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ON THE USAGE OF REFINED LINEAR MODELS FOR DETERMINING N-WAY CLASSIFICATION DESIGNS WHICH ARE OPTIMAL FOR COMPARING TEST TREATMENTS WITH A STANDARD TREATMENT*

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Summary

In this paper we consider experimental settings in which v test treatments are to be compared to some control or standard treatment and where heterogeneity needs to be eliminated in n-directions. Using techniques similar to those used by Kunnert (1983, *Ann. Statist.*, 11, 247-257) concerning the determination of optimal designs under a refined linear model, some methods are given for constructing n-way classification designs which are A- and MV-optimal for estimating elementary treatment differences involving the standard treatment from m-way classification designs, $m<n$, which are A- and MV-optimal for estimating the same treatment differences. Examples are given for the case $n=2$ to show how the results obtained can be applied.

1. Notation and introduction

We begin by giving some matrix notation which is used throughout the sequel;

A'=the transpose of an $m\times n$ matrix A,

$R(A)$=the vector space generated by the column vectors of an $m\times n$ matrix A,

A^-=a generalized inverse of an $m\times n$ matrix A,

$P_A=A(A'A)^-A'$=the orthogonal projector on $R(A)$ where A is an $m\times n$ matrix,

I_n=the $n\times n$ identity matrix,

J_{mn} or $J_{m,n}$=the $m\times n$ matrix of ones,

$\mathrm{tr}\, A$=the trace of an $n\times n$ matrix.

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In this paper we consider experimental situations in which the data obtained from a given design d is to be analyzed via a simple Gauss-Markov (GM) model, i.e., a model of the form $(Y, X_d\beta, \sigma^2 I_n)$ where Y is an $n\times 1$ random vector of observations with expectation $E(Y)=X_d\beta$ and covariance matrix $\sigma^2 I_n$, X_d is a known $n\times s$ design matrix, β is an $s\times 1$ vector of unknown parameters and $\sigma^2>0$ is a known or unknown constant.

In a simple GM model, a linear combination $t'\beta=\sum_{i=1}^{s} t_i\beta_i$ of the model parameters is said to be estimable in d provided there exists a linear combination $a'Y=\sum_{i=1}^{n} a_iY_i$ such that $E(a'Y)=t'\beta$. The best linear unbiased estimator (b.l.u.e.) for any parametric function $t'\beta$ which is estimable is $t'\hat{\beta}$ where $\hat{\beta}$ is any solution to the normal equations

$$X_d' X_d\hat{\beta}=X_d' Y \tag{1.1}$$

which are obtained using ordinary least squares. In the experimental situations considered here, we will only be interested in estimating estimable functions of certain subsets of parameters in β, e.g., if β and X_d are partitioned so that $X_d=(A_d, U_d)$ where A_d is an $n\times p$ matrix, U_d is an $n\times q$ matrix for $p+q=s$, $\beta=\begin{pmatrix}\alpha\\ \theta\end{pmatrix}$, and

$$E(Y)=X_d\beta=A_d\alpha+U_d\theta\ , \tag{1.2}$$

then we will be primarily interested in estimating estimable functions of the form

$$t'\beta=(l', 0')\begin{pmatrix}\alpha\\ \theta\end{pmatrix}=l'\alpha\ .$$

In such cases, the b.l.u.e. for $l'\alpha$ is also given by $l'\hat{\alpha}$ where $\hat{\alpha}$ is any solution to the equations

$$G_d'G_d\hat{\alpha}=C_d\hat{\alpha}=G_d'Y \tag{1.3}$$

where

$$G_d=(I_n-P_{U_d})A_d\ . \tag{1.4}$$

Equations (1.3) are called the reduced normal equations for estimating α and the matrix C_d is called the information matrix of d for estimating α.

