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THE WEAK CONVERGENCE OF LEAST SQUARES RANDOM FIELDS AND ITS APPLICATION

B. L. S. PRAKASA RAO

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1. Introduction

Consider a nonlinear regression model

$$Y_i = g_i(\theta) + \varepsilon_i\,, \qquad i \geqq 1$$

where $\theta \in \Theta \subset R^k$ and $\{\varepsilon_i,\ i \geqq 1\}$ are independent random variables with mean 0 and finite variances under the true model. The true model might be $Y_i = \alpha_i + \varepsilon_i$ where $\{\alpha_i,\ i \geqq 1\}$ might not be in the range space of $\{g_i(\theta),\ i \geqq 1,\ \theta \in \Theta\}$. The problem is to estimate θ by the least squares approach given that the process is observed upto time n and obtain the asymptotic properties of the corresponding least squares estimator (LSE). In contrast to the standard approach via normal equations for the study of asymptotic properties, the author has proposed an alternate approach via the study of weak convergence of the least squares process. This method was used to study the asymptotic behaviour of a LSE in a series of papers (Prakasa Rao [11]-[14]). Further more, in the case when $g_i(\theta)$ is not differentiable with respect to θ, the standard approach is not applicable. An example of a non-regular nonlinear regression model where the new approach is found to be useful is given in Prakasa Rao [12].

The asymptotic theory of the least squares estimator for multi-dimensional parameter is discussed in Prakasa Rao [14] through the study of the weak convergence of the corresponding least squares random field. One of the basic conditions used there in is that $\sup_i \mathrm{E}\,|\varepsilon_i|^m < \infty$ for some $m > k$ and $m \geqq 4$ where k is the dimension of the parameter space. Even though, for practical purposes, this condition is not a severe restriction since the errors are likely to be bounded, however, for theoretical reasons, it would be interesting to relax this restriction. This condition is of a technical nature as it is

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mainly used to derive some fluctuation inequalities for the least squares random field which are useful to obtain the tightness of the corresponding family of measures generated on a suitable function space.

Here we study the weak convergence of least squares random field under a different set of conditions and hence derive the asymtotic properties of the least squares estimator.

In this approach, we relax the restriction on moments of $\{\varepsilon_i\}$ to $\sup_i E|\varepsilon_i|^{2+\delta}<\infty$ for some $\delta>0$. This condition does not depend on the dimension of the parameter space. Our view point is similar to that of Inagaki and Ogata [4] who use the technique of Huber [2]. Prakasa Rao [8] used the method of Huber [2] in his study of the maximum likelihood estimation for Markov processes.

2. Preliminaries

Consider the nonlinear regression model

$$Y_i=g_i(\theta)+\varepsilon_i\,, \qquad i\geqq 1$$

where $\{g_i(\theta),\ i\geqq 1\}$ is a sequence of functions possibly nonlinear in $\theta \in \Theta\subset R^k$ and $\{\varepsilon_i,\ i\geqq 1\}$ are independent random variables with mean 0 and finite variances under the true model. Let

$$Q_n(\theta)=\sum_{i=1}^{n}(Y_i-g_i(\theta))^2\,.$$

An estimator $\hat{\theta}_n$ based on the observations $Y_1, \cdots, Y_n$ is called a *least squares estimator* if $\hat{\theta}_n$ is a measurable solution of the equation

$$Q_n(\hat{\theta}_n)=\inf_{\theta\in\bar{\Theta}} Q_n(\theta)\,.$$

If $g_i(\theta)$, $1\leqq i\leqq n$ are continuous in θ and $\bar{\Theta}$ is compact, then it can be shown that there exists a measurable solution $\hat{\theta}_n$ by using Lemma 3.3 in Schmetterer ([15], p. 307). Here after we assume that $\hat{\theta}_n$ is such an estimator.

