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POSITIVE DEPENDENCE ORDERINGS

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Summary

This paper presents a systematic basis for studying orderings of bivariate distributions according to their degree of positive dependence. The general concept of a positive dependence ordering (PDO) is introduced and its properties discussed. Based on this concept, a new ordering of bivariate distributions according to their degree of total positivity of order two (TP_2) is presented, and is shown to be a PDO. Properties of this TP_2 ordering are derived and numerous applications are presented.

1. Introduction

Two random variables X and Y are said to be positively dependent if either random variable being large probabilistically indicates that the other random variable is large. This concept of positive dependence has played a fundamental role in many recent new ideas in statistics. It has led to a number of different concepts including measuring the degree of positive dependence, describing the properties of the positive dependence, and relating the degree of positive dependence between two pairs of random variables. Much of the research in these areas is concerned with specifics and very little attention appears to be directed to organizing and unifying the various developments. In this paper, we provide a framework for examining and further developing the fundamental concept of one pair of random variable's being

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more positively dependent than another pair. The development of this framework is also related to the more widely considered area of properties of positive dependence. Numerous examples and applications are discussed.

The research concerning various specific positive dependence properties has been fairly recent, but explosive. Some early concepts that were developed include positively quadrant dependence (PQD) (Lehmann [25]), association (Esary, Proschan and Walkup [11]), positively regression dependence (Tukey [43]), and TP_2 (e.g., Karlin [20]). More recently developed ideas include dependence by total positivity of degree (m, n) (Shaked [36]), right corner set increasing (Brindley and Thompson [6]) and some weakened notions of association (Shaked [38]). The implications among many of these properties have been studied, e.g., Barlow and Proschan [3], Shaked [37], and Block and Ting [5].

In the last several years, a new and fundamental approach to studying positive dependence has been started. The basic idea is to compare two bivariate distributions having the same pair of marginals to determine whether one distribution is more positively dependent than the other, thus attempting to partially order the distributions according to degree of positive dependence. The task is a difficult one in that the suitable comparability conditions are often difficult to obtain. Some specific examples of such orderings are: (a) Tchen's [42] more concordant ordering (see also Ahmed, Langberg, Léon, and Proschan's [1] positively quadrant dependence ordering), which measures the direction of increasing positively quadrant dependence; (b) Rinott and Pollak's [31] covariance ordering; and (c) Shaked and Tong's [39] orderings for multivariate exchangeable random variables.

Little work has yet been done in developing a framework of conditions (such as the work of Rényi [30] and Hall [17] for numerical measures of dependence) for orderings of bivariate distributions, although Scarsini [33] presented a class of axioms for measures of concordance. Yet the idea of ordering bivariate distributions by positive dependence is a fundamental concept for studying the notion of positive dependence. In this paper, we develop the general concept of a positive dependence ordering and study its properties. Using this conceptual framework, we introduce and develop a new TP_2 positive dependence ordering for comparing bivariate distributions. The relationships of this new ordering to the more positive quadrant dependent ordering is discussed. Various interesting applications of the TP_2 ordering are presented, including applications to bivariate extreme value distributions.

2. Positive dependence orderings

In this section the concept of a positive dependence ordering is introduced. While there could be other ways for this concept to be defined, we find it useful to adopt the following definition.

DEFINITION 2.1. A relation $\ll$ on the family of all bivariate distributions is a *positive dependence ordering* (*PDO*) if it satisfies the following ten conditions:

(P0) $F\ll G \Rightarrow F(x, \infty)=G(x, \infty)$ and $F(\infty, y)=G(\infty, y)$;
(P1) $F\ll G \Rightarrow F(x, y)\leqq G(x, y)$ for all x, y;
(P2) $F\ll G$ and $G\ll H \Rightarrow F\ll H$;
(P3) $F\ll F$;
(P4) $F\ll G$ and $G\ll F \Rightarrow F=G$;
(P5) $F^{-}\ll F\ll F^{+}$; where F^{+}, the upper Fréchet bound, is given by $F^{+}(x, y)=\min [F(x, \infty), F(\infty, y)]$ and F^{-}, the lower Fréchet bound, is given by $F^{-}(x, y)=\max [F(x, \infty)+F(\infty, y)-1, 0]$;
(P6) $(X, Y)\ll(U, V) \Rightarrow (f(X), Y)\ll(f(U), V)$ for all increasing functions f, where the notation $(X, Y)\ll(U, V)$ means that the relation $\ll$ holds between the corresponding bivariate distributions $F_{X,Y}$ of (X, Y) and $F_{U,V}$ of (U, V), i.e., $F_{X,Y}\ll F_{U,V}$;
(P7) $(X, Y)\ll(U, V) \Rightarrow (-U, V)\ll(-X, Y)$;
(P8) $(X, Y)\ll(U, V) \Rightarrow (Y, X)\ll(V, U)$;
(P9) $F_n\ll G_n$, $F_n \rightarrow F$, $G_n \rightarrow G \Rightarrow F\ll G$, where F_n, F, G_n, G all have the same pair of marginals and $\rightarrow$ denotes convergence in distribution.

