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ψ-CORRECT DECISION FOR SELECTION AND ELIMINATION

EVE BOFINGER

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Summary

The selection of t out of k populations with parameters θ_i ($i=1, \cdots, k$) is said to result in an ψ-correct decision provided

$$\psi \text{ (minimum selected } \theta) > \text{maximum non-selected } \theta$$

where $\psi(\theta)$ ($>\theta$) is an increasing function. For the cases of location or scale parameters the minimum probability of ψ-correct decision over the entire parameter space is shown to be no less than the minimum probability of correct selection over a preference zone determined by $\psi(\theta)$. For other types of parameters this result is shown to be true under certain conditions linking the distribution function and the ψ function.

1. Introduction

Consider k independent random variables X_i ($i=1, \cdots, k$) from populations π_i with continuous distribution functions $F(\cdot; \theta_i)$ which are stochastically increasing in the parameters θ_i. We suppose that the parameter space is given by

$$\Omega = \{\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k) : \theta_i \in \boldsymbol{\Theta}\} \tag{1.1}$$

where $\boldsymbol{\Theta}$ is a subset of the real line.

The ordered θ_i are given by

$$\theta_{\rho(1)} \leqq \theta_{\rho(2)} \leqq \cdots \leqq \theta_{\rho(k-t)} < \theta_{\rho(k-t+1)} \leqq \cdots \leqq \theta_{\rho(k)} \tag{1.2}$$

where $\rho(\cdot)$ is an unknown parametric function, and the ordered X_i are given by

$$X_{R(1)} < X_{R(2)} < \cdots < X_{R(k)} \tag{1.3}$$

where $R(\cdot)$ is a random function.

Key words: Correct selection, correct decision, preference zone.

Bechhofer [2] considers the decision rule:

R: Select populations π_i for all $i \in G$ where

$$G=\{R(k),\ R(k-1),\ \cdots,\ R(k-t+1)\}. \tag{1.4}$$

Notice that R selects the populations with the t largest values of X_i and eliminates those with the $k-t$ smallest values. Since the distributions of the X_i are stochastically increasing in the θ_i, we expect, with this decision rule, to select populations with large θ_i values and to eliminate those with small θ_i.

Let the set of indices of populations with the t largest θ_i values be denoted by

$$\gamma=\{\rho(k),\ \rho(k-1),\ \cdots,\ \rho(k-t+1)\}. \tag{1.5}$$

Correct selection, CS, is defined as the event

$$\mathrm{CS}=\{G=\gamma\}. \tag{1.6}$$

In order to put a lower bound on the probability of correct selection, P (CS), Barr and Rizvi [1] consider a preference zone, Ω_ϕ, taken to be a non-empty subset of Ω and given by

$$\Omega_\phi=\{\boldsymbol{\theta}:\ \phi(\theta_j)\leqq\theta_i \quad \forall i \in \gamma,\ j \in \gamma^c\} \tag{1.7}$$

where c indicates the complementary set and $\phi(\cdot)$ is an increasing function with

$$\phi(\theta)>\theta \qquad \forall \theta \in \boldsymbol{\Theta}.$$

It is usual to take $\phi(\theta)=\theta+\Delta_L$ when the θ_i are location parameters, with Δ_L non-negative, and to take $\phi(\theta)=\theta/\Delta_s$ when the θ_i are scale parameters, with $\Delta_s \in (0, 1)$.

Feigin and Weissman [5] define ϕ-correct selection, $\phi-$CS, as

$$\phi-\mathrm{CS}=\{\min_{i \in G} \phi(\theta_i)>\theta_{\rho(k-t+1)}\} \tag{1.8}$$

and show that

$$\inf_{\boldsymbol{\theta} \in \Omega} \mathrm{P}\,\{\phi-\mathrm{CS}\}=\inf_{\boldsymbol{\theta} \in \Omega_\phi} \mathrm{P}\,(\mathrm{CS}). \tag{1.9}$$

Let us define ϕ-correct decision, $\phi-$CD, as

$$\phi-\mathrm{CD}=\{\min_{i \in G} \phi(\theta_i)>\max_{j \in G^c} \theta_j\}. \tag{1.10}$$

This is considered also by Feigin and Weissman who refer to it as F-CS since it is a particular case of correct selection as studied by Fabian [4]. We prefer the term $\phi-$CD since there is a joint statement about the eliminated as well as the selected populations.