We will henceforth assume that for any design d being considered, $A_dJ_{pn}\in R(U_d)$. We note that this assumption is not restrictive in the sense that when estimating the effects of treatments in any classical design setting, it is usually the case that $A_dJ_{pn}=a_1J_{nn}$ for some constant

a_1 and $U_d J_{qn} = a_2 J_{nn}$ for some constant a_2. Under this assumption, $(I_n - P_{U_d}) A_d J_{pn} = 0$, the row and column sums of C_d are equal to zero, and the only parametric functions of α which are estimable are contrasts, i.e., parametric functions of the form $l'\alpha = \sum_{i=1}^{p} l_i \alpha_i$ where $\sum_{i=1}^{p} l_i = 0$. Any design in which all treatment contrasts are estimable is said to be connected in α and a design which is connected in α has C_d of rank $p-1$. We will also only be considering designs which are connected in the model parameters we wish to estimate.

In most experimental situations there are a number of designs available which can be used to achieve some specified set of experimental objectives. We shall use D to denote a class of designs which are available for use in a given experimental setting. An optimality criterion is often used to help an experimenter select a design from the available class D which best accomplishes the primary goals of the experiment. The optimality criteria of primary interest here are the A- and MV-optimality criteria.

DEFINITION 1.5. Let $L'\alpha = \begin{pmatrix} l_1'\alpha \\ \vdots \\ l_t'\alpha \end{pmatrix}$ be a vector of t estimable functions. Then $d^* \in D$ is said to be A-optimal over D for estimating $L'\alpha$ if for any other $d \in D$,

$$\operatorname{tr} \operatorname{Cov}_{d^*}(L'\hat{\alpha}) \leqq \operatorname{tr} \operatorname{Cov}_d(L'\hat{\alpha})$$

where $\operatorname{Cov}_d(L'\hat{\alpha})$ denotes the covariance matrix of $L'\hat{\alpha}$ under d.

DEFINITION 1.6. Let $L'\alpha$ be as in Definition 1.5. Then $d^* \in D$ is said to be MV-optimal over D for estimating $L'\alpha$ if for any other $d \in D$

$$\max_{1 \leqq i \leqq t} \operatorname{Var}_{d^*}(l_i'\hat{\alpha}) \leqq \max_{1 \leqq i \leqq t} \operatorname{Var}_d(l_i'\hat{\alpha})$$

where $\operatorname{Var}_d(l_i'\hat{\alpha})$ denotes the variance of $l_i'\hat{\alpha}$ under d.

For each $d \in D$, assume $U_d = (U_{d1}, U_{d2})$ where U_{d1} is an $n \times r$ matrix, $r < q$. Then with each $d \in D$, we can associate another design d_1 obtained by ignoring the effects of the parameters in θ corresponding to the columns in U_{d2}. The information matrix of d_1 for estimating α in the simpler model

$$E(Y) = A_{d1}\alpha + U_{d1}\theta \tag{1.7}$$

is given by

$$C_{d1} = G_{d1}' G_{d1} \tag{1.8}$$

where

$$G_{d1}=(I_n-P_{U_{d1}})A_d \; . \tag{1.9}$$

Using the terminology of Kunnert [6], model (1.7) is called finer than model (1.2). We shall use D_1 to denote the set of designs d_1 which are associated with $d \in D$. For $d \in D$, there is of course a close relationship between C_d and C_{d1}. If we let $\bar{G}=(I_n-P_{U_{d1}})U_{d2}$, then it can be verified that

$$P_{U_d}=P_{U_{d1}}+P_{\bar{G}} \; ,$$

hence we see that

$$C_d=G_d'G_d=A_d'(I_n-P_{U_d})A_d=C_{d1}-A_d'P_{\bar{G}}A_d \; . \tag{1.10}$$

Since $A_d'P_{\bar{G}}A_d$ given in (1.10) is positive semi-definite, we have the following proposition which is given in Kunnert [6] and which is also proven in Magda [7].

