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# ON DISCRETE DISTRIBUTIONS OF ORDER $k$

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## Summary

The class of discrete distributions of order $k$ is defined as the class of the generalized discrete distributions with generalizer a discrete distribution truncated at zero and from the right away from $k+1$. The probability function and factorial moments of these distributions are expressed in terms of the (right) truncated Bell (partition) polynomials and several special cases are briefly examined. Finally a Poisson process of order $k$, leading in particular to the Poisson distribution of order $k$, is discussed.

## 1. Introduction

Philippou and Muwafi [15] considered the problem of finding the probability function (p.f.) of the number $N_k$ of trials required until the first occurrence of the $k$-th consecutive success in a sequence of independent trials with constant success probability and expressed it combinatorially in terms of the Fibonacci polynomials of order $k$. The same probability was also derived by Feller ([7], p. 322) as an application of the renewal theory but the origin of this problem can be attributed to De Moivre ([6], p. 254) (see also Todhunter ([17], p. 184)). Uppuluri and Patil [18] provided another derivation of this probability using generating functions. Philippou, A., Georghiou, C. and Philippou, G. [14] named the distribution of the random variable $N_k$ generalized geometric distribution of order $k$ and further studied it along with its $s$-fold convolution, the negative binomial distribution of order $k$; the Poisson distribution of order $k$ was derived as a limiting form of the corresponding negative binomial distribution. Properties of the Poisson and the negative binomial distributions of order $k$ were also discussed by Philippou, A. [13], [14]. Reliability models connected with the preceding

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distributions were studied by Chiang and Niu [4] and Bollinger and Salvia [2]. Hirano, Kuboki, Aki and Kuribayashi [9], Hirano [8] and Aki, Kuboki and Hirano [1] obtained some formulae useful for the calculation of the probabilities and moments of these distributions and derived the logarithmic series distribution of order $k$ as a limiting form of a left truncated negative binomial distribution of order $k$.

In the geometric and the negative binomial distributions of order $k$, the order is the length of the run of occurrence of an event, considered as a success. Since the Poisson and logarithmic series distributions of order $k$ were obtained as limiting forms of the complete and a left truncated negative binomial distribution of order $k$ respectively, this meaning of the order can hardly be extended. Aki, Kuboki and Hirano [1] in an attempt to give a meaning of the order for these distributions introduced a class of $\theta$-generalized discrete distributions. Examining more closely the definition adopted it follows that the class of $\theta$-generalized distributions is nothing but the class of the generalized power series distributions.

In the present paper the class of discrete distributions of order $k$ is more suggestively defined as the class of the generalized discrete distributions with generalizer a discrete distribution truncated at zero and from the right away from $k+1$. Thus the order of the distributions is the order of the probability generating function of the generalizer (Section 2). The probabilty function and the factorial moments of the class of discrete distributions of order $k$ are expressed in terms of the truncated Bell (partition) polynomials and several special cases are discussed. (Sections 3 and 4). Finally modifying a model studied by Janossy, Renyi and Aczel [10], a Poisson process of order $k$ is discussed, which leads in particular to the Poisson distribution of order $k$. In this model $k$ is the maximum number of events occurring in a "small" time interval (Section 5).

## 2. Probability generating function

Aki, Kuboki and Hirano [1] assuming that $\varphi(t)$ and $\psi(t)$ are probability generating functions (p.g.f.'s) and $\theta>0$ such that $\varphi(\theta)<\infty$, called the distribution with p.g.f., $h(t)=\varphi(\theta\psi(t))/\varphi(\theta)$ a $\theta$-generalized distribution. From this definition it follows that $f(t)=\varphi(\theta t)/\varphi(\theta)$ is the p.g.f. of a power series distribution with parameter $\theta$ and series function $\varphi(\theta)$. Hence $h(t)=f(\psi(t))=\varphi(\theta\psi(t))/\varphi(\theta)$ is the p.g.f. of a generalized power series distribution. Note also that the generalizing distribution for the geometric, negative binomial and logarithmic series distributions of order $k$ is a truncated geometric distribution with probability function (p.f.)

$$P(X=r)=(1-p^k)^{-1}qp^{r-1}\,, \qquad r=1, 2, \cdots, k,\ 0<p<1,\ q=1-p\,.$$

and p.g.f.