Feigin and Weissman remark that it is an open question whether or not

$$\inf_{\theta \in \Omega} \mathrm{P}\{\psi-\mathrm{CD}\} = \inf_{\theta \in \Omega_\psi} \mathrm{P}(\mathrm{CS}) . \tag{1.11}$$

In Section 2 it will be shown that (1.11) holds under certain conditions on the distribution functions and the ψ function. This generalizes the work on location and scale parameters in Bofinger [3].

2. ψ-correct decision

Let

$$T = \{(i, j) : i \in \gamma, \; j \in \gamma^c \text{ and } \psi(\theta_j) \leq \theta_i\}$$
$$I = \{i : (i, j) \in T\}$$
$$J = \{j : (i, j) \in T\}$$

and

$$D(\boldsymbol{\theta}) = \{X_j < X_i \quad \forall (i, j) \in T | \boldsymbol{\theta}\} .$$

Before proving the main result we indicate the connection between $D(\boldsymbol{\theta})$ and $\psi-\mathrm{CD}$ with the following lemma:

LEMMA 2.1.

$$D(\boldsymbol{\theta}) \Rightarrow \psi-\mathrm{CD}$$

where the symbol "$\Rightarrow$" is to be interpreted as "implies".

PROOF.

$$\begin{aligned} \overline{\psi-\mathrm{CD}} &\Rightarrow \{i \in G^c \text{ and } j \in G \text{ for some } (i, j) \in T\} \\ &\Rightarrow \{X_j > X_i \text{ for some } (i, j) \in T\} \\ &\Rightarrow \overline{D(\boldsymbol{\theta})} . \end{aligned}$$

Hence the result follows.

THEOREM 2.1.

$$\inf_{\theta \in \Omega} \mathrm{P}\{\psi-\mathrm{CD}\} = \inf_{\theta \in \Omega_\psi} \mathrm{P}(\mathrm{CS})$$

provided that there exists a $\theta^* \in \Theta$: *either*

$$\text{(a)} \quad \forall j \in J , \qquad F(X_j; \psi(\theta_j)) \leq_{st} F(Y; \psi(\theta^*)) \tag{2.1}$$

or

$$\text{(2.2)} \qquad \text{(b)} \quad \forall i \in I, \qquad F(X_i; \psi^{-1}(\theta_i)) \geqq_{st} F(Y; \psi^{-1}(\theta^*))$$

where Y has distribution function $F(\cdot\,; \theta^*)$, $\psi^{-1}(\cdot)$ is the function inverse to $\psi(\cdot)$ and $\leqq_{st}$ or $\geqq_{st}$ indicates stochastic ordering.

PROOF. Since $F(x; \theta_i)$ is increasing in x

$$\begin{aligned} D(\boldsymbol{\theta}) &= \{F(X_j; \theta_i) < F(X_i; \theta_i) \quad \forall (i, j) \in T\} \\ &\Leftarrow \{F(X_j; \psi(\theta_j)) < F(X_i; \theta_i) \quad \forall (i, j) \in T\} \end{aligned} \tag{2.3}$$

since $F(x; \theta)$ is decreasing in θ and

$$\theta_i \geqq \psi(\theta_j) \qquad \forall (i, j) \in T .$$

If condition (a) holds we may replace the LHS of the inequality in (2.3) by

$$F(Y_j; \psi(\theta^*))$$

where, for all $j \in J$, the Y_j are i.i.d. with distribution function $F(\cdot\,; \theta^*)$.

Also, since the $F(X_i; \theta_i)$ are uniformly distributed on $(0, 1)$ we may replace the RHS of the inequality by

$$F(Z_i; \psi(\theta^*))$$

where, for all $i \in I$, the Z_i are i.i.d. with distribution function $F(\cdot\,; \psi(\theta^*))$.

