mathcal{P}$ there is a $Q \in \mathcal{M}$ which is equivalent to $P$, then, by Lemma 2.2 and Condition 3.2 is fulfilled if $W_P = \{ Q \in \alpha_1(\mathcal{A}) : Q \geq_T P, Q(\cdot \cap \{ T \in D \}) < \mathcal{M} \} (P \in \mathcal{P})$, $\Phi = \{ \varphi \in \Lambda : \varphi = 1_{\{ T \leq P \} } \text{ or } E_{\varphi} > 0 \text{ for some } P \in \mathcal{M} \}$ and $\xi_P = 1_{\{ T \leq P \} }$.

c) Suppose $\mathcal{O} = R$, $\mathcal{A}$ is the power set of $R$, $\mathcal{M} = \mathcal{P}$ is the set of all Dirac measures on $\mathcal{A}$, $\Phi$ is the set of all tests of the type $\varphi = a1_{\{ x \geq 1 \} } + 1_{\{ x = 1 \} }$, $a, x \in R$. Then Condition 3.2 is fulfilled with $M_r = P$ for all $P \in \mathcal{P}$, $W_{\varphi} = \{ \varphi : y \geq x \}$, $\xi_{\varphi} = 1_{\{ x \geq 1 \} }$, $\Phi' = \{ \varphi : x \in \mathcal{R} \}$ and $S(1_{x \geq 1}) = x, x \in R$.

**Proposition 3.4.** Suppose that Condition 3.2 holds. If $\varphi, \psi \in \Phi'$, then $\varphi \leq \psi [\mathcal{M}]$ or $\psi \leq \varphi [\mathcal{M}]$.

**Proof.** First we prove that $\xi_{\varphi} \leq \xi_{\psi} [\mathcal{M}]$ if $P, Q \in \mathcal{P}$ and $M_q \in W_{\mathcal{M}}$. By Conditions 3.2b), c) and (1.1), $E_{\mathcal{M}_q} \xi_{\varphi} = 0$. Applying Condition 3.2b) and (1.1) gives $E_{\mathcal{M}} \xi_{\varphi} = 0$ for every $V \in \mathcal{M}$ such that $M_q \in W_{\mathcal{V}_q}$. By (1.4) and Condition 3.2c), $\xi_{\varphi} \leq \xi_{\psi} [V]$ for all $V \in W_{\mathcal{M}} \cap \mathcal{M}$.

Next we suppose $P \in \mathcal{M}$ and $0 < E_{\varphi} \psi < 1$. Then $\psi \leq \varphi [P]$ implies $\psi \leq \varphi [\mathcal{M}]$, and $\varphi \leq \varphi [P]$ implies $\varphi \leq \varphi [\mathcal{M}]$. Indeed, suppose $\psi \leq \varphi [P]$. Then we have $E_{\varphi} \psi \leq E_{\varphi} \varphi < 1$. If $V \in \mathcal{M}$, then $E_{\varphi} \psi = 1$ or, by Condition 3.2b), $P_{\varphi} V$, and (1.4) implies $\psi \leq \varphi [V]$. If $V \in \mathcal{M}$ and $P \in W_{\mathcal{V}_q}$, then $E_{\varphi} \psi = 0$ or, by Condition 3.2b), $P_{\varphi} V$. Then (1.4) implies $\varphi \leq \varphi [V]$. The proof in the case $\varphi \leq \varphi [P]$ is analogous.

It remains to prove that $\varphi \leq \psi [P]$ or $\varphi \leq \psi [P]$ if $\varphi = 1_{\{ dQ/d\mu > cP/d\mu \} } [P]$ and $0 < E_{\varphi} \psi < 1$, where $P \in \mathcal{P}$, $Q \in W_{\mathcal{V}_q}$, $0 \leq c < \infty$ and $\mu = P + Q$. The proof of this assertion is more or less the same as the one of Pfanzagl’s (10), Hilfssatz 2). If $E_{\varphi} \psi = 0$ or $E_{\varphi} \psi = 1$, there is nothing to be proved. Suppose $0 < E_{\varphi} \psi < 1$. It follows from Condition 3.2b) that $\varphi$ is most powerful for testing $P$ against $Q$ at level $E_{\varphi} \psi$. The Neyman-Pearson-Lemma shows that there is a $k \in [0, \infty]$ such that $1_{\{ dQ/d\mu > kdP/d\mu \} } \geq \varphi \geq 1_{\{ dQ/d\mu > dP/d\mu \} } [\mu]$. The rest of the proof is easy.