If $F\ll G$, F is said to be *less positively dependent than* G. Often we write $G\gg F$ and say that G (or a pair of *rv*'s whose distribution is G) is *more positively dependent than* F (or a pair of *rv*'s whose distribution is F).

While the relation $\ll$ is defined on the entire family of bivariate distributions, Property (P0) expresses the condition that only bivariate distributions having the same pair of marginals are compared for positive dependence. We show below how the definition can be extended to allow the comparison of bivariate distributions not having the same pair of marginals. Property (P1) indicates (by using (P0)) that any positive dependence ordering must satisfy the basic intuitive concept that given that X is large, Y is more likely to be large under G than under F. Properties (P2), (P3) and (P4), respectively, are the standard transitivity, reflexivity, and antisymmetry properties characteristic of all partial orderings. Property (P4) additionally is important because by itself it typically eliminates "unidimensional" orderings such as

"$F \ll G$ if and only if $m(F) \leqq m(G)$", where m is a scalar-valued measure of positive dependence.

The upper Fréchet bound is the most positive dependent distribution possible (see Fréchet [13], Dall'Aglio [9], and Kimeldorf and Sampson [24]) and the lower Fréchet bound is the least positive dependent distribution. Hence, the upper (lower) Fréchet bound should be more (less) dependent than *every* comparable distribution. This requirement is expressed as Property (P5). Property (P6) is a weak monotone invariance requirement, in that ordered distributions remain ordered after monotone scale transformations on one of the variables. Requiring instead invariance with respect to one-to-one functions changes the meaning of the ordering and leads to serious difficulties (see Kimeldorf and Sampson [24]), and requiring invariance only with respect to increasing linear functions is too weak. Intuitively, if (U, V) is more positively dependent than (X, Y), then $(-U, V)$ is more negatively dependent than $(-X, Y)$, so that $(-U, V)$ is less positively dependent than $(-X, Y)$. This property is expressed as (P7). Also, (P6) along with (P7) permits the discussion of decreasing functions. Property (P8) is a symmetry condition, and (P9) requires continuity with respect to weak convergence. Note that Properties (P6), (P7) and (P8) yield

$$(X, Y) \ll (U, V) \Rightarrow (f(X), g(Y)) \ll (f(U), g(V)), \tag{2.1}$$

for f, g both increasing or both decreasing.

The following examples provide illustrations of PDO's.

Example 2.1. *PQD Ordering.* Tchen [42] defines a bivariate distribution G to be *more positively quadrant dependent* (PQD) *than* a bivariate distribution F having the same pair of marginals if $G(x, y) \geqq F(x, y)$ for all $(x, y) \in R^2$, in which case we write $F \ll_{PQD} G$; alternatively, G is said to be *more concordant than F*. Fréchet [13] showed (see also Johnson and Kotz [19], pp. 22–23) that (P5) holds, while the remaining properties are obviously satisfied. Hence, the PQD partial ordering is a PDO.

Example 2.2. *Fréchet Ordering.* The following trivial ordering is a PDO: $F \ll_W G$ if and only if $F = G$ or $G(F)$ is the upper (lower) Fréchet bound with the same marginals as $F(G)$. It is obvious that $\ll_W$ is an extreme PDO in the sense that for any other PDO $\ll$ we have $F \ll_W G \Rightarrow F \ll G$.

Example 2.3. *A TP_2 PDO.* Let I_1 and I_2 be real intervals. We say $I_1 < I_2$ if and only if $x_1 \in I_1$ and $x_2 \in I_2$ imply $x_1 < x_2$. Let F and G be bivariate distributions with the same pair of marginals. The distribution G is *more TP_2 than F*, written $F \ll_T G$, if for all intervals $I_1 < I_2$, $J_1 < J_2$,

$$(2.2)\qquad F(I_1, J_1)F(I_2, J_2)G(I_1, J_2)G(I_2, J_1) \leqq G(I_1, J_1)G(I_2, J_2)F(I_1, J_2)F(I_2, J_1),$$

where $F(I_i, J_j)$ represents the probability assigned by F to the rectangle $I_i \times J_j$. In Section 4 we shall prove that the ordering $\ll_T$ is, in fact, a PDO.

Note that if F_0 is the independent distribution having the same pair of marginals as a distribution F, then $F_0 \ll_T F$ if and only if

$$(2.3)\qquad F(I_1, J_1)F(I_2, J_2) \geqq F(I_1, J_2)F(I_2, J_1).$$

In fact, Block, Savits and Shaked [4] define F to be TP_2 if and only if inequality (2.3) holds. In the case that F has density f with respect to some product measure, it is easy to verify that (2.3) holds if and only if the function f is TP_2 in the ordinary sense. The observation that F is TP_2 if and only if $F_0 \ll_T F$ motivates our calling $\ll_T$ a TP_2 ordering. Further properties of the $\ll_T$ ordering will be discussed in Section 5.

