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ON THE INADMISSIBILITY OF PRELIMINARY-TEST ESTIMATORS WHEN THE LOSS INVOLVES A COMPLEXITY COST

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Summary

Estimation-preceded-by-testing is studied in the context of estimating the mean vector of a multivariate normal distribution under squared error loss together with a complexity cost. It is shown that although the preliminary test estimator is admissible for the univariate problem (cf Meeden and Arnold (1979), *J. Amer. Statist. Assoc.*, 74, 872–874), for dimension $p \geqq 3$, the estimator is inadmissible. A new preliminary test estimator is obtained, which depends on the cost for each component and dominates the usual preliminary test estimator.

1. Introduction

A preliminary test estimation procedure is described as follows. After a preliminary test of a certain null hypothesis, estimation is made under the alternative hypothesis if the null hypothesis is rejected, and is made under the null hypothesis otherwise.

Let $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ be iid $N_p(\boldsymbol{\theta}, \sigma^2 \boldsymbol{I}_p)$. For estimating $\boldsymbol{\theta}=(\theta_1, \cdots, \theta_p)$ by $\boldsymbol{a}=(a_1, \cdots, a_p)$, suppose the loss incurred is given by

$$L(\boldsymbol{\theta}, \boldsymbol{a})=\sum_{i=1}^{p} (a_i-\theta_i)^2+\sum_{i=1}^{p} c_i^2 I_{[a_i \neq 0]}\ , \tag{1.1}$$

where $c_1, \cdots, c_p$ are known positive real numbers, and I is the usual indicator function. The constant c_i is described as the "complexity cost" associated with the i-th component (cf Faden and Rausser [4], Meeden and Arnold [5]). First consider the case when σ^2 is known. The generalized Bayes estimator of $\boldsymbol{\theta}$ under the improper prior with pdf $g(\boldsymbol{\theta})=1$ for $\boldsymbol{\theta} \in R^p$, the p-dimensional Euclidean space, is then ob-

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tained by minimizing $\sum_{i=1}^{p} h_i(a_i)$ with respect to $a_1, \cdots, a_p$, where

$$(1.2) \qquad h_i(a_i) = \int_{-\infty}^{\infty} (\theta_i - a_i)^2 (2\pi\sigma^2)^{-1/2} \exp[-(2\sigma^2)^{-1}(\theta_i - \bar{x}_{ni})^2] d\theta_i + c_i^2 I_{[a_i \neq 0]} .$$

In the above $\bar{x}_{ni}$ is the i-th component of $\bar{\boldsymbol{x}}_n = n^{-1} \sum_{i=1}^{n} \boldsymbol{x}_i$. Noting that $h_i(0) = \bar{x}_{ni}^2 + \sigma^2/n$ and $h_i(a_i) = (a_i - \bar{x}_{ni})^2 + \sigma^2/n + c_i^2$ if $a_i \neq 0$, it follows that $h_i(a_i)$ is minimized at 0 if $|\bar{x}_{ni}| \leqq c_i$ and at $\bar{x}_{ni}$ if $|\bar{x}_{ni}| > c_i$. Thus, the generalized Bayes estimator of $\boldsymbol{\theta}$ is given by $\boldsymbol{\delta}(\bar{\boldsymbol{X}}_n) = (\delta_1(\bar{X}_{ni}), \cdots, \delta_p(\bar{X}_{np}))$, where

$$(1.3) \qquad \delta_i(\bar{X}_{ni}) = \bar{X}_{ni} I_{[|\bar{x}_{ni}| > c_i]} , \qquad i = 1, \cdots, p .$$

Such an estimator is easily identified as a preliminary test estimator. For example, when the preliminary test accepts $H_0: \theta_i = 0$, we estimate θ_i to be 0, otherwise we estimate θ_i by $\bar{x}_{ni}$. If the test is performed at the $100\alpha_i\%$ significance level, then we choose $c_i = \sigma n^{-1/2} z_{\alpha_i/2}$, where z_α denotes the upper $100\alpha\%$ point of the $N(0, 1)$ distribution.

For $p=1$, Meeden and Arnold [5] proved the admissibility of the preliminary test estimator given in (1.3) under a loss more general than (1.1) in that the squared error component $(a-\theta)^2$ is replaced by a more general $W(|a-\theta|)$, where $W(u)$ is nondecreasing in u with $W(0) = 0$. Indeed for $p=1, 2$, the admissibility of the preliminary test estimator given in (1.3) under the loss (1.1) can now be proved more easily by a direct appeal to Theorem 3.1 and Corollary 4.1 of Brown and Hwang [3] noting that the estimator is generalized Bayes with respect to the uniform prior on R^p.

The above results are of interest because it is the introduction of the complexity cost in addition to the squared error loss that makes the preliminary test estimator admissible. Under the regular squared error loss which is smooth, the preliminary test estimator, due to its lack of smoothness, cannot be generalized Bayes with respect to any smooth prior, and as such, is inadmissible for all p. The introduction of the complexity cost makes the loss non-smooth, and makes the preliminary test estimator generalized Bayes with respect to some prior, and hence, potentially admissible.