In Proposition 3.4 we have $P(0 < \varphi < 1) = 0$ for every $P \in \mathcal{M}$ and $\varphi \in \Phi'$. Indeed, by (1.2) and Condition 3.2c), $V(0 < \xi_{\varphi} < 1) = 0$ for all $V \in W_{\mathcal{M}} \cap \mathcal{M}$. On the other hand, by Condition 3.2b) and (1.1), $E_{\varphi} \xi_{\varphi} = 0$ for all $V \in \mathcal{M}$ such that $M_q \in W_{\mathcal{V}_q}$. Now suppose $0 < E_{\varphi} \psi < 1$ and $Q(0 < \varphi < 1) = 0$ for some $Q \in \mathcal{M}$. Then Condition 3.2b) and (1.3) imply that $V(0 < \varphi < 1) = 0$ for every $V \in W_{\mathcal{V}_q}$ and $V(0 < \varphi < 1) = 0$ for every $V \in \mathcal{M}$.
such that \( Q \in \mathcal{W}_P \). We conclude that instead of tests \( \varphi \) from \( \Phi' \) we could have used the sets \( \{ \varphi = 1 \} \) or \( \{ \varphi = 0 \} \). In analogous situations, Pfanzagl ([10], p. 114, the set \( C \)) and Dettweiler ([2], p. 250) considered sets of the type \( \{ \varphi = 0 \} \). Proposition 3.4 replaces Pfanzagl's ([10], Hilfsatz 2) and Dettweiler's ([2], assertion (F)). If we replace \( R \) in Example 3.3. c) by the set of all rational numbers, then in this example Pfanzagl's set \( C \) consists of all intervals \( ]-\infty, x[ \) and \( ]-\infty, x] \), \( x \) rational.

4. Conditions of Pfanzagl and Dettweiler

Pfanzagl ([10], p. 110) considered a set of tests \( \Phi \) and a non-empty dominated set \( \mathcal{M} \subset \mathcal{A} \) which bears a total order \( \leq \). He introduced Conditions 3.2. a) to c) where \( \mathcal{M} = \mathcal{P} = Q \), \( \mathcal{W}_P = \{ Q \in \mathcal{M} : Q \geq P \} \), \( \mathcal{M}_P = P \) for all \( P \in \mathcal{P} \). Obviously, this implies that all of Condition 3.2 are fulfilled (we may put \( S(\varphi) = E_\mu (1 - \varphi) \) for all \( \varphi \in \Phi' \), where \( \mu \) is a dominating finite measure). Under the additional assumption that for every \( P \in \mathcal{Q} \) there is a \( \varphi \in \Phi \) such that \( E_\mu \varphi = 1 \) and \( Q \leq P \) for every \( Q \leq P \), Pfanzagl ([10], p. 110) proved that there is a real valued statistic \( T \) such that \( P \leq TQ \) whenever \( P, Q \in \mathcal{Q} \) and \( P \leq Q \). We shall see that only Condition 3.2 is needed.

Dettweiler ([2], Theorem 1) considered a set of tests \( \Phi \) and non-empty subsets \( \mathcal{P} \subset Q \subset \mathcal{A} \) where \( Q \) bears a partial order \( \leq \) with the property that for all \( P \in \mathcal{P} \) and \( Q \in I \) there is a \( K \in \mathcal{P} \) such that \( K \leq Q \) and \( K \leq P \).