$$g_k(t)=(1-p^k)^{-1}qt(1-pt)^{-1}(1-p^kt^k)\ . \tag{2.1}$$

The generalizing distribution for the Poisson distribution of order $k$ is the discrete uniform distribution over the set $\{1, 2,\cdots, k\}$ with p.f.

$$P(X=r)=1/k\ ,\qquad r=1, 2,\cdots, k\ .$$

and p.g.f.

$$g_k(t)=t(1-t)^{-1}(1-t^k)/k\ . \tag{2.2}$$

Therefore the class of discrete distributions of order $k$ can be more suggestively defined as the class of the generalized discrete distributions with generalizer a discrete distribution truncated at zero and from the right away from $k+1$. Letting $f(t)$ be the p.g.f. of the distribution to be generalized and $g_k(t)$ the p.g.f. of the generalizer then the p.g.f. of the class of discrete distributions of order $k$ is given by

$$h(t)=f(g_k(t))\ . \tag{2.3}$$

A random variable obeying a discrete distribution of order $k$ may be represented as

$$S_N=X_1+X_2+\cdots+X_N \tag{2.4}$$

where $X_i$, $i=1, 2,\cdots$ is a sequence of independent and identically distributed discrete random variables with distribution truncated at zero and from the right away from $k+1$ and $N$ is a nonnegative integer valued random variable independent of $X_i$, $i=1, 2,\cdots$.

Letting

$$f(t)=p_k^s(1-q_kt)^{-s}\ ,\qquad p_k=p^k,\ q_k=1-p^k \tag{2.5}$$

which is the p.g.f. of a negative binomial distribution and $g_k(t)$ as in (2.1), (2.3) reduces to the p.g.f. of the negative binomial distribution of order $k$. The special case $s=1$ yields the p.g.f. of the geometric distribution of order $k$. For

$$f(t)=\log(1-\theta_kt)/\log(1-\theta_k)\ ,\qquad \theta_k=1-p^k \tag{2.6}$$

which is the p.g.f. of a logarithmic series distribution and for $g_k(t)$ given by (2.1), (2.3) reduces to the p.g.f. of the logarithmic series distribution of order $k$. Finally taking

$$f(t)=e^{-\lambda_k(1-t)}\ ,\qquad \lambda_k=k\lambda,\ \lambda>0$$

which is the p.g.f. of a Poisson distribution and $g_k(t)$ as in (2.2), (2.3) gives the p.g.f. of the Poisson distribution of order $k$.

The probability function and the factorial moments of the class of discrete distributions of order $k$ may be expressed in terms of the (right) truncated Bell (partition) polynomials which are briefly presented in the next section.

## 3. Truncated Bell polynomials

The Bell (partition) polynomials denoted by $Y_n = Y_n(fg_1, fg_2, \cdots, fg_n)$, $f^r \equiv f_r$, may be defined for every nonnegative integer $n$ by the sum

$$(3.1) \qquad Y_n(fg_1, fg_2, \cdots, fg_n) = \sum \frac{n! f_r}{r_1!\, r_2! \cdots r_n!} \left(\frac{g_1}{1!}\right)^{r_1} \left(\frac{g_2}{2!}\right)^{r_2} \cdots \left(\frac{g_n}{n!}\right)^{r_n}$$

where the summation is extended over all partitions of $n$, that is over all $r_i \geqq 0$, $i = 1, 2, \cdots, n$ such that $r_1 + 2r_2 + \cdots + nr_n = n$; $r_1 + r_2 + \cdots + r_n = r$ is the number of parts (summands) (cf. Riordan [16], Chapter 5).

The partial Bell (partition) polynomials denoted by $Y_{n,r} = Y_{n,r}(g_1, g_2, \cdots, g_n)$ may be defined for $n$ and $r$ nonnegative integers by the sum

$$(3.2) \qquad Y_{n,r}(g_1, g_2, \cdots, g_n) = \sum \frac{n!}{r_1!\, r_2! \cdots r_n!} \left(\frac{g_1}{1!}\right)^{r_1} \left(\frac{g_2}{2!}\right)^{r_2} \cdots \left(\frac{g_n}{n!}\right)^{r_n}$$

where the summation is extended over all partitions of $n$ into $r$ parts, that is over all $r_i \geqq 0$, $i = 1, 2, \cdots, n$ such that $r_1 + 2r_2 + \cdots + nr_n = n$ and $r_1 + r_2 + \cdots + r_n = r$ (cf. Comtet [5], Chapter 3).