PROPOSITION 1.11. *For $d \in D$, let $d_1 \in D_1$ be its associated design. Then $C_d \leqq C_{d1}$ is the sense that $C_{d1}-C_d$ is positive semi-definite. Further, $C_{d1}=C_d$ if and only if $A_d\bar{G}=0$.*

In this paper, we use arguments similar to those used by Kunnert [6] to establish the A- and MV-optimality of certain types of n-way classification designs for comparing a set of test treatments to some standard or control treatment. However, since the results of Kunnert [6] do not seem directly applicable for establishing the A- and MV-optimality of designs for estimating an arbitrary vector $L'\alpha$ of estimable parametric functions, in Section 2 we give some general results. In Section 3, we apply the results given in Section 2 to experimental settings in which heterogeneity is to be eliminated in n-directions and where a primary goal of the experiment is to estimate the difference in the effects between a set of test treatments and a standard treatment with as much precision as possible.

2. Theoretical results

In this section we give some results concerning A- and MV-optimality which yield strategies for finding A- and MV-optimal designs in a given class D which are analogous to those described in Kunnert [6].

THEOREM 2.1. *Let d be some design whose data is to be analyzed via model* (1.2) *and assume d is connected in α. Also let d_1 be some design which can be associated with d as described in Section* 1. *If $l'\alpha$ is estimable in model* (1.2), *then*

$$\mathrm{Var}_d\,(l'\hat{\alpha}) \geqq \mathrm{Var}_{d1}\,(l'\hat{\alpha})\ . \tag{2.2}$$

with equality holding in (2.2) *if* $P_{\bar{G}} A_d \rho = 0$ *where* ρ *is such that* $l = C_d \rho$.

PROOF. It is easy to verify that if $l'\alpha$ is estimable in model (1.2), then $l'\alpha$ is estimable in the finer model (1.7). Further, $l'\alpha$ is estimable in model (1.2) if and only if $l = C_d \rho$ for some vector ρ and

$$(1/\sigma^2)\,\mathrm{Var}_d\,(l'\hat{\alpha}) = l' C_d^- l = \rho' C_d \rho \tag{2.3}$$

where C_d^- is some reflexive generalized inverse of C_d, i.e., C_d^- is a generalized inverse of C_d which also satisfies $C_d^- C_d C_d^- = C_d^-$. Similarly, if $l = C_{d1}\gamma$, then

$$(1/\sigma^2)\,\mathrm{Var}_{d1}\,(l'\hat{\alpha}) = l' C_{d1}^- l = \gamma' C_{d1} \gamma \tag{2.4}$$

where C_{d1}^- is a reflexive generalized inverse of C_{d1}. Since d is connected in α, C_d has rank $p-1$ and one reflexive generalized inverse of C_d is

$$C_d^- = \begin{bmatrix} 0J_{11} & 0J_{1,p-1} \\ 0J_{p-1,1} & C_{d00}^{-1} \end{bmatrix}$$

where C_{d00} is the principal submatrix obtained by eliminating the first row and first column from C_d. Also, since C_d has rank $p-1$, C_{d1} will have rank $p-1$ and a reflexive generalized inverse of C_{d1} is given by

$$C_{d1}^- = \begin{bmatrix} 0J_{11} & 0J_{1,p-1} \\ 0J_{p-1,1} & C_{d100}^{-1} \end{bmatrix}$$

where C_{d100} is the principal submatrix obtained by eliminating the first row and first column from C_{d1}. From Proposition 1.11, we have that $C_d \leqq C_{d1}$ and since

$$C_d = C_{d1} - A_d' P_{\bar{G}} A_d\ , \tag{2.5}$$

we clearly have that $C_{d00} \leqq C_{d100}$, and the inequality part of (2.2) follows from the expressions given in (2.3) and (2.4). The equality statement in the proposition follows from (2.5) and the second expressions for the variance of $l'\hat{\alpha}$ given (2.3) and (2.4).

From Theorem 2.1, we easily obtain the following two theorems concerning A- and MV-optimality.

