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THE ASYMPTOTIC POWER OF RANK TESTS UNDER SCALE-ALTERNATIVES INCLUDING CONTAMINATED DISTRIBUTIONS

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Summary

The alternative hypothesis of translated scale for the classical nonparametric hypothesis of equality of two distribution functions in the two-sample problem is extended to a scale-alternative including contamination. The asymptotic power of rank tests and the two-sample F-test under contiguous sequences of the alternatives is derived and asymptotic relative efficiency of these rank tests with respect to the F-test is investigated. It is found that some of the rank tests have reasonably high asymptotic powers satisfied enough.

1. Introduction

Let $X_{11},\cdots,X_{1n_1}$ and $X_{21},\cdots,X_{2n_2}$ be two samples from populations with continuous distributions F_1 and F_2 respectively. Further we set $X_{ij}=\sigma_i e_{ij}+\mu$ for $i=1,2$ and $j=1,\cdots,n_i$, where $\{e_{ij};\ i=1,2,\ j=1,\cdots,n_i\}$ are unobservable random variables, μ is an unknown common nuisance parameter of F_1 and F_2, and σ_1 and σ_2 are respective scale parameters. Then our null hypothesis which gives $F_1=F_2$ is as follows.

(1.1) H: $\sigma_1=\sigma_2=\sigma$ and $\{e_{ij}:\ i=1,2,\ j=1,\cdots,n_i\}$ are independent and identically distributed with continuous distribution function F (i.i.d. F).

For the null hypothesis H versus the scale alternative

$$A:\ \sigma_1\neq\sigma_2 \text{ and } \{e_{ij};\ i=1,2,\ j=1,\cdots,n_i\} \text{ are i.i.d. } F,$$

Mood [7], Ansari and Bradley [1], and Siegel and Tukey [9] proposed rank tests based on statistics of form $\sum_{j=1}^{n_1} a_N(R_{1j})$ where $N=n_1+n_2$, R_{1j} is the rank of X_{1j} among the pooled observations $\{X_{ij};\ i=1,2,\ j=1,$

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$\cdots, n_i\}$ $(j=1,\cdots,n_1)$ and $a_N(\cdot)$ is a real valued function defined on $\{1, \cdots, N\}$. When F is a normal distribution function, Mood [7] showed that the asymptotic relative efficiency of his test with respect to the parametric F-test is 0.76 and Klotz [4] found that the efficiency for the test of Siegel and Tukey is 0.61. We find that the values are too small. Capon [2] proposed the normal scores test based on $\sum_{j=1}^{n_1} \mathrm{E}\,[\{Z_N^{(R_{1j})}\}^2]$ where $Z_N^{(k)}$ is the k-th order statistic from a sample of N standard normal variables and showed the asymptotic relative efficiency 1 relative to the F-test under normal alternatives. Also Klotz [4] proposed the normal scores test based on $\sum_{j=1}^{n_1} \{\Phi^{-1}(R_{1j}/(N+1))\}^2$ where $\Phi(\cdot)$ is the standard normal distribution function and showed that the test has the same efficiency 1. On the other hand, Rieder [8] considered a null hypothesis possessing contamination versus a location alternative including contamination from the point of view of gross errors and an asymptotic minimax rank test under the hypotheses. Also Lehmann, in [5], considered alternatives of three types including the alternative of a contaminated distribution and investigated the power of linear rank tests under the alternatives and, in Section 7A of Chapter 2 of [6], pointed out that the location alternative may be an oversimplification. So we think that the scale alternative is an oversimplification. Also in many cases, after receiving treatments, we can predict that the observations give rise not only to a variation in scale of a distribution but also to a slight amount of contamination which cannot be represented by the simple scale alternative. We consider in this paper the alternative of the form K: $\sigma_i=\sigma e^{\theta_i}$ and $\{e_{ij};\ j=1,\cdots,n_i\}$ are independent and identically distributed with distribution function $(1-\varepsilon_i)F(x)+\varepsilon_i J(x)$ for $i=1,2$, where $\varepsilon_i \geqq 0$ for $i=1,2$ and $\varepsilon_1 \neq \varepsilon_2$ or $\theta_1 \neq \theta_2$ and $J(x)$ is a distribution function. Here for the possibility of arguments on the asymptotic property, we assume that there exists some increasing function $G(u)$ with derivative $G'(u)=g(u)$ such that $G(0)=0$ and $G(1)=1$ and $J(x)=G(F(x))$. If $F(x)$ is a distribution function such that the density $f(x)=F'(x)>0$ for $x \in (-\infty, \infty)$, there exists $G(u)$ satisfying the above assumption without loss of generality. The distribution functions of the observations under the alternative K are given by