Other authors have introduced orderings, apparently measuring positive dependence, for specific classes of bivariate random variables (e.g., Shaked and Tong's [39] ordering for exchangeable random variables). Because of the specificity of these orderings, the concept of a PDO does not appear to be directly applicable. The definition of a PDO can be extended to allow the comparison of all continuous bivariate distributions, including those not having the same pair of marginals. To do so, consider the *copula* (Sklar [40]) or *uniform representation* (Kimeldorf and Sampson [23]) U_F of a continuous bivariate distribution F defined by

$$U_F(x, y) = F[F_1^{-1}(x), F_2^{-1}(y)], \qquad 0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1,$$

where $F_i^{-1}(x) = \inf\{t: F_i(t) \geqq x\}$ and F_1 and F_2 are the marginals of F. The bivariate distribution U_F has uniform marginals on (0, 1) and the "same dependence structure" as F. If F and G are continuous bivariate distributions and $\ll$ is a PDO, define the relation $\ll^*$ by

$$F \ll^* G \qquad \text{iff } U_F \ll U_G.$$

It is clear that the relation $\ll^*$ satisfies (P2), (P3), (P5), (P6), (P7), (P8), (P9), and

(P4)* $F \ll^* G$ and $G \ll^* F \Rightarrow U_F = U_G$.

(Mardia [26] discusses in some detail the situation where $U_F = U_G$, i.e., F and G are translates of each other.) Note that if F and G have the same pair of marginals, then $F \ll G$ if and only if $F \ll^* G$. Thus, a PDO $\ll$ induces an extended PDO $\ll^*$ defined on all continuous distributions. This extended PDO satisfies (P1) through (P9) with (P4) replaced by (P4)* and (P1) replaced by

(P1)* $F \ll^* G \Rightarrow U_F(x, y) \leqq U_G(x, y)$ for all x, y.

Where ambiguity is unlikely, we write $\ll$ in place of $\ll^*$. The concept of using copulas or uniform representations to compare bivariate distributions with different pairs of marginals was also considered by Scarsini [33] for the PQD ordering. Scarsini also considered the problem of comparing the concordance of pairs of discrete bivariate distributions.

In studying the dependence structure of a family of fixed-marginal bivariate distributions indexed by a real parameter it is natural to inquire whether the bivariate distributions are more positively dependent according to some PDO as the parameter increases. For example, Ahmed Langberg, Léon and Proschan [1] discuss families of bivariate distributions that are increasingly PQD as the parameter increases. They show that the family of Marshall-Olkin bivariate exponential distributions (with suitable constraints) and the Farlie-Gumbel-Morgenstern family (see (5.3) below) are PQD ordered. Slepian [41] showed that the standardized bivariate normal family is PQD ordered by ρ, and Das Gupta et al. [10] obtained similar results for elliptically symmetric families. Also, the bivariate logistic family of Ali, Mikhail and Haq [2] can be shown to be PQD ordered. Section 5 below considers families which are TP_2 ordered.

3. Alternative properties

In evaluating which properties should define a PDO, a larger number of possible properties were considered. Among those which were excluded, three warrant discussion.

DEFINITION 3.1. A PDO $\ll$ has the *generalized monotone invariance property* if for independent pairs (X_1, Y_1), (X_2, Y_2), (U_1, V_1), (U_2, V_2) of *rv*'s and increasing functions f and g

$$(U_1, V_1) \ll (X_1, Y_1) \text{ and } (U_2, V_2) \ll (X_2, Y_2) \Rightarrow (f(U_1, U_2), g(V_1, V_2)) \ll (f(X_1, X_2), g(Y_1, Y_2)) \,. \tag{3.1}$$

This condition is substantially stronger than (P6) and implies the special case of a type of closure under convolution. Let $f(s, t) = g(s, t) = s + t$, so that (3.1) reduces to

$$F_1 \ll G_1 \text{ and } F_2 \ll G_2 \Rightarrow F_1 * F_2 \ll G_1 * G_2 \,, \tag{3.2}$$

where $*$ denotes convolution.

Ahmed, Langberg, Léon, and Proschan [1] show that the PQD PDO satisfies (3.2); moreover, as we now show, it also satisfies the stronger condition (3.1).

THEOREM 3.1. *The PQD PDO has the generalized monotone invariance property* (3.1).

We first prove the following lemma.

LEMMA 3.1. *Suppose* $(U, V) \ll_{\mathrm{PQD}} (X, Y)$; (Z_1, Z_2), (X, Y), *and* (U, V) *are independent pairs of rv's; and* $f: R^2 \to R$ *and* $g: R^2 \to R$ *are increasing. Then* $(f(U, Z_1), g(V, Z_2)) \ll_{\mathrm{PQD}} (f(X, Z_1), g(Y, Z_2))$.