THEOREM 2.6. *Assume the data to be obtained in a given experimental situation is to be analyzed via a model of the form given in* (1.2) *and let D denote a class of designs which are available for usage such that all $d \in D$ are connected in α. Let D_1 be the class of designs which can be associated with $d \in D$ via the finer model given in* (1.7) *and let* $L'\alpha = \begin{pmatrix} l_1'\alpha \\ \vdots \\ l_t'\alpha \end{pmatrix}$ *be a vector of t parametric functions which are estimable in* *model* (1.2). *If* $d^* \in D$ *is such that* d_1^* *is A-optimal in* D_1 *for estimating* $L'\alpha$ *and if* $\operatorname{tr}\operatorname{Cov}_{d^*}(L'\hat{\alpha})=\operatorname{tr}\operatorname{Cov}_{d_1^*}(L'\hat{\alpha})$ (tr A *denotes the trace of an* $n\times n$ *matrix* A), *then* d^* *is A-optimal in* D *for estimating* $L'\alpha$.

THEOREM 2.7. *Assume the same conditions as in Theorem* 2.6. *If* $d^* \in D$ *is such that* d_1^* *is MV-optimal in* D_1 *for estimating* $L'\alpha$ *and if* $\max_{1\leq i\leq t}\operatorname{Var}_{d^*}(l_i'\hat{\alpha})=\max_{1\leq i\leq t}\operatorname{Var}_{d_1^*}(l_i'\hat{\alpha})$, *then* d^* *is MV-optimal in* D *for estimating* $L'\alpha$.

Clearly the results given in Theorems 2.6 and 2.7 yield a method for finding a design which is A- or MV-optimal within a given class D which is similar to that suggested by Kunnert [6]. Namely, find a design d_1^* which is A- or MV-optimal in an associated class D_1 under the simpler model given in (1.7), then construct from d_1^* a design d^* which satisfies the conditions given in the theorems under model (1.7). Of course, the simplest situation in which this method will work is when $d_1^* \in D_1$ can be found such that the design d^* which can be constructed has $C_{d^*}=C_{d1^*}$ which implies that the orthogonality condition given in Proposition 1.11 must be satisfied.

3. Applications

In this section we apply the results given in Section 2 to experimental situations in which the effects of v treatments are to be compared to the effect of some control or standard treatment and where it is necessary to eliminate heterogeneity in n directions. In such a setting there are n classification factors and we shall let b_i denote the number of levels of the i-th classification factor ($1\leq i\leq n$). Altogether, there are $b=b_1\times b_2\times\cdots\times b_n$ different combinations of levels of the classification factors under which the treatments are to be tested. Any combination of these factor levels is called a cell and we can coordinatize these cells by the n-tuples of integers $(i_1,\cdots,i_n)$ with $1\leq i_j\leq b_j$, $j=1,\cdots,n$. The usual model used to analyze the data obtained from an n-way classification design specifies that the expectation of an observation on treatment i in cell $(j_1,\cdots,j_n)$ is $\alpha_i+\sum_{k=1}^{n}B_{j_k}^{(k)}$ where α_i and $B_{j_k}^{(k)}$ are the effects of treatment i and the j_k-th level of factor k, respectively. All observations obtained are assumed to satisfy the usual GM model assumptions. A design d to be used in the setting of n-way heterogeneity just described is some allocation of the $v+1$ treatments, denoted by $0,1,\cdots,v$, 0 *denoting the standard treatment and* $1,2,\cdots,v$ *the test treatments*, into the b available cells.

For some design d which can be used in such an experimental setting, let $N_{di}=(n_{dsu}^{(i)})$, $i=1,\cdots,n$, be the incidence matrix between the