From the above definitions it follows that

$$(3.3) \qquad Y_n(fg_1, fg_2, \cdots, fg_n) = \sum_{r=0}^{n} f_r Y_{n,r}(g_1, g_2, \cdots, g_n) .$$

For a given positive integer $k$ and for every positive integer $n > k$ consider the polynomials

$$(3.4) \qquad Y_{n;k}(fg_1, fg_2, \cdots, fg_k) = \sum \frac{n! f_r}{r_1!\, r_2! \cdots r_k!} \left(\frac{g_1}{1!}\right)^{r_1} \left(\frac{g_2}{2!}\right)^{r_2} \cdots \left(\frac{g_k}{k!}\right)^{r_k}$$

where the summation is extended over all partitions of $n$ with parts not greater than $k$, that is over all $r_i \geqq 0$, $i = 1, 2, \cdots, k$ such that $r_1 + 2r_2 + \cdots + kr_k = n$; $r_1 + r_2 + \cdots + r_k = r$ is the number of parts.

The polynomials $T_{n;k} = T_{n;k}(fg_1, fg_2, \cdots, fg_m)$, $m = \min\{n, k\}$, defined by

$$(3.5) \qquad T_{n;k}(fg_1, fg_2, \cdots, fg_m) = \begin{cases} Y_n(fg_1, fg_2, \cdots, fg_n) , & n \leqq k \\ Y_{n;k}(fg_1, fg_2, \cdots, fg_k) , & n > k \end{cases}$$

may be called (right) truncated Bell (partition) polynomials.

For a given positive integer $k$ and for every pair of positive integers $n \geqq r$, $n > k$ consider the polynomials

$$Y_{n,r;k}(g_1, g_2, \cdots, g_k) = \sum \frac{n!}{r_1!\, r_2! \cdots r_k!} \left(\frac{g_1}{1!}\right)^{r_1} \left(\frac{g_2}{2!}\right)^{r_2} \cdots \left(\frac{g_k}{k!}\right)^{r_k} \tag{3.6}$$

where the summation is extended over all partitions of $n$ into $r$ parts each of which is not greater than $k$, that is over all $r_i \geqq 0$, $i = 1, 2, \cdots, k$ such that $r_1 + 2r_2 + \cdots + kr_k = n$, $r_1 + r_2 + \cdots + r_k = r$.

The polynomials $T_{n,r;k} = T_{n,r;k}(g_1, g_2, \cdots, g_m)$, $m = \min\{n, k\}$, defined by

$$T_{n,r;k}(g_1, g_2, \cdots, g_m) = \begin{cases} Y_{n,r}(g_1, g_2, \cdots, g_n)\,, & n \leqq k \\ Y_{n,r;k}(g_1, g_2, \cdots, g_k)\,, & n > k \end{cases} \tag{3.7}$$

may be called (right) truncated partial Bell (partition) polynomials.

Definitions (3.5) and (3.7) imply the relation

$$T_{n;k}(fg_1, fg_2, \cdots, fg_m) = \sum_{r=0}^{n} f_r T_{n,r;k}(g_1, g_2, \cdots, g_m)\,, \quad m = \min\{n, k\}\,. \tag{3.8}$$

The generating functions of the truncated Bell and partial Bell polynomials may easily be obtained as

$$\sum_{n=0}^{\infty} T_{n;k}(fg_1, fg_2, \cdots, fg_m) t^n/n! = \sum_{r=0}^{\infty} f_r [g_k(t)]^r/r! = e^{fg_k(t)}\,, \qquad f^r \equiv f_r \tag{3.9}$$

$$\sum_{n=0}^{\infty} T_{n,r;k}(g_1, g_2, \cdots, g_m) t^n/n! = [g_k(t)]^r/r! \tag{3.10}$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} T_{n,r;k}(g_1, g_2, \cdots, g_m) u^r t^n/n! = e^{ug_k(t)} \tag{3.11}$$

where

$$g_k(t) = \sum_{j=1}^{k} g_j t^j/j!\,.$$

Some special cases of the truncated Bell and partial Bell polynomials useful in expressing the probabilities and factorial moments of the Poisson, logarithmic series, geometric and negative binomial distributions of order $k$ are briefly presented.