PROOF. By the monotonicity of f and g, the set $\{(u, v): f(u, z_1) \leqq a,\ g(v, z_1) \leqq b\}$ is a lower rectangle and hence

$$\begin{aligned}\Pr[f(U, Z_1) \leqq a,\ g(V, Z_2) \leqq b] &= \iint \Pr[f(U, z_1) \leqq a,\ g(V, z_2) \leqq b] dH(z_1, z_2) \\ &\geqq \iint \Pr[f(X, z_1) \leqq a,\ g(Y, z_2) \leqq b] dH(z_1, z_2) \\ &= \Pr[f(X, Z_1) \leqq a,\ g(Y, Z_2) \leqq b],\end{aligned}$$

where H is the cdf of (Z_1, Z_2).

PROOF OF THEOREM 3.1. Assume that the left hand side of implication (3.1) holds and fix increasing functions f and g. Define $f'(s, t) = f(t, s)$ and $g'(s, t) = g(t, s)$. Apply the lemma to deduce that $(f(X_1, X_2), g(Y_1, Y_2))$ is more PQD than $(f(U_1, X_2), g(V_1, X_2)) = (f'(X_2, U_1), g'(X_2, V_1))$, which by Lemma 3.1 is more PQD than $(f'(U_2, U_1), g'(V_2, V_1)) = (f(U_1, U_2), g(V_1, V_2))$.

The following example shows that the TP_2 PDO of Example 2.3 does not satisfy (3.2) and, a fortiori, does not have the generalized monotone invariance property (3.1).

Example 3.1. Let G_1 be a distribution with probability function g_1 such that $g_1(2, 1) = 1/3$, $g_1(4, 3) = g_1(6, 5) = 2/9$, $g_1(4, 5) = g_1(6, 3) = 1/9$, $g_1(x, y) = 0$ elsewhere. Let G_2 be a distribution with probability function g_2 such that $g_2(2, 1) = 1/3$, $g_2(2, 2) = g_2(3, 3) = 1/4$, $g_2(2, 3) = g_2(3, 2) = 1/12$, $g_2(x, y) = 0$ elsewhere. Let F_1 and F_2 be the distributions of independent rv's having the same respective marginals as G_1 and G_2. Then it is easy to verify that (3.2) is violated.

DEFINITION 3.2. A PDO $\ll$ has the *mixture property* if

$$F \ll G \Rightarrow F \ll aF + (1-a)G \ll G \quad \text{for } 0 \leqq a \leqq 1. \tag{3.3}$$

It is clear that the PQD PDO satisfies (3.3). On the other hand, the TP_2 PDO fails to satisfy (3.3), as is shown by the following example: Let $S=\{1, 2, 3\}$ and let G assign mass 1/3 to each of the points (1, 1), (2, 2), (3, 3). Let F assign mass 1/9 to each of the nine points in $S\times S$. The distribution $H=(F+G)/2$ assigns mass 2/9 to (2, 2), but assigns mass 1/18 to each of (2, 1), (3, 1) and (3, 2); hence, it is not the case that $F\ll_T H$.

Let $F(\rho)$ denote the standardized bivariate normal distribution with correlation ρ.

DEFINITION 3.3. A PDO $\ll$ has the *normal-agreeing property* if

$$F(\rho_1)\ll F(\rho_2) \text{ if and only if } \rho_1\leqq\rho_2 . \tag{3.4}$$

This property requires essentially that the bivariate normal distribution be well-ordered by ρ. An analogous property for numerical measures of dependence was included by Rényi [30], Schweitzer and Wolff [35], and Scarsini [33]. While there are obvious merits in including the normal-agreeing property as part of the definition of a PDO, we feel it is too restrictive by placing an undue emphasis on normality. (For instance, why not focus on elliptically symmetric distributions?) Note that the PQD PDO has the normal-agreeing property (see Slepian [41]), but the Fréchet PDO (Example 2.2) does not have the property. It is not known whether the TP_2 PDO has the normal-agreeing property.

4. The TP_2 PDO

In this section the new ordering $\ll_T$ defined by (2.2) is proved to be, in fact, a PDO. It is interesting to note that Kemperman [22] as well as Karlin and Rinott [21] suggested an ordering, which could be viewed as related to total positivity of order 2 in pairs. This ordering, which we denote by $\ll_{KKR}$, is defined as follows: $F\ll_{KKR}G$ if F and G have densities f and g, respectively, with respect to the same product measure for which

$$\begin{aligned} &g(\max(x_1, x_2), \max(y_1, y_2))f(\min(x_1, x_2), \min(y_1, y_2)) \\ &\quad\geqq f(x_1, y_1)g(x_2, y_2) \end{aligned} \tag{4.0}$$

for all real x_1, x_2, y_1, y_2. Since the total positivity of order 2 (TP_2) of the density function is a positive dependence property and (4.0) can be viewed as a TP_2-like inequality, it might be conjectured that (4.0) can serve as a means of comparing the degree of total positivity of order 2 for two distributions with the same pair of marginals. However, the partial ordering $\ll_{KKR}$ obviously does not have Property (P7) and less obviously also fails (P5). Thus, this partial ordering is not a PDO.