$v+1$ treatments and the b_i levels of the i-th factor, i.e., $n_{dsu}^{(i)}$ is the total number of times that treatment s occurs in the cells having i-th coordinate equal to u. Under the n-way classification model given above and using $\operatorname{diag}(a_1, \cdots, a_n)$ to denote an $n \times n$ diagonal matrix, the information matrix for estimating the treatment effects $\alpha_0, \alpha_1, \cdots, \alpha_v$ corresponding to (1.3) can also be expressed as (see Cheng [1], Theorem 2.1)

$$\begin{aligned} C_d &= \operatorname{diag}(r_{d0}, r_{d1}, \cdots, r_{dv}) - (1/b) b_1 N_{d1} N'_{d1} - (1/b) \sum_{h=2}^{n} b_h N_{dh} (I_{b_h} - b_h^{-1} J_{b_h, b_h}) N'_{dh} \\ &= \operatorname{diag}(r_{d0}, r_{d1}, \cdots, r_{dv}) - (1/b) \sum_{h=1}^{n} b_h N_{dh} N'_{dh} + ((n-1)/b) [r_{di} r_{dj}]_{(v+1)\times(v+1)} \end{aligned}$$

where r_{di} = the number of replications of treatment i in d, $b = b_1 \times b_2 \times \cdots \times b_n$, and $[r_{di} r_{dj}]_{(v+1)\times(v+1)}$ is the $(v+1)\times(v+1)$ matrix whose (i, j)-th entry is $r_{di} r_{dj}$. C_d is also referred to as the C-matrix of d and is well known to have zero row and column sums.

We shall only be considering designs which are treatment connected, i.e., those designs d for which all treatment contrasts $l'\alpha = \sum_{i=0}^{v} l_i \alpha_i$ are estimable and for which C_d has rank v. We shall also use $D(v+1; b_1, \cdots, b_n)$ to denote the class of all treatment connected designs having $v+1$ treatments arranged in the b cells described previously.

Suppose M is some subset of m subscripts out of $\{1, \cdots, n\}$. With each $d \in D(v+1; b_1, \cdots, b_n)$, we can associate the design $d_1(M)$ which is that m-way classification design obtained from the model for d by ignoring the effects of those classification factors corresponding to subscripts not contained in M. The relationship between the information matrices of d and $d_1(M)$ as given in (1.10) can also be expressed as

$$C_d = C_{d_1(M)} - (1/b) \sum_{h \notin M} b_h N_{dh} (I_{b_h} - b_h^{-1} J_{b_h, b_h}) N'_{dh} \,. \tag{3.1}$$

We will henceforth use $D_1(M)$ to denote the class of all designs $d_1(M)$ corresponding to $d \in D(v+1; b_1, \cdots, b_n)$. If we take $M = \{i\}$ for some subscript i in $\{1, \cdots, n\}$ the resulting design, denoted by $d_1(i)$, is a block design having $v+1$ treatments arranged in b_i blocks of size (b/b_i), incidence matrix N_{di}, and concurrence matrix $N_{di} N'_{di} = (\lambda_{dsu}^{(i)})_{(v+1)\times(v+1)}$.

Since our primary interest in this section is to find designs which are optimal for estimating differences in the effects between the v test treatments being studied and a standard treatment, we want to find designs which estimate elementary treatment differences of the form $\alpha_i - \alpha_0$ for $i = 1, \cdots, v$ with as much precision as possible.

LEMMA 3.2. *Consider the class of designs $D(v+1; b_1, \cdots, b_n)$ and let $d \in D(v+1; b_1, \cdots, b_n)$ be arbitrary. Now, in the n-way classification* *model, let* $L'\alpha=\begin{pmatrix} l_1'\alpha \\ \vdots \\ l_v'\alpha \end{pmatrix}$ *where* $l_i'\alpha=\alpha_i-\alpha_0$. *Then*

$$(1/\sigma^2)\operatorname{Cov}_d(L'\hat{\alpha})=C_{d00}^{-1}$$

where C_{d00} *is the principal submatrix obtained by eliminating the first row and first column of* C_d.