**Proposition 4.1.** Suppose that for every \( a \in [0, 1] \) and \( P \in \mathcal{P} \), there is a \( \varphi \in \Phi \) such that \( E_\varphi \varphi = a \) and that \( P, Q \) if \( P \in \mathcal{P} \), \( Q \in I \), \( P \leq Q \) and \( \varphi \in \Phi \). Then all probability measures from \( \mathcal{P} \) are equivalent and there is a \( G \in \mathcal{A} \) such that \( P(G) = 1 \), \( Q(\cdot \cap G) \ll P \) and \( \chi(1 - 1_G) = 1 - 1_G \) [Q] for all \( P \in \mathcal{P}, Q \in I \) and \( \chi \in \Phi \). Furthermore, Condition 3.2 is fulfilled with \( \mathcal{W}_P = \{ Q \in I : Q \geq P \} \), \( \mathcal{A} = \{ K \} \), \( M_P = K \) and \( S(\varphi) = E_K (1 - \varphi) \) for all \( P \in \mathcal{P} \) and \( \varphi \in \Phi' \), where \( K \in \mathcal{P} \) is arbitrary.

**Proof.** Suppose \( K, P \in \mathcal{P} \). We will show that \( K \) and \( P \) are equivalent. There is a \( Q \in \mathcal{P} \) such that \( Q \leq P \) and \( Q \leq K \). We can find a \( \varphi \in \Phi \) such that \( E_\varphi \varphi = 1 \), \( Q \leq K \) and \( Q \leq P \). We can find a \( \varphi \in \Phi \) such that \( E_\varphi \varphi = 1 \), \( Q \leq K \) and \( Q \leq P \). Now a simple application of (1.2) gives \( Q \ll P \) and \( Q \ll K \). On the other hand, there exist \( \varphi, \chi \in \Phi \) such that \( E_\varphi \psi = 0 \), \( Q \ll K \), \( E_\varphi \chi = 0 \) and \( Q \ll P \). Then (1.2) shows that \( K \ll Q \) and \( P \ll Q \). Thus \( K \) and \( P \) are equivalent.

Let \( K \in \mathcal{P} \) be fixed. There are tests \( \varphi, \varphi \in \Phi \) such that \( E_\varphi \varphi = 0 \), \( E_\varphi \psi = 1 \), \( P \ll Q \) and \( P \ll K \) whenever \( P \in \mathcal{P} \), \( Q \in I \), \( P \ll Q \) and \( P \ll K \). Define \( G = \{ \varphi = 0 \} \cap \{ \varphi = 1 \} \).

For a fixed \( Q \in I \) there is a \( P \in \mathcal{P} \) such that \( P \ll Q \) and \( P \ll K \). If \( K(B) = 0 \), then from (1.2) we get \((1 - \varphi)_P \neq 1 \) [P]. Using again (1.2),
this implies $\phi_{1_{[\phi=1]} \cap B}=1_{[\phi=1]} \cap B \ [Q]$. Hence $Q(G \cap B)=0$. From $P \ll K$, we get $P(\{\phi>0\} \cup \{\phi<1\})=0$. If $\chi \in \Phi$, then $P_{\chi}Q$. Applying (1.2) gives $\chi_{1_{[\phi>0]} \cup \{\phi<1\}}=1_{[\phi>0]} \cup \{\phi<1\} \ [Q]$. The rest of the proof is obvious.

Dettweiler ([2], Theorem 1) showed that under the premise of Proposition 4.1 and two additional assumptions there is a real valued statistic $T$ such that $P \leq T Q$ whenever $P \in \mathcal{P}$, $Q \in \mathcal{Q}$ and $P \leq Q$. We shall see that this result follows from Condition 3.2 alone. Dettweiler assumed that there is a sequence $(P_n)$ in $\mathcal{P}$ which has the property that for each $P \in \mathcal{P}$ there is a positive integer $m$ such that $P_m \leq P$, and that $Q \leq P$ if $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, $Q \leq P$ and $\varphi \in \Phi$.