We assume that the following regularity conditions are satisfied.

$$(\mathrm{A}_1) \qquad \sum_{i=1}^{\infty}[g_i(\theta_1)-g_i(\theta_2)]^2>0 \qquad \text{if } \theta_1\neq\theta_2 \text{ in } \Theta\,.$$

(A_2) $g_i(\theta)$ has partial derivatives with respect to θ for $i\geqq 1$ and, for any $\theta_0\in\Theta$, we denote the vector

$$\left(\frac{\partial g_i}{\partial\theta_1}, \cdots, \frac{\partial g_i}{\partial\theta_k}\right)$$

evaluated at θ_0 by $\nabla g_i(\theta_0)$. Assume that there exists a neighbourhood U_{θ_0} of θ_0 in Θ such that, for all $i \geqq 1$,

$$|g_i(\theta)-g_i(\theta_0)-(\theta-\theta_0)'\nabla g_i(\theta_0)| \leqq \alpha_i(\theta_0)\|\theta-\theta_0\| \qquad \text{for all } \theta \in V_{\theta_0}.$$

Furthermore

$$\overline{\lim}\, \frac{1}{n} \sum \alpha_i^2(\theta_0) < \infty .$$

(A_3) Observe that

$$\frac{1}{2}\nabla Q_n(\theta) = \sum_{i=1}^{n} [Y_i - g_i(\theta)]\nabla g_i(\theta)$$

$$= \sum_{i=1}^{n} \eta_i(Y_i, \theta) \qquad \text{(say)} .$$

Suppose that

$$\mathrm{E}\, \eta_i(Y_i, \theta) = \lambda_i(\theta)$$

exists where E denotes expectation under the true model. Suppose there exists $\theta_0 \in \theta$ such that $\lambda_i(\theta_0)=0$, $i=1, 2, \cdots$.

(A_4) Suppose there exists a neighbourhood U_{θ_0} of θ_0 in Θ such that the following conditions hold.

(i) $\bar{\lambda}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i(\theta) \rightarrow \lambda(\theta) \qquad \text{as } n \rightarrow \infty$

where $\lambda(\theta) \neq \lambda(\theta_0)$ if $\theta \neq \theta_0$, $\theta \in U_{\theta_0}$.

(ii) $\lambda_i(\theta)$ are continuously differentiable for $i \geqq 1$. Let

$\Lambda_i(\theta) = \dfrac{\partial \lambda_i(\theta)}{\partial \theta}$. Suppose

$$\bar{\Lambda}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \Lambda_i(\theta) \rightarrow \Lambda(\theta)$$

uniformly in $\theta \in U_{\theta_0}$.

(A_5) Let

$$u_i(y, \theta, d) = \sup_{\|\tau-\theta\| \leq d} |\eta_i(y, \tau) - \eta_i(y, d)| .$$

Suppose that for every compact $C \subset \Theta$, there are positive numbers d_0, H_1 and H_2 such that, for any $0<d<d_0$ and any $\theta \in C$,

$$\mathrm{E}\, u_i(Y_i, \theta, d) < H_1 d , \qquad i \geqq 1$$

and

$$\mathrm{E}\, u_i^2(Y_i, \theta; d) < H_2 d\,, \qquad i \geqq 1\,.$$

(A_6) $\{\varepsilon_i, i \geqq 1\}$ are independent random variables with $\mathrm{E}\,\varepsilon_i = 0$, $i \geqq 1$ and there exists $\delta > 0$ such that $\sup_i \mathrm{E}\,|\varepsilon_i|^{2+\delta} < \infty$. Let $\sigma_i^2 = \mathrm{E}\,\varepsilon_i^2$. Suppose $\inf_i \sigma_i^2 \geqq \sigma^2 > 0$.

(A_7) For any fixed ϕ_1 and ϕ_2

$$\text{(i)} \quad \sum_{i=1}^{n} \sigma_i^2 [g_i(\theta_0) - g_i(\theta_0 + \phi_1 n^{-1/2})][g_i(\theta_0) - g_i(\theta_0 + \phi_2 n^{-1/2})] = \phi_1' K(\theta_0) \phi_2 + o(1)$$

and

$$\text{(ii)} \quad \sum_{i=1}^{n} [g_i(\theta_0) - g_i(\theta_0 + \phi_1 n^{-1/2})]^2 = \phi_1' K^*(\theta_0) \phi_1 + o(1)$$

where $K(\theta_0)$ and $K^*(\theta_0)$ are positive definite matrices.