Note that for unknown σ^2, the preliminary test estimator given in (1.3) should be modified as $\tilde{\boldsymbol{\delta}}(\bar{\boldsymbol{X}}_n, S_n^2) = (\tilde{\delta}_1(\bar{\boldsymbol{X}}_n, S_n^2), \cdots, \tilde{\delta}_p(\bar{\boldsymbol{X}}_n, S_n^2))$ with

$$(1.4) \qquad \tilde{\delta}_i(\bar{X}_n, S_n^2) = \bar{X}_{ni} I_{[|\bar{x}_{ni}|/S_n > c_i]} , \qquad i = 1, \cdots, p ,$$

where $S_n^2 = ((n-1)p+2)^{-1} \sum_{i=1}^{n} \|\boldsymbol{X}_i - \bar{\boldsymbol{X}}_n\|^2$ is the best scale invariant estimator of σ^2.

It is thus of some interest to know whether the Stein effect persists in this case, i.e. whether or not the preliminary test estimators given in (1.3) or (1.4) are admissible in three or higher dimensions. In Section 2 of this paper, we produce explicit estimators dominating the preliminary test estimators given in (1.3) and (1.4). For known σ^2, as an important special case of the class of estimators to be proposed in Section 2, we get the James-Stein type estimator $\boldsymbol{\delta}^0(\bar{\boldsymbol{X}}_n)=(\delta_1^0(\bar{\boldsymbol{X}}_n), \cdots, \delta_p^0(\bar{\boldsymbol{X}}_n))$, where

$$\delta_i^0(\bar{\boldsymbol{X}}_n)=\left(1-\frac{\sigma^2(N(\bar{\boldsymbol{X}}_n)-2)^+}{n\|\bar{\boldsymbol{X}}_n\|^2}\right)\bar{X}_{ni}I_{[|\bar{X}_{ni}|>c_i]}\,, \qquad i=1,\cdots,p\,, \tag{1.5}$$

$N(\bar{\boldsymbol{X}}_n)=\#\{i:\ |\bar{X}_{ni}|>c_i\}$, and $a^+=\max(a, 0)$. For unknown σ^2, we get the estimator $\tilde{\boldsymbol{\delta}}^0(\bar{\boldsymbol{X}}_n, S_n^2)$ with its i-th component given by

$$\tilde{\delta}_i^0(\bar{\boldsymbol{X}}_n, S_n^2)=\left(1-\frac{S_n^2(N(\bar{\boldsymbol{X}}_n)-2)^+}{n\|\bar{\boldsymbol{X}}_n\|^2}\right)\bar{X}_{ni}I_{[|\bar{X}_{ni}|/S_n>c_i]}\,, \qquad i=1,\cdots,p\,. \tag{1.6}$$

Estimators dominating the preliminary test estimators under squared error loss have been obtained by Sclove et al. [7] (see also Bock et al. [2] in the more general regression model). Specialized to our situation, such estimators are given as follows. For known σ^2, let

$$\tilde{\delta}_i^1(\bar{\boldsymbol{X}}_n)=\left(1-\frac{(p-2)\sigma^2}{n\|\bar{\boldsymbol{X}}_n\|^2}\right)^+\bar{X}_{ni}I_{[\|\bar{X}_n\|>c]}\,, \qquad i=1,\cdots,p\,. \tag{1.7}$$

In the above, the choice of c depends on the level of significance of the preliminary chisquare test for $H_0:\ \boldsymbol{\theta}=\boldsymbol{0}$ against $H_1:\ \boldsymbol{\theta}\neq\boldsymbol{0}$. Then under the squared error loss, for $p\geqq 3$, the estimator $\tilde{\boldsymbol{\delta}}^1(\bar{\boldsymbol{X}}_n)=(\tilde{\delta}_1^1(\bar{\boldsymbol{X}}_n), \cdots, \tilde{\delta}_p^1(\bar{\boldsymbol{X}}_n))$ dominates the preliminary test estimator $\boldsymbol{\delta}^1(\bar{\boldsymbol{X}}_n)=(\delta_1^1(\bar{\boldsymbol{X}}_n), \cdots, \delta_p^1(\bar{\boldsymbol{X}}_n))$ with

$$\delta_i^1(\bar{\boldsymbol{X}}_n)=\bar{X}_{ni}I_{[\|\bar{X}_n\|>c]}\,, \qquad i=1,\cdots,p\,. \tag{1.8}$$

For unknown σ^2, the estimators $\boldsymbol{\delta}^1$ and $\tilde{\boldsymbol{\delta}}^1$ are replaced respectively by $\boldsymbol{\delta}^2$ and $\tilde{\boldsymbol{\delta}}^2$ with

$$\delta_i^2(\bar{\boldsymbol{X}}_n, S_n^2)=\bar{X}_{ni}I_{[\|\bar{X}_n\|/S_n>c_0]}\,, \qquad i=1,\cdots,p\,; \tag{1.9}$$

$$\tilde{\delta}_i^2(\bar{\boldsymbol{X}}_n, S_n^2)=\left(1-\frac{(p-2)S_n^2}{n\|\bar{\boldsymbol{X}}_n\|^2}\right)^+\bar{X}_{ni}I_{[\|\bar{X}_n\|/S_n>c_0]}\,, \qquad i=1,\cdots,p\,. \tag{1.10}$$

The choice of c_0 depends on the level of significance of the preliminary F-test for testing $H_0:\ \boldsymbol{\theta}=\boldsymbol{0}$ against $H:\ \boldsymbol{\theta}\neq\boldsymbol{0}$.