Example 4.2. a) In Example 3.3. c) $\mathcal{P}$ is not dominated. Therefore neither Pfanzagl’s nor Dettweiler’s conditions hold.

b) If in Example 3.3. c) we take the set of rational numbers instead of $\mathcal{R}$, then Pfanzagl’s conditions are fulfilled, but Dettweiler’s ones are not, since the premise of Proposition 4.1 does not hold: If $0<\gamma<1$, $P=\varepsilon_1$, $Q=\varepsilon_2$ and $\varphi=1_{[1]}+\gamma 1_{[1,\infty[}$, then $P_{\varphi}Q$ is not true.

5. Main results

Suppose that Condition 3.2 holds. The aim of this section is to introduce a statistic $T$ which has the property that $P \leq T Q$ whenever $Q \in \mathcal{W}_P$ and $P \in \mathcal{P}$. Let $F \subset \mathcal{R}$ be the image of $\Phi'$ under $S$. Put

$$F_0 = \bigcup_{n \geq 1} \{ b \in F : |b| > b + 1/n \} \mathcal{R} \setminus F \} \quad \text{and} \quad F_1 = F \setminus F_0 .$$

Then $F_0$ is countable. Furthermore, let $F_2$ denote a countable dense subset of $F_1$. For every $b \in F_0 \cup F_2$ we choose a $\varphi \in \Phi'$ such that $S(\varphi)=b$; in this way, we get a countable subset $\Upsilon \subset \Phi'$. The definition of $\Upsilon$ differs from the definition of certain countable sets which were used by Pfanzagl ([8], p. 171) and Dettweiler ([2], p. 251) for the same purpose. It resembles more Dettweiler’s definition than Pfanzagl’s one. Dettweiler’s condition (H), (C2) makes no sense; this is certainly due to misprints.

The following definition is completely analogous to those given by Pfanzagl ([8], p. 171) and Dettweiler ([2], p. 252). For each $\omega \in \Omega$ we put

$$T(\omega) = \inf \{ S(\varphi) : \varphi \in \Upsilon ; \varphi(\omega)=0 \}$$

if $\omega \in \bigcup \{ \chi=0 \}$ and otherwise $T(\omega)=\infty$.

Remark 5.1. Without loss of generality we could have assumed $S$ to be real valued and bounded. In this case, in the definition of $T$,
\( \infty \) could be replaced by a real number which is greater than every \( S(\varphi) \), and \( T \) would become a bounded real valued function.

The next result replaces Pfanzagl's [8], Hilfssatz 1 (see also Pfanzagl [10], p. 114, (10)). The proof is almost the same as part (a) and (b) of Pfanzagl's proof. Part (c) of that proof is not needed because of our definition of \( F_\varphi \). Moreover, the argument of part (c) cannot be used under our conditions. This can be seen by means of Eexample 3.3. c), since then, roughly speaking, \( \sup \{ S(\varphi) \mid \varphi \in \Gamma \} = S(\varphi) \) does not imply \( \inf \{ E_\varphi \varphi : \varphi \in \Gamma \} = E_\varphi \varphi \).

**Lemma 5.2.** Suppose that Condition 3.2 holds. Let \( S \) be real valued and \( T(\omega) > -\infty \) for all \( \omega \in \Omega \). Then \( T \) is measurable and, if \( \varphi \in \Phi' \), then \( \varphi = 1_{\{ T > S(\varphi) \}} \) [\( Q \)].

**Proof.** The measurability follows from

\[
\{ T < a \} = \bigcup_{\varphi \in \Phi, S(\varphi) < a} \{ \varphi = 0 \} .
\]

It suffices to show that \( \{ \varphi = 0 \} = \{ T \leq S(\varphi) \} \) [\( Q \)]. From (5.1), Conditions 3.2. e) and d) we get

\[
\{ T < S(\varphi) \} \subset \{ \varphi = 0 \} \quad [Q]
\]

for all \( \varphi \in \Phi' \). For each \( \varphi \in \Phi \) we have

\[
\{ \varphi = 0 \} \subset \{ T \leq S(\varphi) \} .
\]

Suppose \( \varphi \in \Phi' \).