The truncated exponential Bell polynomials corresponding to $f_r = 1$, $r = 0, 1, 2, \cdots$ and denoted by $T_{n;k}(g_1, g_2, \cdots, g_m)$ have generating function

$$T_k(t) = \sum_{n=0}^{\infty} T_{n,k}(g_1, g_2, \cdots, g_m) t^n/n! = e^{g_k(t)}\,, \qquad g_k(t) = \sum_{j=1}^{k} g_j t^j/j!\,. \tag{3.12}$$

The truncated logarithmic polynomials corresponding to $f_0 = 0$, $f_r = (r-1)!$, $r = 1, 2, \cdots$ and denoted by $L_{n;k}(g_1, g_2, \cdots, g_m)$ have generating function

$$(3.13)\qquad L_k(t)=\sum_{n=1}^{\infty} L_{n;k}(g_1, g_2, \cdots, g_m)t^n/n!$$
$$=-\log\{1-g_k(t)\}\ , \qquad g_k(t)=\sum_{j=1}^{k} g_j t^j/j!\ .$$

The truncated potential polynomials corresponding to $f_r=(s)_r$, $r=0, 1, 2, \cdots$ and denoted by $P_{n;k}^{(s)}(g_1, g_2, \cdots, g_m)$ have generating function

$$(3.14)\qquad P_{k,s}(t)=\sum_{n=0}^{\infty} P_{n;k}^{(s)}(g_1, g_2, \cdots, g_m)t^n/n!$$
$$=[1+g_k(t)]^s\ , \qquad g_k(t)=\sum_{j=1}^{k} g_j t^j/j!\ .$$

From the recurrence relations for the complete exponential, logarithmic and potential polynomials (cf. Charalambides [3]), the following recurrence relations for the corresponding truncated polynomials are deduced:

$$(3.15)\qquad T_{n;k}(g_1, g_2, \cdots)=\sum_{j=1}^{m}\binom{n-1}{j-1} g_j T_{n-j;k}(g_1, g_2, \cdots)\ , \qquad T_{0,k}=1$$

$$(3.16)\qquad L_{n+1;k}(g_1, g_2, \cdots)=g_{n+1}\zeta(n, k)+\sum_{j=1}^{m}\binom{n}{j} g_j L_{n-j+1;k}(g_1, g_2, \cdots)\ ,$$
$$L_{1,k}=g_1$$

where the zeta function $\zeta(n, k)=1$, $n\leqq k$ and $\zeta(n, k)=0$, $n>k$,

$$(3.17)\qquad P_{n;k}^{(s)}(g_1, g_2, \cdots)=\sum_{j=1}^{m}\binom{n-1}{j-1}\frac{sj-n+j}{j} g_j P_{n-j;k}^{(s)}(g_1, g_2, \cdots)\ ,$$
$$P_{0;k}^{(s)}=1\ .$$

Putting in (3.10), $g_j=j!ab^j$, $j=1, 2, \cdots, k$, $a^{-1}=\sum_{j=1}^{k} b^j$ it follows that

$$\sum_{n=0}^{\infty} T_{n,r;k}(ab, 2ab^2, \cdots, m!ab^m)t^n/n!=a^r b^r t^r\left(\sum_{i=0}^{k-1} b^i t^i\right)^r\Big/ r!$$

and

$$(3.18)\qquad T_{n,r;k}(ab, 2ab^2, \cdots, m!ab^m)=n!a^r b^n N(n-r, r, k-1)/r!\ ,$$
$$r\leqq n\leqq rk$$

where

$$N(n, r, k)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{n-k(j+1)+r-1}{r-1}$$

is the number of ways of distributing $n$ like objects into $r$ different cells with no cell containing no more than $k$ objects.

Letting $g_j=\frac{j!}{k}\binom{k+1}{j+1}$, $j=1,2,\cdots,k$ from (3.10) it follows that

$$\sum_{n=0}^{\infty} T_{n,r;k}(g_1, g_2,\cdots, g_m)t^n/n! = k^{-r}t^{-r}\left(\sum_{j=1}^{k}\binom{k+1}{j+1}t^{j+1}\right)^r\Big/ r!$$

and

$$T_{n,r;k}(g_1, g_2,\cdots, g_m)=n!k^{-r}Q(n+r, r, k+1)/r! \tag{3.19}$$

where $Q(n, r, k)$ is the number of ways of distributing $n$ like objects into $r$ different cells, each having $k$ different compartments with one object capacity, such that each cell contains at least two objects.