PROOF. As in the proof of Theorem 2.1,

$$(1/\sigma^2)\operatorname{Cov}_d(L'\hat{\alpha})=L'\begin{pmatrix} 0J_{11} & 0J_{1v} \\ 0J_{v1} & C_{d00}^{-1} \end{pmatrix}L$$

where C_{d00} is the principal submatrix obtained by eliminating the first row and first column of C_d. The result now follows since

$$L=\begin{pmatrix} -J_{1v} \\ I_v \end{pmatrix}.$$

By the results given in Section 2, one way to find a design d^* which is A- or MV-optimal in $D(v+1;b_1,\cdots,b_n)$ for estimating elementary treatment differences of the form $\alpha_i-\alpha_0$ is to let $M\subseteqq\{1,\cdots,n\}$ and then find a design $d_1^*(M)\in D_1(M)$ which is A- or MV-optimal in $D_1(M)$ for estimating the same contrasts and for which $\operatorname{Cov}_{d^*}(L'\hat{\alpha})=\operatorname{Cov}_{d_1^*(M)}(L'\hat{\alpha})$. As also indicated in Section 2, one way of doing this is to find $d^*\in D(v+1;b_1,\cdots,b_n)$ such that $d_1^*(M)\in D_1(M)$ is A- or MV-optimal in $D_1(M)$ and such that $C_{d^*}=C_{d_1^*(M)}$.

LEMMA 3.3. *Consider the class of designs* $D(v+1;b_1,\cdots,b_n)$ *and let* $M\subseteqq\{1,\cdots,n\}$. *Now let* $d\in D(v+1;b_1,\cdots,b_n)$ *be arbitrary and let* $d_1(M)\in D_1(M)$ *be its associated design. If for all* $i\notin M$,

$$N_{di}'=(a_1J_{b_i,1},a_2J_{b_i,1},\cdots,a_{v+1}J_{b_i,1}),$$

then $C_d=C_{d_1(M)}$.

PROOF. Without loss of generality, assume that $M=\{1,\cdots,m\}$. Then from (3.1), we see that

$$C_d=C_{d_1(M)}-(1/b)\sum_{h=m+1}^{n} b_hN_{dh}(I_{b_h}-b_h^{-1}J_{b_h,b_h})N_{dh}'.$$

But now, since $(I_{b_h}-b_h^{-1}J_{b_h,b_h})J_{b_h,1}=0$, it follows that for all $h\geqq m+1$,

$$(I_{b_h}-b_h^{-1}J_{b_h,b_h})N_{dh}=0,$$

hence that $C_d=C_{d_1(M)}$.

THEOREM 3.4. *Consider the class of designs* $D(v+1; b_1, \cdots, b_n)$ *and let* $M \subseteq \{1, \cdots, n\}$. *Now consider* $L'\alpha = \begin{pmatrix} l_1'\alpha \\ \vdots \\ l_v'\alpha \end{pmatrix}$ *where* $l_i'\alpha = \alpha_i - \alpha_0$. *If* $d^* \in D(v+1; b_1, \cdots, b_n)$ is *such that* $d_1^*(M)$ *is A- or MV-optimal in* $D_1(M)$ *for estimating* $L'\alpha$ *and is such that for all* $i \notin M$,

$$N_{di}' = (a_1 J_{b_i,1}, \cdots, a_{v+1} J_{b_i,1}) ,$$

then d^* *is A- or MV-optimal in* $D(v+1; b_1, \cdots, b_n)$ *for estimating* $L'\alpha$.

PROOF. The theorem clearly follows from Lemma 3.3 and Theorems 2.6 and 2.7.

A problem which has received considerable attention in the last several years has been that of finding designs which are A-optimal and MV-optimal for estimating test treatment-control treatment differences in experimental settings where heterogeneity is to be eliminated in a single direction. In particular, a good deal of study has been done concerning the construction of balanced treatment block designs and group divisible treatment block designs which are A-optimal and MV-optimal for estimating test treatment-control treatment differences.