$$F_i(x)=(1-\varepsilon_i)F((x-\mu)/(\sigma e^{\theta_i}))+\varepsilon_i G(F((x-\mu)/(\sigma e^{\theta_i}))) \qquad \text{for } i=1,2\,.$$

In order to compare tests by the Pitman asymptotic relative efficiency, we discuss the sequence of the above alternatives

$$\text{(1.2)} \qquad K_N:\ F_i(x)=(1-\lambda_i/\sqrt{N})F((x-\mu)/(\sigma \exp(\Delta_i/\sqrt{N}))) + (\lambda_i/\sqrt{N})G(F((x-\mu)/(\sigma \exp(\Delta_i/\sqrt{N}))))$$

for $i=1,2$, where $\lambda_i \geqq 0$ for $i=1,2$ and $\lambda_1 \neq \lambda_2$ or $\Delta_1 \neq \Delta_2$.

In Section 2, we shall derive asymptotic normal distributions of the rank test statistics and the parametric F-test statistic under K_N and obtain asymptotic relative efficiency of the tests. We shall be able to verify that, when $F(x)$ is a normal distribution function, the asymptotic relative efficiency of the normal scores test with respect to the F-test is 1 irrespective of $G(u)$. In Section 3, we shall investigate the numerical values of the asymptotic relative efficiency given by Section 2 and find that the asymptotic power of some rank tests including normal scores test is satisfying enough, but that the Ansari-Bradley rank test and the quartile test do not have desirable asymptotic powers.

2. Asymptotic properties

As we consider the tests invariant with respect to location parameter μ and scale parameter σ, we set $\mu=0$ and $\sigma=1$ for the rest of this paper; then we may assume that the observations X_{ij}'s have the joint density under K_N defined by (1.2)

$$\begin{aligned}(2.1)\quad q_N(x)=\prod_{i=1}^{2}\prod_{j=1}^{n_i}\{&(1-\lambda_i/\sqrt{N})(\exp(-\Delta_i/\sqrt{N}))f(x_{ij}/\exp(\Delta_i/\sqrt{N}))\\ &+(\lambda_i/\sqrt{N})(\exp(-\Delta_i/\sqrt{N}))g(F(x_{ij}/\exp(\Delta_i/\sqrt{N})))\\ &\times f(x_{ij}/\exp(\Delta_i/\sqrt{N}))\}\ .\end{aligned}$$

The joint density under the null hypothesis H defined by (1.1) is given by

$$(2.2)\qquad p_N(x)=\prod_{i=1}^{2}\prod_{j=1}^{n_i}f(x_{ij})\ .$$

We set Assumptions (1) to prove the contiguity introduced by VI. 1.1 of Hájek and Šidák [3] and to derive the asymptotic local power of tests.

ASSUMPTIONS (1)

(2.3) The derivative of $f(x)$ exists and

$$0<I(f)=\int_{-\infty}^{\infty}\{-1-xf'(x)/f(x)\}^2f(x)dx<+\infty\ ,$$

(2.4) $g(u)$ is bounded and its derivative exists,

$$\begin{aligned}(2.5)\quad &\lim_{\theta\to0}\int_{-\infty}^{\infty}[\{g(F(x/e^{\theta}))-g(F(x))\}/\theta]f(x)dx\\ &=-\int_{-\infty}^{\infty}xg'(F(x))f^2(x)dx\qquad\text{holds}\ ,\end{aligned}$$