THEOREM 4.1. *The* TP_2 *ordering is a* PDO.

The proof consists in showing that $\ll_T$ satisfies (P1) through (P9). Properties (P3), (P7) and (P8) are obvious. The others are considered in the following lemmas. For notational convenience $F(I_i, J_j)$ in (2.2) is denoted by $F(i, j)$.

LEMMA 4.1. $F \ll_T G$ *and* $G \ll_T H$ *imply* $F \ll_T H$.

PROOF. Fix $I_1 < I_2$ and $J_1 < J_2$. Hence,

$$F(1,1)F(2,2)G(1,2)G(2,1) \leqq G(1,1)G(2,2)F(1,2)F(2,1) \tag{4.1}$$

and

$$G(1,1)G(2,2)H(1,2)H(2,1) \leqq H(1,1)H(2,2)G(1,2)G(2,1)\,. \tag{4.2}$$

If $G(1,1)G(2,2)G(1,2)G(2,1) > 0$, multiplication of inequalities (4.1) and (4.2) yields

$$F(1,1)F(2,2)H(1,2)H(2,1) \leqq H(1,1)H(2,2)F(1,2)F(2,1)\,, \tag{4.3}$$

and the lemma is proved. Otherwise, there are three cases.

Case 1. Assume $G(1,1)G(2,2) > G(1,2)G(2,1) = 0$. Then (4.2) implies $H(1,2)H(2,1) = 0$ and, hence, (4.3) holds.

Case 2. Assume $G(1,2)G(2,1) > G(1,1)G(2,2) = 0$. Then (4.1) implies $F(1,1)F(2,2) = 0$ and, hence, (4.3) holds.

Case 3. Assume $G(1,2)G(2,1) = G(1,1)G(2,2) = 0$. Assume without loss of generality that $G(1,1) = 0$ and $G(2,1) = 0$. Let $I_{2'}$ be any interval for which $I_2 \subset I_{2'}$, $I_1 < I_{2'}$, and $G(2',1) > 0$. (If no such $I_{2'}$ exists a similar argument can be applied to an interval $I_{1'}$ for which $I_1 \subset I_{1'}$, $I_{1'} \leqq I_{2'}$ and $G(1',1) > 0$. If neither such $I_{1'}$ nor $I_{2'}$ exists, then $G(R, J_1) = 0$, and hence (since F and G have the same marginals) $F(1,1) \leqq F(R, J_1) = G(R, J_1) = 0$, so that (4.3) holds.)

Case 3A. Assume $G(1,2) > 0$. Then we have $G(1,1) = 0$, $G(2',1) > 0$, and $G(1,2) > 0$. Thus, (4.1) implies $0 = F(1,1)F(2',2) \geqq F(1,1)F(2,2)$. Therefore, (4.3) holds.

Case 3B. Assume $G(1,2) = 0$. Expand J_1 or J_2 to get either $G(1,1') > 0$ or $G(1,2') > 0$.

Case 3B1. Assume $G(1,2') > 0$. Then we have $G(1,1) = 0$, $G(1,2') > 0$, $G(2',1) > 0$. Thus, (4.1) implies $0 = F(1,1)F(2',2') \geqq F(1,1)F(2,2)$. Therefore, (4.3) holds.

Case 3B2. Assume $G(1,1') > 0$. There are two cases:

Case 3B2a. Assume $G(2',2) > 0$. Thus, $G(1,2) = 0$, $G(1,1') > 0$, $G(2',2) > 0$. Thus, (4.2) implies $0 = H(1,2)H(2',1') \geqq H(1,2)H(2,1)$. Therefore, (4.3) holds.

Case 3B2b. Assume $G(2',2) = 0$. Then we have $G(2',2) = G(1,2) =$

0, $G(1, 1')>0$, $G(2', 1')>0$. As before, we expand $I_{2'}$ or I_1 either to get $G(2'', 2)>0$, $G(2'', 1')>0$, $G(1, 2)=0$, $G(1, 1')>0$, in which case (4.2) implies (4.3), or to get $G(1', 2)>0$, $G(2', 2)=0$, $G(1', 1')>0$, in which case (4.1) implies (4.3).