DEFINITION 3.5. Let $d \in D(v+1; b_1, \cdots, b_n)$. Then $d_1(i)$ is called a group divisible treatment design with $s+1$ classes (GDTD $(s+1)$) if $r_{d1} = \cdots = r_{dv}$ and if the subscripts $1, 2, \cdots, v$ corresponding to the test treatments can be partitioned into s mutually disjoint sets $V_1, \cdots, V_s$, all of size $\bar{v} = v/s$, such that $N_{di} N_{di}' = (\lambda_{dpq}^{(i)})$ has

i) $\lambda_{d11}^{(i)} = \cdots = \lambda_{dvv}^{(i)}$,
ii) $\lambda_{d10}^{(i)} = \cdots = \lambda_{dv0}^{(i)} = \lambda_0$ for some constant λ_0,
iii) for $p, q \in V_x$, $p \neq q$, $\lambda_{dpq}^{(i)} = \lambda_1$ for some constant λ_1,
iv) for $p \in V_x$, $q \in V_y$, $x \neq y$, $\lambda_{dpq}^{(i)} = \lambda_2$ for some constant λ_2.

DEFINITION 3.6. If $d \in D(v+1; b_1, \cdots, b_n)$ and $d_1(i)$ is a GDTD $(s+1)$ such that $\lambda_1 = \lambda_2$, then $d_1(i)$ is called a balanced treatment block design (BTBD).

A number of results have been discovered recently concerning the A-optimality and MV-optimality of BTBD's and GDTD $(s+1)$'s for estimating treatment differences of the form $\alpha_i - \alpha_0$. Majumdar and Notz [9] give an algorithm which can sometimes be used to establish the A-optimality and MV-optimality of BTBD's under certain conditions. Hedayat and Majumdar ([3], [4]) have catalogued a number of designs as well as characterized several families of BTBD's which satisfy the conditions of the algorithm given by Majumdar and Notz [9]. Jacroux [5] has generalized the results given by Majumdar and Notz [9] with respect to MV-optimality and has developed an algorithm which can often be used to establish the MV-optimality of certain GDTD$(s+1)$'s. In many cases, the designs which can be proven to be A-optimal and MV-optimal in the setting of one-way heterogeneity can be used to construct designs which are A-optimal and MV-optimal in experimental settings which require the elimination of heterogeneity in more than one direction.

THEOREM 3.7. *Let $L'\alpha$ be as defined in Theorem 3.4 and let $d^* \in D(v+1; b_1, \cdots, b_n)$ be a design such that $d_1^*(i) \in D_1(M)$ where $M=\{i\}$ is a GDTD$(s+1)$ or BTBD which is A-optimal or MV-optimal in $D_1(M)$ for estimating $L'\alpha$. If for all $h \notin M$, N_{d^*h} has the form $N'_{d^*h}=(a_1 J_{b_h,1}, \cdots, a_{v+1} J_{b_h,1})$, then d^* is A-optimal or MV-optimal in $D(v+1; b_1, \cdots, b_n)$.*

PROOF. Simply apply Theorem 3.4 with $M=\{i\}$, and we obtain the desired result.

Of course, in general, certain conditions must be satisfied in order to construct a design d^* which satisfies the conditions given in Theorem 3.7. In particular, using arguments exactly analogous to those used to prove Proposition 2.2 in Constantine [2], one can prove the following theorem.

THEOREM 3.8. *Let $L'\alpha$ be as defined in Theorem 3.4 and consider the class of designs $D(v+1; b_1, \cdots, b_n)$. If $d_1^*(i) \in D_1(M)$ where $M=\{i\}$ is a GDTD$(s+1)$ or a BTBD which is A-optimal or MV-optimal in $D_1(M)$ for estimating $L'\alpha$ and if for all $h \neq i$, r_{dp}/b_h is an integer for $p=0, 1, \cdots, v$, then there exists a design $d^* \in D(v+1; b_1, \cdots, b_n)$ satisfying the conditions of Theorem 3.7.*

We now give several examples in the case $n=2$ to illustrate some of the results given in this section.