$$(2.6)\quad 1+\int_{-\infty}^{\infty}xg(F(x))f'(x)dx=-\int_{-\infty}^{\infty}xg'(F(x))f^2(x)dx\ ,$$

and

$$(2.7)\qquad \int_{-\infty}^{\infty} \{xf'(x)/f(x)\} f(x)dx = -1 .$$

LEMMA 2.1. *Suppose that Assumptions* (1) *and* $\lim_{N\to\infty}(n_i/N)=\alpha_i$ $(0<\alpha_i<1)$ *for* $i=1, 2$ *are satisfied. Then we get*

$$(2.8)\qquad \Big|\log\{q_N(X)/p_N(X)\} - (1/\sqrt{N})\sum_{i=1}^{2}\sum_{j=1}^{n_i}[\lambda_i\{g(F(X_{ij}))-1\} - \Delta_i\{1+X_{ij}f'(X_{ij})/f(X_{ij})\}] + \sum_{i=1}^{2}(\alpha_i/2)\operatorname{Var}[\lambda_i\{g(F(X_{11}))-1\} - \Delta_i\{1+X_{11}f'(X_{11})/f(X_{11})\}]\Big| \xrightarrow[P]{} 0$$

where $\xrightarrow[P]{}$ *denotes convergence in probability. Further the family of densities* $\{q_N(x)\}$ *defined by* (2.1) *is contiguous to* $\{p_N(x)\}$ *defined by* (2.2).

PROOF. Throughout this proof, we assume that $(X_{11},\cdots,X_{1n_1},X_{21},\cdots,X_{2n_2})$ has the joint density $p_N(x)$. The logarithm of the likelihood ratio transforms the following expression.

$$(2.9)\qquad \log\{q_N(X)/p_N(X)\} = \log\prod_{i=1}^{2}\prod_{j=1}^{n_i}\{(1-\lambda_i/\sqrt{N})+(\lambda_i/\sqrt{N})g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))\} + \log\prod_{i=1}^{2}\prod_{j=1}^{n_i}\{\exp(-\Delta_i/\sqrt{N})f(X_{ij}/\exp(\Delta_i/\sqrt{N}))/f(X_{ij})\} .$$

Taylor's series expansion of the first term of the above right expression yields

$$(2.10)\qquad \sum_{i=1}^{2}\sum_{j=1}^{n_i}\log[1+(\lambda_i/\sqrt{N})\{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))-1\}] = (1/\sqrt{N})\sum_{i=1}^{2}\lambda_i\sum_{j=1}^{n_i}\{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))-1\} - \{1/(2N)\} \times\sum_{i=1}^{2}\lambda_i^2\sum_{j=1}^{n_i}\{g(F(X_{11}/\exp(\Delta_i/\sqrt{N})))-1\}^2 + \{1/(3N\sqrt{N})\} \times\sum_{i=1}^{2}\lambda_i^3\sum_{j=1}^{n_i}(\{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))-1\}^3/ [1+(\delta_{ij}\Delta_i/\sqrt{N})\{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))-1\}]^3) ,$$

where $0<\delta_{ij}<1$. The condition (2.4) gives

$$(2.11)\qquad (1/N)\operatorname{Var}\Big(\sum_{i=1}^{2}\lambda_i\sum_{j=1}^{n_i}[g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))-1-\{g(F(X_{ij}))-1\}]\Big) = \sum_{i=1}^{2}\lambda_i^2(n_i/N)(E\{g(F(X_{i1}/\exp(\Delta_i/\sqrt{N})))-g(F(X_{i1}))\}^2 - [E\{g(F(X_{i1}/\exp(\Delta_i/\sqrt{N})))-g(F(X_{i1}))\}]^2)\to 0 \quad \text{as } N\to\infty .$$