(A_8)
$$\sup_{i \geqq 1} |g_i(\theta_0) - g_i(\theta_0 + \phi n^{-1/2})| \leqq M \|\phi\| n^{-1/2}$$

for some $M < \infty$ uniformly in ϕ and n.

(A_9) Let

$$J_n(\phi) = Q_n(\theta_0 + \phi n^{-1/2}) - Q_n(\theta_0)\,. \tag{2.0}$$

Suppose that there exists $\eta > 0$

$$\lim_{M \to \infty} \overline{\lim_{n \to \infty}}\, \mathrm{P}\,[\inf_{\|\phi\| \geqq M} J_n(\phi) \geqq \eta] = 1\,. \tag{2.1}$$

We denote by $N_k(\mu, \Sigma)$ the k-dimensional multivariate normal distribution with mean vector μ and covariance matrix Σ. We will comment on the regularity conditions listed above later in Section 4 of this paper. Note that (A_7) implies the relations

$$\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \nabla g_i(\theta_0) \nabla g_i(\theta_0)' \to K(\theta_0) \qquad \text{as } n \to \infty \tag{2.2}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \nabla g_i(\theta_0) \nabla g_i(\theta_0)' \to K^*(\theta_0) \qquad \text{as } n \to \infty\,. \tag{2.3}$$

Observe that the condition (A_9) implies that $\hat{\theta}_n \in B(\theta_0, Mn^{-1/2})$ with probability approaching one as $n \to \infty$ and $M \to \infty$ where $B(\theta_0, M)$ is the closed sphere with centre at θ_0 and radius M in R^k. Let B_M denote the sphere with centre at 0 and radius M in R^k. We assume that M and n are sufficiently large so that $B(\theta_0, Mn^{-1/2}) \subset U_{\theta_0}$ given by (A_4).

3. Weak convergence

Clearly, if $\hat{\theta}_n$ is a LSE minimizing $Q_n(\theta)$, then $\hat{\theta}_n$ minimizes $Q_n(\theta)-Q_n(\theta_0)$ for any fixed $\theta_0 \in \Theta$. Let θ_0 be as given by (A_3). Define

$$J_n(\phi)=Q_n(\theta_0+\phi n^{-1/2})-Q_n(\theta_0)$$

as in (2.0). The following theorem can be proved by arguments analogous to those given in Prakasa Rao ([14], Theorem 3.1) using central limit theorem due to Eicker [1]. We omit the details.

THEOREM 3.1. *Under the conditions* (A_2), (A_6) *and* (A_7), *the finite dimensional distributions of the random field* $\{J_n(\phi), \|\phi\|\leqq\tau\}$ *converge weakly to the finite dimensional distributions of the random field* $\{J(\phi), \|\phi\|\leqq\tau\}$ *for any fixed* $\tau>0$ *where*

$$J(\phi)=\phi' K^*\phi+2\phi'\xi \, , \tag{3.0}$$

and ξ *is* $N_k(0, K)$. *Here* $K(\theta_0)\equiv K$, $K^*(\theta_0)=K^*$ *are as given by* (A_7).

Let

$$Z_n(\phi)=\sum_{i=1}^{n} \varepsilon_i[g_i(\theta_0)-g_i(\theta_0+n^{-1/2}\phi)]$$

and

$$T_n(\phi)=\sum_{i=1}^{n} [g_i(\theta_0)-g_i(\theta_0+n^{-1/2}\phi)]^2 \, .$$

In the rest of this section we prove some lemmas leading to the tightness and hence the weak convergence of the family of probability measures generated by the random fields $\{J_n(\phi), \phi \in B_M\}$ on a suitable function space to be defined later.

THEOREM 3.2. *Under the assumptions* (A_1)-(A_9) *stated above,*

$$\lim_{d\to 0} \overline{\lim_{n\to\infty}} \mathrm{P}\{ \sup_{\substack{\|\phi_1-\phi_2\|<d \\ \|\phi\|\leqq M}} |J_n(\phi_1)-J_n(\phi_2)|>\varepsilon\} =0 \tag{3.1}$$

for every $\varepsilon>0$.