Hence

$$\begin{aligned} D(\boldsymbol{\theta}) &\Leftarrow \{F(Y_j; \psi(\theta^*)) < F(Z_i; \psi(\theta^*)) \quad \forall (i, j) \in T\} \\ &\Leftarrow \{Y_j < Z_i \quad \forall i \in \gamma,\ j \in \gamma^c\} . \end{aligned}$$

Now

$$\inf \mathrm{P}\{Y_j < Z_i \quad \forall i \in \gamma,\ j \in \gamma^c\} \geqq \inf_{\boldsymbol{\theta} \in \Omega_\psi} \mathrm{P}\,(\mathrm{CS})$$

which, using Lemma (2.1), shows that

$$\inf_{\boldsymbol{\theta} \in \Omega} \mathrm{P}\,(\psi - \mathrm{CD}) \geqq \inf_{\boldsymbol{\theta} \in \Omega_\psi} \mathrm{P}\,(\mathrm{CS}) .$$

However,

$$\inf_{\boldsymbol{\theta} \in \Omega_\psi} \mathrm{P}\,(\psi - \mathrm{CD}) = \inf_{\boldsymbol{\theta} \in \Omega_\psi} \mathrm{P}\,(\mathrm{CS})$$

which completes the theorem when condition (a) holds.

When condition (b) holds a similar argument is used, where the expression (2.3) is replaced by

$$\{F(X_j; \theta_j) < F(X_i; \psi^{-1}(\theta_i)) \quad \forall (i, j) \in T\} .$$

Remarks. Condition (a) is satisfied if there exists a $\theta^* \in \Theta$ such that, for X and Y random variables with distribution functions $F(\cdot\,;\theta)$ and $F(\cdot\,;\theta^*)$ respectively,

$$(2.4)\qquad \text{(a}'\text{)}\quad F(X;\psi(\theta))\leqq_{st} F(Y;\psi(\theta^*)) \qquad \forall\,\theta<\psi^{-1}(U)$$

where $U=\max_{\theta}\theta$.

Similarly, condition (b) is satisfied if there exists a $\theta^*\in\Theta$ such that

$$\text{(b}'\text{)}\quad F(X;\psi^{-1}(\theta))\geqq_{st} F(Y;\psi^{-1}(\theta^*)) \qquad \forall\,\theta>\psi(L)$$

where $L=\min_{\theta}\theta$.

In Section 3 these conditions will be considered for location and scale parameters.

3. Location and scale parameters

It will be shown that, for location and scale parameters, the conditions of Theorem (2.1) are satisfied for any choice of the function $\psi(\cdot)$.

Location. Suppose

$$F(x;\theta)=H(x-\theta)\,.$$

Then condition (a) of Theorem 2.1 may be written:

$$\forall\, j\in J \qquad X_j-\psi(\theta_j)\leqq_{st} Y-\psi(\theta^*)\,.$$

Since $X_j-\theta_j$ and $Y-\theta^*$ have the same distribution we see that condition (a) becomes

$$(3.1)\qquad \forall\, j\in J \qquad \psi(\theta_j)-\theta_j\geqq\psi(\theta^*)-\theta^*$$

and a possible value for θ^* is that θ_j which gives the minimum value of $\psi(\theta_j)-\theta_j$. Hence, although we may not be able to specify θ^* (since the θ_j are unknown), we know that an appropriate θ^* exists and therefore the theorem is true for any choice of the function $\psi(\cdot)$.

In the particular case where

$$\psi(\theta)=\theta+\Delta_L$$

we see that any value of θ^* satisfies (3.1).

Scale. Suppose

$$F(x;\theta)=H(x/\theta)\,.$$

Then condition (a) of Theorem 2.1 may be written

$$\forall j \in J \qquad X_j/\psi(\theta_j) \leqq_{st} Y/\psi(\theta^*) .$$

Since X_j/θ_j and Y/θ^* have the same distribution the condition becomes

$$\forall j \in J \qquad \psi(\theta_j)/\theta_j \geqq \psi(\theta^*)/\theta^* \tag{3.2}$$

and a possible value for θ^* is that θ_j which gives the minimum value of $\psi(\theta_j)/\theta_j$. Hence condition (a) is satisfied for any choice of $\psi(\cdot)$.

In the particular case where

$$\psi(\theta) = \theta/\Delta,$$

any value of θ^* satisfies (3.2).

THE UNIVERSITY OF NEW ENGLAND

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