Here from (2.11) and the condition (2.5) we get

$$
\begin{aligned}
&(1/\sqrt{N}) \sum_{i=1}^{2} \lambda_i \sum_{j=1}^{n_i} \{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))-1\} \\
&\quad \sim (1/\sqrt{N}) \sum_{i=1}^{2} \lambda_i \sum_{j=1}^{n_i} [g(F(X_{ij}))+E\{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))\}-2] \\
&\quad =(1/\sqrt{N}) \sum_{i=1}^{2} \lambda_i \sum_{j=1}^{n_i} (\{g(F(X_{ij}))-1\}+[E\{g(F(X_{ij}/\exp(\Delta_i/\sqrt{N})))\} \\
&\qquad -E\{g(F(X_{ij}))\}]) \\
&\quad \sim (1/\sqrt{N}) \sum_{i=1}^{2} \lambda_i \sum_{j=1}^{n_i} \{g(F(X_{ij}))-1\} - \sum_{i=1}^{2} \alpha_i \lambda_i \Delta_i E\{X_{11} g'(F(X_{11})) f(X_{11})\} ,
\end{aligned}
$$

where $Y \sim Z$ denotes that $Y-Z$ converges to zero in probability. Further the condition (2.4) and the law of large numbers show that the sum of the remainder term of the right-hand side of (2.10) converges to $-\sum_{i=1}^{2} (\alpha_i \lambda_i^2/2) \operatorname{Var}\{g(F(X_{11}))\}$ in probability as $N \to \infty$. Also Theorem VI.2.2 of Hájek and Šidák [3] gives, under the condition (2.7), the second term of the right-hand side of (2.9),

$$
\begin{aligned}
&\log \prod_{i=1}^{2} \prod_{j=1}^{n_i} \{\exp(-\Delta_i/\sqrt{N}) f(X_{ij}/\exp(\Delta_i/\sqrt{N}))/f(X_{ij})\} \\
&\quad \sim -\sum_{i=1}^{2} (\Delta_i/\sqrt{N}) \sum_{j=1}^{n_i} \{1+X_{ij} f'(X_{ij})/f(X_{ij})\} - I(f) \sum_{i=1}^{2} \alpha_i \Delta_i^2/2 .
\end{aligned}
$$

Hence we get (2.8) from the above arguments of the two terms on the right expression of (2.9). Further applying the central limit theorem to (2.8), we get $\log\{q_N(X)/p_N(X)\} \xrightarrow[L]{} N(-\sigma_0^2/2, \sigma_0^2)$, where $\xrightarrow[L]{}$ denotes convergence in law and

$$
(2.12) \qquad \sigma_0^2 = \sum_{i=1}^{2} \alpha_i \operatorname{Var}[\lambda_i \{g(F(X_{11}))-1\} - \Delta_i \{1+X_{11} f'(X_{11})/f(X_{11})\}] .
$$

By Corollary VI.1.2 of Hájek and Šidák [3], we find the contiguity.

The normalized two-sample rank test statistic is given by

$$
S=\left\{\sum_{j=1}^{n_1} a_N(R_{1j}) - n_1 \bar{a}_N\right\} \Big/ \sqrt{[n_1 n_2/\{N(N-1)\}] \sum_{k=1}^{N} \{a_N(k)-\bar{a}_N\}^2}
$$

where $\bar{a}_N = \sum_{k=1}^{N} a_N(k)/N$.

THEOREM 2.2. *Assume that, for some square integrable function* $\phi(u)$ *such that* $\int_0^1 \left\{\phi(u) - \int_0^1 \phi(v) dv\right\}^2 du > 0$,

$$
(2.13) \qquad \lim_{N \to \infty} \int_0^1 \{a_N(1+[uN]) - \phi(u)\}^2 du = 0
$$

holds. Further suppose that the assumptions of Lemma 2.1 *are satisfied.*

Then the two-sample rank statistic S has asymptotically a normal distribution with mean ν and variance 1 *under* $\{q_N(x)\}$, *where*

$$\nu=\sqrt{\alpha_1\alpha_2}\,\mathrm{Cov}\,\{\phi(F(X)),\,(\lambda_1-\lambda_2)g(F(X)) -(\Delta_1-\Delta_2)Xf'(X)/f(X)\}/\sqrt{\mathrm{Var}\,\{\phi(F(X))\}} \tag{2.14}$$

and random variable X has the distribution function $F(x)$.