LEMMA 4.2. *$F \ll_T G$ and $G \ll_T F$ imply $F=G$.*

PROOF. Assume F and G are not identical. Then there exists (x, y) for which $F(x, y) \neq G(x, y)$. Without loss of generality, assume $F(x, y)<G(x, y)$. Let $I_1=(-\infty, x]$, $I_2=(x, \infty)$, $J_1=(-\infty, y]$, $J_2=(y, \infty)$. Thus,

$$F(1, 1)<G(1, 1) . \tag{4.4}$$

Since F and G have the same marginals, we have

$$G(2, 1)<F(2, 1) , \tag{4.5}$$

$$G(1, 2)<F(1, 2) , \tag{4.6}$$

$$F(2, 2)<G(2, 2) . \tag{4.7}$$

Multiplying (4.4), (4.5), (4.6) and (4.7) yields a contradiction to $G \ll F$.

LEMMA 4.3. *$F \ll_T G \Rightarrow F(x, y) \leqq G(x, y)$ for all x, y.*

PROOF. Suppose for some (x, y), $F(x, y)<G(x, y)$. Then using the notation and techniques of the proof of Lemma 4.2, we conclude that (4.4) holds and, hence, that (4.5), (4.6) and (4.7) hold. Multiplying these four inequalities contradicts the assumption that $G \ll_T F$.

LEMMA 4.4. *$F^- \ll_T F \ll_T F^+$.*

PROOF. Fix $I_1 \leqq I_2$ and $J_1 \leqq J_2$. Since F^+ assigns mass 1 to some increasing function and F^- assigns mass 1 to some decreasing function, $F^+(1, 2)F^+(2, 1)=F^-(1, 1)F^-(2, 2)=0$.

LEMMA 4.5. *For all increasing functions f, $(X, Y) \ll_T (U, V)$ implies $(f(X), Y) \ll_T (f(U), V)$.*

PROOF. Fix $I_1 \leqq I_2$, $J_1 \leqq J_2$ and fix f. Define $f^{-1}(I_i)=\{x: f(x) \in I_i\}$. Thus, $f^{-1}(I_1) \leqq f^{-1}(I_2)$ and $\Pr[f(X) \in I_i, Y \in J_j]=\Pr[X \in f^{-1}(I_i), Y \in J_j]$ and $\Pr[f(U) \in I_i, V \in J_j]=\Pr[U \in f^{-1}(I_i), V \in J_j]$. The result follows.

LEMMA 4.6. *If $\{F_n\}$ converges in distribution to F, $\{G_n\}$ converges in distribution to G, $F_n \ll_T G_n$ for all n, then $F \ll_T G$.*

PROOF. Let S denote the subset of R^2 at which both F and G are continuous. For rectangles $I \times J$ whose corners belong to S, $F_n(I, J) \rightarrow F(I, J)$ and $G_n(I, J) \rightarrow G(I, J)$, since F and G assigns zero probability to the boundary of $I\times J$. Hence, (2.2) holds when I_1, I_2, J_1, J_2 are such rectangles. But since S is dense in R^2 and F and G are increasing functions, (2.2) holds everywhere.

5. Properties of the TP$_2$ PDO and examples

Lemma 4.3 shows that the TP$_2$ PDO is stronger than the PQD PDO; i.e., $F\ll_T F'\Rightarrow F\ll_{\text{PQD}}F'$. However, for the class of 2×2 contigency tables with fixed marginals, the PQD and TP$_2$ PDO's are identical: $F\ll_{\text{PQD}}F'\Leftrightarrow F\ll_T F'\Leftrightarrow p_{11}\leqq p'_{11}$, where p_{11} and p'_{11} are the respective probabilities of the (1, 1) cell under F and F'. For larger contingency tables the PQD and TP$_2$ PDO's obviously cannot coincide, because there are tables which are PQD, but not TP$_2$. For example, Let F_δ be distributed on $\{0, 1, 2\}\times\{0, 1\}$ as follows: $f_\delta(0, 0)=.3-\delta$, $f_\delta(0, 1)=.2+\delta$, $f_\delta(1, 0)=.1$, $f_\delta(1, 1)=.15$, $f_\delta(2, 0)=.1+\delta$, and $f_\delta(2, 1)=.15-\delta$. Then for all δ, $0<\delta<.025$, F_δ is PQD, but not TP$_2$.

Note that if F and G have respective densities f and g with respect to some product measure, then $F\ll_T G$ clearly implies that

$$f(x_1, y_1)f(x_2, y_2)g(x_1, y_2)g(x_2, y_1)\leqq g(x_1, y_1)g(x_2, y_2)f(x_1, y_2)f(x_2, y_1)\,, \tag{5.1}$$

whenever $x_1\leqq x_2$ and $y_1\leqq y_2$. Suppose now in addition that $\partial^2 f(x, y)/\partial x\partial y$ and $\partial^2 g(x, y)/\partial x\partial y$ both exist. Then (5.1) is equivalent to

$$f^2\Delta_g-g^2\Delta_f\geqq 0\,, \tag{5.2}$$

where

$$\Delta_f=f\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\,,$$

and Δ_g is defined similarly. (This result follows from the standard differential version of TP$_2$; see Karlin ([20], p. 11).)