Example 3.9. Consider the class of designs $D(4; 9, 3)$ and consider the BTBD $d_1^*(1) \in D_1(M)$ where $M_1=\{1\}$ and whose incidence matrix is given by

$$N_{d^*1}=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Using the results of Majumdar and Notz [9], it follows that $d_1^*(1)$ is both A-optimal and MV-optimal in $D_1(M)$. Now consider the row-column design $d^* \in D(4; 9, 3)$ given by

$$d^*=\begin{bmatrix} 0 & 3 & 1 & 0 & 2 & 1 & 0 & 3 & 2 \\ 1 & 0 & 2 & 1 & 0 & 3 & 2 & 0 & 3 \\ 2 & 1 & 0 & 3 & 1 & 0 & 3 & 2 & 0 \end{bmatrix}.$$

Then d^* has N_{d^*2} where

$$N_{d^*2}=\begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Since N_{d^*2} has the form specified in Theorem 3.7, it follows that d^* is both A-optimal and MV-optimal in $D(4; 9, 3)$ for estimating elementary treatment differences involving the test treatments and the standard treatment.

Example 3.10. Consider the class of designs $D(10; 24, 3)$ and the BTBD $d_1^*(1) \in D_1(M)$ where $M=\{1\}$ whose incidence matrix is given by

$$N_{d1}^*=\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
1&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0\\
0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&1&0&1&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&1&1&0&0&0&0&1
\end{bmatrix}.$$

It is shown in Majumdar and Notz ([9], pp. 263–264) that $d_1^*(1)$ is both A-optimal and MV-optimal in $D_1(M)$. Now consider the row-column design d^* given by

$$d^*1=\begin{bmatrix}
0&4&1&0&2&5&7&8&3&0&9&6&0&0&8&0&7&9&1&6&2&3&4&5\\
1&0&5&8&0&2&0&0&5&3&0&4&4&6&6&9&0&7&2&1&3&8&7&9\\
3&1&0&1&4&0&2&2&0&7&3&0&9&5&0&6&8&0&9&7&6&4&5&8
\end{bmatrix}.$$

Then d^* has N_{d^*2} where

$$N_{d^*2}=\begin{matrix}\begin{bmatrix} 6 & 6 & 6 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\end{matrix}.$$

Since N_{d^*2} has the form specified in Theorem 3.7, it follows that d^* is both A-optimal and MV-optimal for estimating treatment differences of the form $\alpha_i - \alpha_0$.

Example 3.11. Consider the class of designs $D(7; 18, 3)$ and the design $d_1^*(1) \in D_1(M)$ where $M = \{1\}$ and whose incidence matrix is given by

$$N_{d^*1} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&1&1&0&0&1&1&0&0 \\ 0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1&1 \\ 0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&1&1 \end{bmatrix}.$$

It is shown in Jacroux [5] that $d_1^*(1)$ is a GDTD(4) with $V_1 = \{1, 2\}$, $V_2 = \{3, 4\}$, $V_3 = \{5, 6\}$, $\lambda_1 = 2$, $\lambda_2 = 1$ and which is MV-optimal in $D_1(M)$. Now consider the row-column design d^* given by

$$d^* = \begin{bmatrix} 0&2&1&0&5&1&0&4&2&0&4&3&0&3&5&0&6&6 \\ 1&0&3&1&0&6&2&0&5&2&0&4&3&0&4&6&0&5 \\ 2&1&0&4&1&0&3&2&0&6&3&0&5&6&0&4&5&0 \end{bmatrix}.$$

Then d^* has N_{d^*2} where

$$N_{d^*2} = \begin{bmatrix} 6&6&6 \\ 2&2&2 \\ 2&2&2 \\ 2&2&2 \\ 2&2&2 \\ 2&2&2 \\ 2&2&2 \end{bmatrix}.$$

Since N_{d^*2} has the form specified in Theorem 3.7, it follows that d^* is MV-optimal for estimating elementary treatment differences involving the test treatments and the standard treatment in $D(7; 18, 3)$.

Comment. It has come to the author's attention that Professor D. Majumdar has obtained several results which are related to some of those presented here. For further information concerning these results, the reader is referred to Majumdar [8].

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