PROOF. If we set

$$T=\left\{\sum_{j=1}^{n_1}\phi(F(X_{1j}))-n_1\bar{\phi}\right\}\Big/\sqrt{n_1\alpha_2\,\mathrm{Var}\,\{\phi(F(X))\}}\ ,$$

where $\bar{\phi}=\sum_{i=1}^{2}\sum_{j=1}^{n_i}\phi(F(X_{ij}))/N$, from Theorems V.1.5a and V.1.6a of Hájek and Šidák [3], we get $S-T\xrightarrow[P]{}0$ under $\{p_N(x)\}$. Further from (2.8), the central limit theorem shows that $(\log\{q_N(X)/p_N(X)\},\,T)'$ has asymptotically a bivariate normal distribution with mean $(-\sigma_0^2/2,\,0)'$ and covariance matrix $\begin{pmatrix}\sigma_0^2, & \nu\\ \nu, & 1\end{pmatrix}$ under $\{p_N(x)\}$, where σ_0^2 is defined by (2.12). So using LeCam's third lemma stated in Lemma VI.1.4 of Hájek and Šidák [3], we get that T has asymptotically a normal distribution with mean ν and variance 1 under $\{q_N(x)\}$, which implies the conclusion.

When we put $a_N(k)=E\{\phi(U_N^{(k)})\}$, $N\int_{(k-1)/N}^{k/N}\phi(u)du$ or $\phi(k/(N+1))$ where $U_N^{(k)}$ is the k-th order statistic, among a sample of size N from the uniform distribution on $(0, 1)$, the equation (2.13) is satisfied.

Next in order to derive the asymptotic power of the two-sample F-test, we consider the test based on the following statistic which is equivalent to the discussed F-test.

$$U=\sqrt{N}\left\{\sum_{j=1}^{n_1}(X_{1j}-\bar{X}_{1\cdot})^2/(n_1-1)-\sum_{j=1}^{n_2}(X_{2j}-\bar{X}_{2\cdot})^2/(n_2-1)\right\}\Big/\left\{\sum_{j=1}^{n_2}(X_{2j}-\bar{X}_{2\cdot})^2/(n_2-1)\right\}$$

where $\bar{X}_{i\cdot}=\sum_{j=1}^{n_i}X_{ij}/n_i$ for $i=1, 2$.

THEOREM 2.3. *If the assumptions of Lemma* 2.1 *are satisfied, the statistic U has asymptotically a normal distribution with mean ν' and variance* $\mathrm{Var}\,\{(X-E(X))^2\}/[\alpha_1\alpha_2\{\mathrm{Var}\,(X)\}]$ *under* $\{q_N(x)\}$, *where*

$$\nu'=\mathrm{Cov}\,\{(X-E(X))^2,\,(\lambda_1-\lambda_2)g(F(X)) -(\Delta_1-\Delta_2)Xf'(X)/f(X)\}/\{\mathrm{Var}\,(X)\}\ . \tag{2.15}$$

PROOF. If we set

$$V=\sqrt{N}\left\{\sum_{j=1}^{n_1}(X_{1j}-E(X))^2/n_1-\sum_{j=1}^{n_2}(X_{2j}-E(X))^2/n_2\right\}\Big/\mathrm{Var}\,(X)\,,$$

after a simple argument, we find that $U-V\xrightarrow[P]{}0$ under $\{p_N(x)\}$. From the way similar to the proof of Theorem 2.2, we can show that $(\log\{q_N(X)/p_N(X)\}, V)'$ has asymptotically a bivariate normal distribution with mean $(-\sigma_0^2/2, 0)'$ and covariance matrix $\begin{pmatrix}\sigma_0^2, & \nu' \\ \nu', & 1\end{pmatrix}$ under $\{p_N(x)\}$. So LeCam's third lemma gives the result.

The square of the ratio of the asymptotic mean (2.14) stated in Theorem 2.2 to (2.15) in Theorem 2.3 gives the asymptotic relative efficiency of the rank test with respect to the F-test.