It is interesting to note that if f and g are standardized bivariate normal densities with correlations ρ_1 and ρ_2, respectively, where $-1<\rho_1\leqq\rho_2<1$, then (5.1) is satisfied. To verify this assertion, note that $g(x, y)/f(x, y)$ is proportional to $a(x)b(y)\exp[K(\rho_1, \rho_2)xy]$, where $K(\rho_1, \rho_2)=\rho_2(1-\rho_2^2)^{-1}-\rho_1(1-\rho_1^2)^{-1}$, and a and b are positive functions of x and y, respectively, for all ρ_1 and ρ_2; since $K(\rho_1, \rho_2)\geqq 0$, it follows that g/f is TP$_2$.

For any pair F and G of continuous univariate cdf's, consider the following family of bivariate distributions studied by Farlie [12], Gumbel [15] and Morgenstern [28]:

$$F(x, y; a)=F(x)G(y)\{1+a[1-F(x)][1-G(y)]\}\,,\qquad -1\leqq a\leqq 1\,. \tag{5.3}$$

Ahmed, Langberg, Léon, and Proschan ([1], Example 3.2.3) show that if $0 \leqq a_1 < a_2 < 1$, then $F(x, y; a_1) \ll_{\mathrm{PQD}} F(x, y; a_2)$. We now show the following stronger result:

THEOREM 5.1. *For the Farlie-Gumbel-Morgenstern family* $a_1 < a_2$ *implies that* $F(x, y; a_1) \ll_T F(x, y; a_2)$.

PROOF. By Properties (P6) and (P8) it is sufficient to consider the case when F and G are uniform distributions on $(0, 1)$, in which case $f(x, y; a) = 1 + a(2x-1)(2y-1)$ and $\Delta_{f(a)} = 4a$. Furthermore, if I is the interval (x_1, x_2) and J is the interval (y_1, y_2), then in the notation of (2.2)

$$\begin{aligned} F(I, J) &= (x_2 - x_1)(y_2 - y_1)[1 + a(x_1 + x_2 - 1)(y_1 + y_2 - 1)] \\ &= (x_2 - x_1)(y_2 - y_1) f(\bar{x}, \bar{y}), \end{aligned}$$

where $\bar{x} = (x_2 + x_1)/2$ and $\bar{y} = (y_2 + y_1)/2$. Thus, (2.2) is equivalent to (5.1), which is equivalent to (5.2), the left hand side of which reduces to $4(a_2 - a_1)[1 - a_1 a_2 (2\bar{x} - 1)^2 (2\bar{y} - 1)^2] \geqq 0$.

One of the more interesting questions concerning the PQD PDO that has been addressed by several authors is to determine the class Γ of functions h of two variables for which

$$(X, Y) \ll_{\mathrm{PQD}} (U, V) \Leftrightarrow E[h(X, Y)] \leqq E[h(U, V)] \qquad \text{for all } h \in \Gamma .$$

Under certain regularity conditions, Γ consists essentially of all functions h for which $e^{h(x,y)}$ is TP_2. See, for example, Cambanis and Simons [7], Cambanis, Simons and Stout [8], Rüschendorf [32] and Tchen [42]. The same question can be raised for any PDO. In particular, it would be interesting to determine the class Γ' for the TP_2 PDO. Although we have not been able to determine Γ', one can at least conclude by Lemma 4.3 that $\Gamma \subset \Gamma'$.

6. Applications to extreme value theory

We now consider some applications to the study of dependence properties for bivariate extreme value distributions. A general summary of the results for multivariate extreme value distributions is given by Galambos [14] and a recent presentation of some positive dependence properties of multivariate extreme value distributions is given by Marshall and Olkin [27]. Our discussion of this material leads us to generalize a result of Marshall and Olkin, for the bivariate case.

LEMMA 6.1. *For* $i = 1, 2$, *suppose* $(X_{i1}, Y_{i1}), \cdots, (X_{in}, Y_{in})$ *are iid having bivariate cdf* F_i, *where* F_1 *and* F_2 *have the same pair of mar-*

ginals. For $i=1, 2$, *let* $W_i^{(n)}=\max_{1\le j\le n} X_{ij}$, $Z_i^{(n)}=\max_{1\le j\le n} Y_{ij}$, $A_i^{(n)}=\min_{1\le j\le n} X_{ij}$, *and* $B_i^{(n)}=\min_{1\le j\le n} Y_{ij}$. *If* $F_1\ll F_2$, *then for all* n, $(W_1^{(n)}, Z_1^{(n)})\ll(W_2^{(n)}, Z_2^{(n)})$ *and* $(A_1^{(n)}, B_1^{(n)})\ll(A_2^{(n)}, B_2^{(n)})$, *where* $\ll$ *is either of the* PDO's $\ll_{\mathrm{PQD}}$ *or* $\ll_T$.