Before we give a proof of Theorem 3.2, we shall prove two lemmas.

LEMMA 3.1. *For any* $M>0$,

$$\sup_{\|\phi\|\leqq M} \left| J_n(\phi)-n^{-1/2}\phi' \left\{\sum_{i=1}^{n} \eta_i(Y_i, \theta_0)\right\} -\frac{1}{2}\phi' \Lambda(\theta_0)\phi \right| \xrightarrow{p} 0 \qquad \text{as } n\to\infty \, . \tag{3.2}$$

PROOF. It is clear that

$$(3.3)\quad J_n(\phi)-n^{-1/2}\phi'\sum_{i=1}^{n}\eta_i(Y_i,\theta_0)-\frac{1}{2}\phi'\Lambda(\theta_0)\phi$$
$$=Q_n(\theta_0+\phi n^{-1/2})-Q_n(\theta_0)-n^{-1/2}\phi'\sum_{i=1}^{n}\eta_i(Y_i,\theta_0)-\frac{1}{2}\phi'\Lambda(\theta_0)\phi$$
$$=\int_0^1\left\{\frac{\phi'}{\sqrt{n}}\sum_{i=1}^{n}\eta_i\left(Y_i,\theta_0+t\frac{\phi}{\sqrt{n}}\right)-\frac{\phi'}{\sqrt{n}}\sum_{i=1}^{n}\eta_i(Y_i,\theta_0)-t\phi'\Lambda(\theta_0)\phi\right\}dt$$

and hence

$$(3.4)\quad \sup_{\|\phi\|\le M}\left|J_n(\phi)-n^{-1/2}\phi'\sum_{i=1}^{n}\eta_i(Y_i,\theta_0)-\frac{1}{2}\phi'\Lambda(\theta_0)\phi\right|$$
$$\le M\sup_{\|\phi\|\le M}\left\|n^{-1/2}\sum_{i=1}^{n}\eta_i\left(Y_i,\theta_0+\frac{\phi}{\sqrt{n}}\right)-n^{-1/2}\sum_{i=1}^{n}\eta_i(Y_i,\theta_0)-\Lambda(\theta_0)\phi\right\|.$$

Therefore Lemma 3.1 is proved provided

$$(3.5)\quad \sup_{\|\phi\|\le M}\left\|n^{-1/2}\sum_{i=1}^{n}\eta_i\left(Y_i,\theta_0+\frac{\phi}{\sqrt{n}}\right)-n^{-1/2}\sum_{i=1}^{n}\eta_i(Y_i,\theta_0)-\Lambda(\theta_0)\phi\right\|\xrightarrow{p}0$$

as $n\to\infty$. This in turn holds provided

$$(3.6)\quad \sup_{\|\phi\|\le M}\left\|n^{-1/2}\sum_{i=1}^{n}\left\{\eta_i\left(Y_i,\theta_0+\frac{\phi}{\sqrt{n}}\right)-\eta_i(Y_i,\theta_0)-\lambda_i\left(\theta_0+\frac{\phi}{\sqrt{n}}\right)\right\}\right\|\xrightarrow{p}0 \quad \text{as } n\to\infty$$

and

$$(3.7)\quad \sup_{\|\phi\|\le M}\left\|n^{-1/2}\sum_{i=1}^{n}\lambda_i\left(\theta_0+\frac{\phi}{\sqrt{n}}\right)-\Lambda(\theta_0)\phi\right\|\to 0$$

as $n\to\infty$. In view of (A_5), the relation (3.6) can be proved by arguments analogous to those given in Inagaki [3], p. 8. Observe that