**Case 1.** \( S(\varphi) \notin F_\varphi \). Then there exists a \( \chi \in \Phi \) such that \( S(\chi) > S(\varphi) \), and

\[
S(\varphi) = \inf \{ S(\varphi) \mid \varphi \in \Phi, S(\varphi) > S(\varphi) \} .
\]

Now Condition 3.2. f) implies

\[
E_\varphi \varphi = \sup \{ E_\varphi \varphi : \varphi \in \Phi, S(\varphi) > S(\varphi) \} \quad \text{for all } Q \in Q .
\]

This is equivalent to

\[
Q_{\{ \varphi = 0 \}} = \inf \{ Q_{\{ \varphi = 0 \}} : \varphi \in \Phi, S(\varphi) > S(\varphi) \} \quad \text{for all } Q \in Q .
\]

Using Conditions 3.2. e) and d), we have \( \{ \varphi = 0 \} \supseteq \{ \varphi = 0 \} \) [\( Q \)] whenever \( S(\varphi) > S(\varphi) \). Therefore we conclude

\[
\{ \varphi = 0 \} = \bigcap_{\varphi \in \Phi, S(\varphi) > S(\varphi)} \{ \varphi = 0 \} \quad [Q] .
\]

On the other hand,
\[ \{ T \leq S(\varphi) \} = \bigcap_{\varphi \in \mathcal{W}, S(\varphi) > S(\varphi)} \{ T < S(\varphi) \} = \bigcap_{\varphi \in \mathcal{W}, S(\varphi) > S(\varphi)} \{ T \leq S(\varphi) \}, \]

and by (5.2) and (5.3), we get

\[ \{ T \leq S(\varphi) \} = \bigcap_{\varphi \in \mathcal{W}, S(\varphi) > S(\varphi)} \{ \varphi = 0 \} [\mathcal{Q}] . \]

**Case 2.** \( S(\varphi) \in R. \) Then there is a \( \xi \in \mathcal{W} \) such that \( \xi = \varphi [\mathcal{P}] \). Hence \( \xi = \varphi [\mathcal{Q}] \), by Condition 3.2. d). From the definition of \( T \) and \( \mathcal{W} \) it follows that \( \{ T = S(\varphi) \} \subset \{ \xi = 0 \} \). Hence \( \{ T = S(\varphi) \} \subset \{ \varphi = 0 \} [\mathcal{Q}] \). By (5.2) and (5.3), we get \( \{ T \leq S(\varphi) \} = \{ \varphi = 0 \} [\mathcal{Q}] \).

We need one more lemma which easily follows from Condition 3.2 and (1.2).

**Lemma 5.3.** Suppose that Condition 3.2 holds, \( P \in \mathcal{P} \), \( Q \in \mathcal{W}_P \) and \( \mu = P + Q \).

a) If \( P[\{ dQ/d\mu > cdP/d\mu \} = 0 \) for some \( c \in [0, \infty] \), then

\[ 1_{\{ dQ/d\mu > cdP/d\mu \}} = \xi_\mathcal{P} \]

b) If \( P[\{ dQ/d\mu > cdP/d\mu \} = 1 \) for some \( c \in [0, \infty] \), then

\[ \{ dQ/d\mu > cdP/d\mu \} = \Omega [\mu] . \]

Now we can prove our main result. One essential difference to the methods used in an analogous situation by Pfanzagl ([8], pp. 174–176 and [10], p. 115) and Dettweiler ([2], pp. 152–153) is that we make use of a well-known factorization theorem due to Doob.

**Proposition 5.4.** Suppose that Condition 3.2 holds. Then there is a statistic \( T \) such that \( P \leq_T Q \) for all \( P \in \mathcal{P} \) and \( Q \in \mathcal{W}_P \).