PROOF. The cdf of $(W_i^{(n)}, Z_i^{(n)})$ is F_i^n, and $F_1\ll F_2$ clearly implies that $F_1^n\ll F_2^n$ for the PQD and TP_2 orderings. The results for $(A_i^{(n)}, B_i^{(n)})$ follow from (2.1).

COROLLARY 6.1. *Suppose* $(W_1^{(n)}, Z_1^{(n)})$ *and* $(W_2^{(n)}, Z_2^{(n)})$ *each converges to an extreme value distribution, where the corresponding limiting rv's are denoted by* (W_1, Z_1) *and* (W_2, Z_2), *respectively. If* $F_1\ll F_2$, *then* $(W_1, Z_1)\ll(W_2, Z_2)$, *where* $\ll$ *is either of the* PDO's $\ll_{\mathrm{PQD}}$ *or* $\ll_T$.

PROOF. By Lemma 6.1, $F_1\ll F_2$ implies that $(W_1^{(n)}, Z_1^{(n)})\ll(W_2^{(n)}, Z_2^{(n)})$ for all n, which by (2.1) implies that for all n

$$(a_n W_1^{(n)}+b_n,\ c_nZ_1^{(n)}+d_n)\ll(a_n W_2^{(n)}+b_n,\ c_nZ_2^{(n)}+d_n)\,, \tag{6.1}$$

where $a_n>0$ and $c_n>0$. To complete the proof, note first that the constants which provide the limiting extreme value distribution are the same because F_1 and F_2 have the same pair of marginals. Note second that in showing that the PQD and TP_2 PDO's satisfy (P9), it was not necessary to assume that the marginal distributions did not vary with n.

Similar limiting results apply to $(A_i^{(n)}, B_i^{(n)})$. If F and G are bivariate extreme value distributions, then $F^\beta G^{1-\beta}$ is a bivariate extreme value distribution for $0\le\beta\le1$. (See Gumbel and Goldstein [16] or Johnson and Kotz ([19], p. 251).) Suppose for $i=1, 2$ that F_i and G_i are extreme value distributions, that F_1, F_2 have the same pair of marginals, and that G_1, G_2 have the same pair of marginals. Clearly for the PQD and TP_2 PDO's if $F_1\ll F_2$ and $G_1\ll G_2$, then $F_1^\beta G_1^{1-\beta}\ll F_2^\beta G_2^{1-\beta}$.

Observe that Corollary 6.1 immediately shows that if the underlying distribution is TP_2, then the corresponding extreme value distribution, if it exists, must be TP_2. Marshall and Olkin ([27], Proposition 5.1) show that regardless of the underlying distribution, if (X, Y) has an extreme value distribution, then X and Y are associated. In fact, the following theorem exhibits a stronger dependence property for *rv*'s having an extreme value distribution.

THEOREM 6.1. *If* (X, Y) *has a bivariate extreme value distribution, then* X *and* Y *are right corner set increasing, i.e., the survival function is* TP_2.

PROOF. Pickands [29] (also see Marshall and Olkin ([27], p. 173)) has shown that (X, Y) has a min extreme value distribution if and only if $X=g_1(U)$ and $Y=g_2(V)$ for certain increasing functions g_1 and g_2, where (U, V) has survival function of the form

$$\bar{G}(u, v)=\exp\left\{-\int_0^1 \max\,[tu, (1-t)v]d\mu(t)\right\},$$

where μ is a finite measure on [0, 1]. It thus suffices to show that $\bar{G}$ is TP_2. Observe that for $u_1<u_2$, $v_1<v_2$,

$$\begin{aligned}(6.2)\qquad &\ln \bar{G}(u_1, v_1)+\ln \bar{G}(u_2, v_2)-\ln \bar{G}(u_1, v_2)-\ln \bar{G}(u_2, v_1)\\ &=\int_0^1 M(t; u_1, u_2, v_1, v_2)d\mu(t),\end{aligned}$$

where

$$\begin{aligned}M(t; u_1, u_2, v_1, v_2)=&\max\,[tu_1, (1-t)v_2]+\max\,[tu_2, (1-t)v_1]\\ &-\max\,[tu_1, (1-t)v_1]-\max\,[tu_2, (1-t)v_2].\end{aligned}$$

For $0\leqq t<1$, let $\theta=t/(1-t)\geqq 0$, so that

$$\begin{aligned}(6.3)\qquad &(1-t)^{-1}M(t; u_1, u_2, v_1, v_2)\\ &=\max\,[u_1^*, v_2]+\max\,[u_2^*, v_1]-\max\,[u_1^*, v_1]-\max\,[u_2^*, v_2],\end{aligned}$$

where $u_1^*\equiv\theta u_1<\theta u_2\equiv u_2^*$. Direct calculation shows that the right side of (6.3) is nonnegative for $u_1^*<u_2^*$, $v_1<v_2$. For the case $t=1$, $M=0$. Hence the left side of (6.2) is nonnegative. The derivation for max extreme value distributions is similar.

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