## 4. Probabilities and factorials moments

The probability function $P_n=P(S_N=n)$ of a discrete distribution of order $k$ with p.g.f. given by (2.3) on using (3.9) may be obtained as

$$\begin{aligned} &P_0=P(N=0)\,,\\ &P_n=\frac{1}{n!}T_{n;k}(fg_1, fg_2,\cdots, fg_m)\,,\\ &\qquad n=1,2,\cdots,\ f^r\equiv f_r,\ m=\min\{n,k\}\,. \end{aligned} \tag{4.1}$$

where

$$f_r=r!Q_r=r!P(N=r)\,,\qquad g_r=r!p_r=r!P(X=r)\,,$$

or if one prefers, on using (3.8), as

$$\begin{aligned} &P_0=P(N=0)\,,\\ &P_n=\frac{1}{n!}\sum_{r=0}^{n} r!Q_rT_{n,r;k}(p_1, 2!p_2,\cdots, m!p_m)\,,\qquad n=1,2,\cdots\,. \end{aligned} \tag{4.2}$$

Similarly the factorial moments $M_{(n)}=E[(S_N)_n]$, may be obtained as

$$\begin{aligned} M_{(n)}&=T_{n;k}(\alpha\mu_{(1)}, \alpha\mu_{(2)},\cdots, \alpha\mu_{(m)})\,,\qquad \alpha^r\equiv\alpha_{(r)}\,,\\ &=\sum_{r=0}^{n}\alpha_{(r)}T_{n,r;k}(\mu_{(1)}, \mu_{(2)},\cdots, \mu_{(m)})\,,\qquad m=\min\{n,k\} \end{aligned} \tag{4.3}$$

where

$$\alpha_{(r)}=E[(N)_r]\,,\qquad \mu_{(r)}=E[(X)_r]\,.$$

Next, the three major sub-classes of discrete distributions of order $k$ are briefly discussed.

4.1. *Poisson distributions of order k*

In this case $f(t)=e^{-\theta(1-t)}$ and (2.3) reduces to

$$h(t)=e^{-\theta+\theta g_k(t)}$$

which on using (3.12) yields

$$(4.4)\quad \begin{aligned} P_n &= e^{-\theta}T_{n;k}(\theta p_1, 2!\theta p_2, \cdots, m!\theta p_m)/n!\ , \qquad n=0, 1, 2, \cdots \\ &= \frac{1}{n!}e^{-\theta}\sum_{r=0}^{n}\theta^r T_{n,r;k}(p_1, 2!p_2, \cdots, m!p_m)\ , \qquad m=\min\{n, k\} \end{aligned}$$

and

$$(4.5)\quad \begin{aligned} M_{(n)} &= T_{n;k}(\theta\mu_{(1)}, \theta\mu_{(2)}, \cdots, \theta\mu_{(m)}) \\ &= \sum_{r=0}^{n}\theta^r T_{n,r;k}(\mu_{(1)}, \mu_{(2)}, \cdots, \mu_{(m)})\ , \qquad m=\min\{n, k\}\ . \end{aligned}$$

Note that the recurrence relation (3.15) implies the following recurrence relations

$$(4.6)\quad P_n=\frac{\theta}{n}\sum_{j=1}^{m} jp_j P_{n-j}\ , \qquad m=\min\{n, k\}$$

$$(4.7)\quad M_{(n)}=\theta\sum_{j=1}^{m}\binom{n-1}{j-1}\mu_{(j)}M_{(n-j)}\ , \qquad m=\min\{n, k\}\ .$$

For the special case of the Poisson distribution of order $k$, $\theta=k\lambda$ and $g_k(t)$ is given by (2.2). Then on using (3.18) and (3.19)

$$(4.8)\quad P_n=e^{-k\lambda}\sum_{r=1}^{n}\lambda^r N(n-r, r, k-1)$$

$$(4.9)\quad M_{(n)}=n!\sum_{r=0}^{n}\lambda^r Q(n+r, r, k+1)/r!$$

the first of which was also derived by Papastavridis [11].