COROLLARY 2.4. *If the assumptions of Theorem 2.2 are satisfied, the asymptotic relative efficiency of the rank test based on S with respect to the two-sample F-test based on U for H versus K_N is given by*

$$\begin{aligned}\mathrm{ARE}\,(S, U)=\mathrm{Var}\,\{(X-E(X))^2\}\,[&\mathrm{Cov}\,\{\phi(F(X)), (\lambda_1-\lambda_2)g(F(X))\\ &-(\Delta_1-\Delta_2)Xf'(X)/f(X)\}\,]^2/(\mathrm{Var}\,\{\phi(F(X))\}\\ &\times[\mathrm{Cov}\,\{(X-E(X))^2, (\lambda_1-\lambda_2)g(F(X))\\ &-(\Delta_1-\Delta_2)Xf'(X)/f(X)\}\,]^2)\,.\end{aligned}$$

3. Numerical results of relative efficiency

We state types of scores, their forms and $\phi(u)$ induced by the scores in Table 1. To compare the rank tests based on these scores with the F-test, we use the expression of the asymptotic relative efficiency (ARE) given by Corollary 2.4. Then if $a_N(\cdot)$ is a normal scores

Table 1. Table of scores functions and functions induced by their scores functions

Type of scores	Form	Function $\phi(u)$ induced by the scores
Normal scores	$E[\{Z_N^{(k)}\}^2]$ or $\{\Phi^{-1}(k/(N+1))\}^2$	$\{\Phi^{-1}(u)\}^2$
Asymptotically optimum scores against scale alternatives for a logistic distribution	$\{2k/(N+1)-1\}\times\log\{k/(N-k+1)\}$	$(2u-1)\log\{u/(1-u)\}$
Asymptotically optimum scores against scale alternatives for a double exponential distribution	$-\log(1-\|2k/(N+1)-1\|)$	$-\log(1-\|2u-1\|)$
Ansari-Bradley's scores	$1/2-\|k/(N+1)-1/2\|$	$1/2-\|u-1/2\|$
Quartile scores	$\mathrm{sign}\,(\|k/(N+1)-1/2\|-1/4)$	$\mathrm{sign}\,(\|u-1/2\|-1/4)$

Table 2. Values of the asymptotic relative efficiency of the rank tests with respect to the F-test under $\{q_N(x)\}$

k	$G(u)=u^k,\ 1-(1-u)^k$		$G(u)=2^{k-1}\|u-1/2\|^k\ \mathrm{sign}\,(u-1/2)+1/2$	
	$\eta=1$	$\eta=\infty$	$\eta=1$	$\eta=\infty$
(1) Normal scores				
(i) F is logistic		ARE=1.118 under scale alternatives ($\eta=0$)		
1.1	1.118	1.103	1.130	1.457
1.5	1.113	1.209	1.163	1.420
3	1.143	1.316	1.191	1.316
10	1.166	1.215	1.104	1.096
(ii) F is double exponential		ARE=1.221 under scale alternatives ($\eta=0$)		
1.1	1.221	1.212	1.249	2.018
1.5	1.209	1.411	1.324	1.910
3	1.278	1.641	1.383	1.641
10	1.323	1.417	1.198	1.186
(2) Asymptotically optimum scores against scale alternatives for a logistic distribution				
(i) F is normal		ARE=0.977 under scale alternatives ($\eta=0$)		
1.1	0.977	1.009	0.987	1.272
1.5	0.973	1.075	1.014	1.239
3	0.997	1.150	1.037	1.150
10	1.019	1.068	0.975	0.974
(ii) F is logistic		ARE=1.144 under scale alternatives ($\eta=0$)		
1.1	1.145	1.113	1.168	1.853
1.5	1.136	1.300	1.231	1.759
3	1.190	1.514	1.278	1.514
10	1.219	1.298	1.096	1.067
(iii) F is double exponential		ARE=1.244 under scale alternatives ($\eta=0$)		
1.1	1.245	1.223	1.288	2.566
1.5	1.228	1.517	1.406	2.367
3	1.330	1.887	1.489	1.887
10	1.384	1.514	1.184	1.155
(3) Asymptotically optimum scores against scale alternatives for a double exponential distribution				
(i) F is normal		ARE=0.977 under scale alternatives ($\eta=0$)		
1.1	0.976	1.004	0.990	1.389
1.5	0.973	1.063	1.022	1.306
3	0.996	1.142	1.034	1.142
10	1.007	1.042	0.961	0.949
(ii) F is logistic		ARE=1.139 under scale alternatives ($\eta=0$)		
1.1	1.140	1.107	1.168	2.024
1.5	1.131	1.285	1.239	1.854
3	1.184	1.503	1.271	1.503
10	1.201	1.267	1.077	1.041