**Proof.** Because of Remark 5.1, we assume without loss of generality that \( S \) is real valued and \( T(\omega) > -\infty \) for all \( \omega \in \Omega \). Put \( \mu = P + Q \). If \( c \in [0, \infty] \) and \( 0 < P[\{ dQ/d\mu > cdP/d\mu \} = 1 \), then there is a test \( \varphi_\mathcal{W} \in \mathcal{W}_\mathcal{P} \) such that \( \varphi_\mathcal{W} = 1_{\{ dQ/d\mu > cdP/d\mu \}} [\mu] \). This follows from Condition 3.2. a) and the Neyman-Pearson-Lemma.

If \( c \in [0, \infty] \) and \( P[\{ dQ/d\mu > cdP/d\mu \} = 0 \), we define \( \varphi_\mathcal{W} = \xi_\mathcal{P} \) (see Lemma 5.3. a)). If \( P[\{ dQ/d\mu > cdP/d\mu \} = 1 \), we put \( \varphi_\mathcal{W} = 1_\mathcal{Q} \) (see Lemma 5.3. b)).

Obviously, \( \varphi_\mathcal{W} \leq \varphi_\mathcal{W} [\mathcal{P}] \) for \( c \leq c' \). As in the proof of Proposition 3.4, we get \( \varphi_\mathcal{W} \leq \varphi_\mathcal{W} [\mathcal{P}] \) for \( c \leq c' \). Therefore a non-decreasing map \( m \) from \( [0, \infty] \) to \( R \cup \{ -\infty \} \) is defined by \( m(c) = S(\varphi_\mathcal{W}) \) if \( P[\{ dQ/d\mu > cdP/d\mu \} = 1 \) and \( m(c) = -\infty \) otherwise. Put

\[
g(\omega) = \begin{cases} 
\frac{dQ}{d\mu}(\omega)/\frac{dP}{d\mu}(\omega) & \text{if } \frac{dP}{d\mu}(\omega) > 0 \\
\infty & \text{otherwise}.
\end{cases}
\]
If \( P(dQ/d\mu > cdP/d\mu) < 1 \) and \( c < \infty \), by Lemmas 5.2 and 5.3 a), we get
\[
\{g > c\} = \{dQ/d\mu > cdP/d\mu\} = \{\varphi_c = 1\} = \{T > m(c)\} \quad [\mu].
\]
If \( P(dQ/d\mu > cdP/d\mu) = 1 \), then \( c < \infty \) and, by Lemma 5.3 b),
\[
\{g > c\} = \{dQ/d\mu > cdP/d\mu\} = \Omega = \{T > -\infty\} = \{T > m(c)\} \quad [\mu].
\]
If \( c = \infty \), then, by Lemmas 5.2 and 5.3 a),
\[
\{g = \infty\} = \{dP/d\mu = 0, dQ/d\mu > 0\} = \{\xi_0 = 1\} = \{T > m(\infty)\} \quad [\mu].
\]

Now we use the proof of the well-known factorization theorem due to Doob (see Dellacherie and Meyer [1], 13-I-18): If we define
\[
g_n = \sum_{k=1}^{\infty} k2^{-n} 1_{\{k2^{-n} < \varphi \leq (k+1)2^{-n}\}} + \infty 1_{\{\varphi = \infty\}}
\]
for each positive integer \( n \), then \( g = \sup_n g_n \). We have \( g_n = h_n \circ T \quad [\mu] \), where
\[
h_n(t) = \sum_{k=1}^{\infty} k2^{-n} 1_{\{m(k2^{-n}) < m((k+1)2^{-n})\}}(t) + \infty 1_{\{m(\infty) = \infty\}}(t).
\]
Since \( m \) is non-decreasing, each \( h_n \) is non-decreasing. Arguing similar as Dellacherie and Meyer [1], we see that there is a non-decreasing map \( h: \tilde{R} \to \tilde{R} \) such that \( g = h \circ T \quad [\mu] \).

**Remark 5.5.** Proposition 5.4 obviously implies that \( \mathcal{M} \) is totally ordered with respect to \( \leq_T \).

**REFERENCES**