4.2. *Logarithmic series distributions of order k*

The p.g.f. (2.3) reduces in this case to

$$h(t)=\log\{1-\theta g_k(t)\}/\log(1-\theta)\ , \qquad 0<\theta<1\ .$$

Therefore, by virtue of (3.13), it follows that

$$(4.10)\quad \begin{aligned} P_n &= [-\log(1-\theta)]^{-1}L_{n;k}(\theta p_1, 2!\theta p_2, \cdots, m!\theta p_m)/n!\ , \\ &\qquad\qquad n=1,2,\cdots \\ &= \frac{1}{n!}[-\log(1-\theta)]^{-1}\sum_{r=1}^{n}(r-1)!\theta^r T_{n,r;k}(p_1, 2!p_2, \cdots, m!p_m)\ , \\ &\qquad\qquad m=\min\{n, k\} \end{aligned}$$

and

$$M_{(n)}=[-\log(1-\theta)]^{-1}L_{n;k}(\theta(1-\theta)^{-1}\mu_{(1)},\ \theta(1-\theta)^{-1}\mu_{(2)},\cdots,\ \theta(1-\theta)^{-1}\mu_{(m)})$$
$$=[-\log(1-\theta)]^{-1}\sum_{r=1}^{n}(r-1)!\theta^r(1-\theta)^{-r}\cdot T_{n,r;k}(\mu_{(1)},\mu_{(2)},\cdots,\mu_{(m)})\ ,\qquad m=\min\{n,k\}\ . \tag{4.11}$$

The recurrence relation (3.16) implies the following recurrence relations

$$P_{n+1}=[-\log(1-\theta)]^{-1}\theta p_{n+1}\zeta(n,k)+\frac{\theta}{n+1}\sum_{j=1}^{m}(n-j+1)p_j P_{n-j+1}\ ,\qquad m=\min\{n,k\} \tag{4.12}$$

and

$$M_{(n+1)}=\theta(1-\theta)^{-1}\left\{[-\log(1-\theta)]^{-1}\mu_{(n+1)}\zeta(n,k)+\sum_{j=1}^{m}\binom{n}{j}\mu_{(j)}M_{(n-j+1)}\right\}\ ,\qquad m=\min\{n,k\}\ . \tag{4.13}$$

For the special case of the logarithmic series distribution of order $k$, $\theta=1-p^k$ and $g_k(t)$ is given by (2.2). Hence by (3.18)

$$P_n=[-k\log p]^{-1}\sum_{r=1}^{n}\frac{1}{r}q^r p^{n-r}N(n-r,r,k-1)\ , \tag{4.14}$$

which was also obtained by Papastavridis [11].

4.3. *Negative binomial distributions of order $k$*

In this case $f(t)=(1-\theta)^s(1-\theta t)^{-s}$, $0<\theta<1$, $s>0$ and (2.3) reduces to

$$h(t)=(1-\theta)^s[1-\theta g_k(t)]^{-s}$$

which on using (3.14) yields

$$P_n=(1-\theta)^s P_{n;k}^{(-s)}(-\theta p_1,\ -2\theta p_2,\cdots,\ -m!\theta p_m)/n!\ ,\qquad n=0,1,2,\cdots$$
$$=\frac{1}{n!}(1-\theta)^s\sum_{r=0}^{n}(s+r-1)_r\theta^r T_{n,r;k}(p_1,2p_2,\cdots,m!p_m)\ ,\qquad m=\min\{n,k\} \tag{4.15}$$

and

$$M_{(n)}=P_{n;k}^{(-s)}(-\theta(1-\theta)^{-1}\mu_{(1)},\ -\theta(1-\theta)^{-1}\mu_{(2)},\cdots,\ -\theta(1-\theta)^{-1}\mu_{(m)})$$
$$=\sum_{r=0}^{n}(s+r-1)_r\theta^r(1-\theta)^{-r}T_{n,r;k}(\mu_{(1)},\mu_{(2)},\cdots,\mu_{(m)})\ ,\qquad m=\min\{n,k\}\ . \tag{4.16}$$