$$(3.8)\quad \sup_{\|\phi\|\le M}\left\|n^{-1/2}\sum_{i=1}^{n}\lambda_i\left(\theta_0+\frac{\phi}{\sqrt{n}}\right)-\Lambda(\theta_0)\phi\right\|$$
$$\le\sup_{\|\phi\|\le M}\left\|\left\{n^{-1}\sum_{i=1}^{n}\Lambda_i(\theta_n')\right\}\phi-\Lambda(\theta_0)\phi\right\| \qquad \text{(By } (A_4))$$
$$\le M\sup_{\|\theta_n'-\theta_0\|\le Mn^{-1/2}}\|\bar{\Lambda}_n(\theta_n')-\Lambda(\theta_0)\|$$
$$\le M\{\sup_{\|\tau-\theta_0\|\le Mn^{-1/2}}\|\bar{\Lambda}_n(\tau)-\Lambda(\tau)\|+\sup_{\|\tau-\theta_0\|\le Mn^{-1/2}}\|\Lambda(\tau)-\Lambda(\theta_0)\|\}.$$

Assumption (A_4) (ii) implies that

$$\sup_{\|\tau-\theta_0\|\le Mn^{-1/2}}\|\bar{\Lambda}_n(\tau)-\Lambda(\tau)\|\to 0 \qquad \text{as } n\to\infty$$

and continuity of $\Lambda(\tau)$ in a neighbourhood of θ_0 implies that

$$\sup_{\|\tau-\theta_0\|\leq Mn^{-1/2}} \|\Lambda(\tau)-\Lambda(\theta_0)\| \to 0 \qquad \text{as } n\to\infty .$$

This completes the proof of Lemma 3.1.

LEMMA 3.2.

$$n^{-1/2}\sum_{i=1}^{n} \eta_i(Y_i, \theta_0) \xrightarrow{\mathcal{L}} N_k(0, K^*) \qquad as\ n\to\infty . \tag{3.9}$$

PROOF. This result follows from a central limit theorem due to Eicker [1] in view of (A_3), (A_6), (A_7) and (A_8).

Remark. In particular, it follows that

$$n^{-1/2}\sum_{i=1}^{n} \eta_i(Y_i, \theta_0) = O_p(1) . \tag{3.10}$$

PROOF OF THEOREM 3.2. We now estimate the probability

$$P\{\sup_{\substack{\|\phi_1-\phi_2\|<d\\ \|\phi\|\leq M}} |J_n(\phi_1)-J_n(\phi_2)|>\varepsilon\} . \tag{3.11}$$

Note that

$$\begin{aligned}
(3.12)\quad &\sup_{\substack{\|\phi_1-\phi_2\|\leq d\\ \|\phi_i\|\leq M}} |J_n(\phi_1)-J_n(\phi_2)| \\
&\leq 2\sup_{\|\phi\|\leq M}\left| J_n(\phi)-n^{-1/2}\phi'\sum_{i=1}^{n}\eta_i(Y_i, \theta_0)-\frac{1}{2}\phi'\Lambda(\theta_0)\phi\right| \\
&\quad +d\left\| n^{-1/2}\sum_{i=1}^{n}\eta_i(Y_i, \theta_0)\right\| + \frac{1}{2}\sup_{\substack{\|\phi_1-\phi_2\|\leq d\\ \|\phi_i\|\leq M}} |\phi_1'\Lambda(\theta_0)\phi_1-\phi_2'\Lambda(\theta_0)\phi_2| \\
&\leq 2\sup_{\|\phi\|\leq M}\left| J_n(\phi)-n^{-1/2}\phi'\sum_{i=1}^{n}\eta_i(Y_i, \theta_0)-\frac{1}{2}\phi'\Lambda(\theta_0)\phi\right| \\
&\quad +d\left\| n^{-1/2}\sum_{i=1}^{n}\eta_i(Y_i, \theta_0)\right\| + dM\gamma
\end{aligned}$$

where γ is the maximum eigenvalue of $\Lambda(\theta_0)$. Lemmas 3.1 and 3.2 and the remark (3.10) show that

$$\lim_{d\to 0}\overline{\lim_{n\to\infty}} P\{\sup_{\substack{\|\phi_1-\phi_2\|<d\\ \|\phi_i\|\leq M}} |J_n(\phi_1)-J_n(\phi_2)|>\varepsilon\} = 0 . \tag{3.13}$$

This completes the proof.