Table 2. (Continued)

(iii) F is double exponential		ARE=1.250 under scale alternatives (η=0)		
1.1	1.251	1.217	1.300	2.803
1.5	1.235	1.500	1.427	2.494
3	1.333	1.874	1.488	1.874
10	1.368	1.477	1.166	1.126
		(4) Ansari-Bradley's scores		
(i) F is normal		ARE=0.608 under scale alternatives (η=0)		
1.1	0.606	0.696	0.652	2.415
1.5	0.595	0.910	0.762	1.994
3	0.672	1.234	0.805	1.234
10	0.706	0.830	0.551	0.512
(ii) F is logistic		ARE=0.838 under scale alternatives (η=0)		
1.1	0.840	0.767	0.906	3.519
1.5	0.825	1.100	1.071	2.831
3	0.927	1.623	1.105	1.623
10	0.920	1.008	0.658	0.562
(iii) F is double exponential		ARE=0.938 under scale alternatives (η=0)		
1.1	0.941	0.843	1.036	4.873
1.5	0.918	1.284	1.279	3.808
3	1.070	2.024	1.328	2.024
10	1.061	1.176	0.709	0.608
		(5) Quartile scores		
(i) F is normal		ARE=0.369 under scale alternatives (η=0)		
1.1	0.367	0.437	0.398	1.592
1.5	0.359	0.619	0.482	1.425
3	0.423	0.925	0.538	0.925
10	0.442	0.537	0.299	0.254
(ii) F is logistic		ARE=0.547 under scale alternatives (η=0)		
1.1	0.549	0.482	0.592	2.320
1.5	0.537	0.749	0.716	2.024
3	0.621	1.218	0.770	1.218
10	0.598	0.653	0.368	0.279
(iii) F is double exponential		ARE=0.607 under scale alternatives (η=0)		
1.1	0.610	0.530	0.672	3.212
1.5	0.592	0.874	0.853	2.722
3	0.714	1.518	0.927	1.518
10	0.687	0.761	0.391	0.302

function and $F(x)$ is a normal distribution function, we get ARE(S, U) $=1$ irrespective of $G(u)$. For the other ARE, we show the values in Table 2, restricting the scores functions appeared in Table 1; $F(x)$ $=$normal, logistic, double exponential; $G(u)=u^k$, $1-(1-u)^k$, $2^{k-1}|u-1/2|^k \operatorname{sign}(u-1/2)+1/2$ with $k=1.1, 1.5, 3, 10$; and $\eta=(\lambda_1-\lambda_2)/(\Delta_1-\Delta_2)=0, 1, +\infty$, where sign$(x)$ denotes the sign of x. From Table 2, we can see that, in the case of normal scores, the values of the ARE are always larger than 1, and that, in the case of the asymptotically optimum rank tests against scale alternatives of a logistic distribution and a double exponential distribution, the values are nearly equal to 1 if $F(x)$ is normal and are larger than 1 for the other distributions, and that the ARE of the Ansari-Bradley test is not stable and that the one of the quartile test is smaller than 1 in many cases investigated. Further we can find that, when $F(x)$ is a normal distribution function, the normal scores test has the highest asymptotic power among those rank tests in many cases investigated and that the power of the two asymptotically optimum rank tests against scale alternatives for a logistic distribution and a double exponential distribution is nearly same and, when $F(x)$ is a logistic distribution or a double exponential distribution, these two rank tests have high asymptotic power.

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