The recurrence relation (3.17) implies the following recurrence relations

$$P_n=\frac{\theta}{n}\sum_{j=1}^{m}(n+sj-j)p_jP_{n-j}\ , \tag{4.17}$$

$$M_{(n+1)}=\theta(1-\theta)^{-1}\sum_{j=1}^{m}\binom{n-1}{j-1}\frac{n+sj-j}{j}\mu_{(j)}M_{(n-j)}\ . \tag{4.18}$$

For the special case of the negative binomial distribution of order $k$, $\theta=1-p^k$ and $g_k(t)$ is given by (2.1). Hence by (3.18)

$$P_n=\sum_{r=0}^{n}\binom{s+r-1}{r}q^rp^{n+sk-r}N(n-r,\,r,\,k-1)\ . \tag{4.19}$$

The geometric distribution of order $k$ corresponds to $s=1$. The expression (4.19) was also obtained by Papastavridis [11].

## 5. A Poisson process of order $k$

Consider a homogeneous stochastic process $\{X(t),\ t\geqq 0\}$ with independent increments and let $P_n(t)=P[X(t)=n]$, $n=0,1,2,\cdots$. Assume that for small $\Delta t>0$,

$$P_r(\Delta t)=P[X(\Delta t)=r]=\lambda_r\Delta t+O(\Delta t)\ ,\qquad r=1,2,\cdots,k\ . \tag{5.1}$$

$$1-\sum_{r=0}^{k}P_r(\Delta t)=P[X(\Delta t)>k]=O(\Delta t)$$

where the function $O(t)$ has the property

$$\lim_{\Delta t\to 0}\frac{O(\Delta t)}{\Delta t}=0\ .$$

Note that the preceding conditions imply

$$P_0(\Delta t)=1-\lambda\Delta t+O(\Delta(t)\ ,\qquad \lambda=\sum_{r=1}^{k}\lambda_r\ .$$

Then the following difference-differential equations for the probabilities $P_n(t)$, $n=0,1,2,\cdots$, can be easily derived

$$\frac{dP_0(t)}{dt}=-\lambda P_0(t)$$

$$\frac{dP_n(t)}{dt}=\sum_{r=1}^{m}\lambda_rP_{n-r}(t)-\lambda P_n(t)\ ,\qquad n=1,2,\cdots,\ m=\min\{n,k\}\ .$$

Multiplying the $n$-th equation by $u^n$ and summing for all $n=0,1,2,\cdots$ we get for the generating function

$$h(u,t)=\sum_{n=0}^{\infty}P_n(t)u^n$$

the differential equation

$$\frac{\partial h(u,t)}{\partial t}=-\lambda[1-g_k(u)]h(u,t)\,, \qquad g_k(u)=\sum_{r=1}^{k} p_r u^r,\ p_r=\lambda_r/\lambda\,.$$

which implies

$$h(u,t)=C(u)\exp\{-\lambda t[1-g_k(u)]\}$$

where the function $C(u)$ is determined by the initial condition

$$P_0(0)=P[X(0)=0]=1\,.$$

Thus

$$C(u)=h(u,0)=\sum_{n=0}^{\infty} P_n(0)s^n=P_0(0)=1\,,$$

and

$$h(u,t)=\exp\{-\lambda t[1-g_k(u)]\} \tag{5.2}$$

which is the p.g.f. of a Poisson distribution of order $k$ with p.f.

$$\begin{aligned} P_n&=\frac{1}{n!}e^{-\lambda t}T_{n;k}(\lambda_1 t, 2\lambda_2 t,\cdots, m!\lambda_m t) \\ &=\frac{1}{n!}e^{-\lambda t}\sum_{r=0}^{n} t^r T_{n,r;k}(\lambda_1, 2\lambda_2,\cdots, m!\lambda_m)\,, \qquad m=\min\{n,k\} \end{aligned} \tag{5.3}$$

and factorial moments

$$\begin{aligned} M_{(n)}&=T_{n;k}(\lambda t\mu_{(1)}, \lambda t\mu_{(2)},\cdots, \lambda t\mu_{(m)}) \\ &=\sum_{r=0}^{n}(\lambda t)^r T_{n,r;k}(\mu_{(1)}, \mu_{(2)},\cdots, \mu_{(m)})\,, \qquad m=\min\{n,k\} \end{aligned} \tag{5.4}$$

where

$$\mu_{(j)}=\sum_{r=j}^{k}(r)_j\lambda_r/\lambda\,, \qquad j=1,2,\cdots,k\,.$$

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