Let $C(B_M)$ be the space of real-valued continuous functions f on B_M endowed with the supremum norm. Clearly the random field $\{J_n(\phi),\ \phi\in B_M\}$ has sample paths in $C(B_M)$ with probability one. Define $J(\phi)$ by (3.0). The random field $\{J(\phi),\ \phi\in B_M\}$ has sample paths in $C(B_M)$ with probability one. Theorems 3.1 and 3.2 prove the following result from Prakasa Rao [9] or from the general theory of weak convergence of probability measures in metric spaces (cf. Parthasarathy [6]).

THEOREM 3.3. *Under the assumptions* (A_1) *to* (A_8), *the sequence of random fields* $\{J_n(\phi),\ \phi \in B_M\}$ *with sample paths in the space* $C(B_M)$ *converge in distribution to the random field* $\{J(\phi),\ \phi \in B_M\}$ *given by* (3.0).

In view of Theorem 3.3 and the remarks made at the end of Section 2 it can be shown that

$$n^{1/2}(\hat{\theta}_n-\theta_0) \xrightarrow{\mathcal{L}} N_k(0, K^{*-1}KK^{*-1}) \qquad \text{as } n \to \infty .$$

by arguments as given in Prakasa Rao [7] by using the theory of weak convergence for probability measures on metric spaces.

THEOREM 3.4. *Under the conditions* (A_1) *to* (A_9), *the least squares estimator is asymptotically normal i.e.,*

$$\sqrt{n}(\hat{\theta}_n-\theta_0) \xrightarrow{\mathcal{L}} N_k(0, K^{*-1}KK^{*-1}) \qquad \textit{as } n \to \infty .$$

4. Remarks

We now comment on the regularity conditions stated in Section 2. (A_1) deals with identifiability. (A_2) is a smoothness condition which is used in Prakasa Rao [13] and similar related works of other authors. (A_3) clearly holds if the true model belongs to the family. Conditions (A_4) (i) and (ii) appear in view of the non i.i.d. nature of the random variables Y_i, $i \geqq 1$. Condition (A_5) is similar to the condition assumed by Huber [2], Prakasa Rao [8] and Inagaki [3]. Condition (A_6) is an improvement over the condition that $\sup_i E|\varepsilon_i|^m < \infty$ for some $m > k$ and $m \geqq 4$ assumed in Prakasa Rao [14]. If ε_i are i.i.d. random variables then the result in Theorem 3.4 holds even if $\delta = 0$ i.e., if $E(\varepsilon_i^2) = \sigma^2$ is greater than zero and finite. Conditions (A_7) and (A_8) occur in Prakasa Rao [14] in the study of asymptotic properties of LSE. Relations (2.2) and (2.3) are assumed in Jennrich [5] or Wu [16]. The condition (A_9) can be checked in linear regression models with Gaussian errors. Similar condition is verified in a non-regular non-linear regression model in (2.11) of Prakasa Rao [12] and an analogous condition is proved in Prakasa Rao [10] in non-linear regression model when the errors are Gaussian in the scalar parameter case. We point out that the rate of convergence of LSE is also obtained in Prakasa Rao [13] under the stronger condition (C_3) in that paper. It is interesting to find whether (A_9) follows as a consequence of (A_1)-(A_8). The usual approach adopted to prove an analogue of (A_9) for likelihood ratio process in Inagaki and Ogata [4] does not give the required result in this problem. It can be shown that, given $d>0$, there exists $c_i>0$, $i=1, 2$ and $c>0$ such that

$$P\,[|J_n(\phi)|>c_1\|\phi\|^2]\leqq c\|\phi\|^{-2-2\delta} \qquad \text{if } \|\phi\|<dn^{1/2}\,,\ n\geqq 1$$

and

$$P\left[\sup_{l\leqq\|\phi\|\leqq l+1}|J_n(\phi)|>c_1\frac{l^2}{2}\right]\leqq\frac{c_2}{l^2} \qquad \text{provided } l+1\leqq\sqrt{n}\,d\,.$$

However these estimates do not give the bounds needed to prove (A_9). Finally, we point out that an innovation in our approach here is that the true model need not be a member of the parametric family with which we started.

INDIAN STATISTICAL INSTITUTE, NEW DELHI

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