With the introduction of the complexity cost as given (1.1), the estimators given in (1.7) and (1.8) (for known σ^2) and (1.9) and (1.10) (for unknown σ^2) do not seem appropriate. Noting that

$$\|\bar{X}_n\|^2 = \sup_{a:\, \|a\|=1} (a'\bar{X}_n)^2 ,$$

according to (1.7) and (1.8), we do not estimate θ_i to be zero even if $|\bar{X}_{ni}|^2$ is sufficiently small, but $\|\bar{X}_n\|^2$ is significantly large. This is because in order for $\|\bar{X}_n\|^2$ to be significantly small, all $(a'\bar{X}_n)^2$ with $\|a\|=1$ should be significantly small, and not just $\bar{X}_{ni}^2$. This can lead to the complexity c_i^2 in addition to the squared error loss for estimating the i-th coordinate, when indeed this should not be the case.

Recently Nagata [6] considered admissibility and inadmissibility of preliminary test estimators using Akaike's Information Criterion. He showed that the preliminary test estimator was admissible for $p=1$ and inadmissible for $p \geqq 3$ under a loss function based on the Kullback-Leibler information measure. Our loss function is different from the one given in Nagata as is our incorporation of the complexity cost in the loss.

2. Inadmissibility results

In this section, we first consider the case when σ^2 is known. We propose a class of estimators including the ones given in (1.5) as special case, which dominate $\boldsymbol{\delta}(\bar{X}_n)$ given in (1.3). The level of generality of the proposed estimators is comparable to that of Baranchik [1] and Strawderman [9]. Like most other recent results of similar nature, our proof involves the integration by parts technique of Stein [8].

The first main result of this section is given as follows.

THEOREM 2.1. *Let $M_n = \|\bar{X}_n\|^2$, and suppose τ is a real-valued function satisfying*

(i) $0 < \tau(m) < 2$

(ii) *$\tau(m)$ is differentiable and increasing in m. Then the estimator $\boldsymbol{\delta}_i^B(\bar{X}_n) = (\delta_i^B(\bar{X}_n), \cdots, \delta_p^B(\bar{X}_n))$ with*

$$\delta_i^B(\bar{X}_n) = \left(1 - \frac{\sigma^2(N(\bar{X}_n)-2)^+\tau(M_n)}{nM_n}\right)\bar{X}_{ni} I_{[|\bar{X}_{ni}|>c_i]} , \tag{2.1}$$

$i=1, \cdots, p$, dominates the estimator $\boldsymbol{\delta}$ given in (1.3). *A lower bound of the risk improvement is given by*

$$(\sigma^4/n^2)\, \mathrm{E}\, [\{(N(\bar{X}_n)-2)^+\}^2\tau(M_n)(2-\tau(M_n))/M_n] . \tag{2.2}$$

PROOF. First write the risk difference

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^B) - R(\boldsymbol{\theta}, \boldsymbol{\delta}) \tag{2.3}$$

$$= -2\sigma^2 n^{-1} \sum_{i=1}^{p} \mathrm{E}\left[\frac{(N(\bar{X}_n)-2)^+\tau(M_n)}{M_n} \bar{X}_{ni}(\bar{X}_{ni}-\theta_i) I_{[|\bar{X}_{ni}|>c_i]}\right]$$

$$+\mathrm{E}\left[\frac{\sigma^4}{n^2} \frac{\{(N(\bar{X}_n)-2)^+\}^2\tau^2(M_n)}{M_n^2} \sum_{i=1}^{p} \bar{X}_{ni}^2 I_{[|\bar{X}_{ni}|>c_i]}\right].$$

Note that second term in the rhs of (2.3)

$$\leqq \frac{\sigma^4}{n^2} \mathrm{E}\,[\{(N(\bar{X}_n)-2)^+\}^2\tau^2(M_n)/M_n]\,. \tag{2.4}$$

Next

$$\mathrm{E}\left[\frac{(N(\bar{X}_n)-2)^+\tau(M_n)}{M_n} \bar{X}_{ni}(\bar{X}_{ni}-\theta_i) I_{[|\bar{X}_{ni}|>c_i]}\right. \tag{2.5}$$

$$=\mathrm{E}\mathrm{E}\left[\frac{(N(\bar{X}_n)-2)^+\tau(M_n)}{M_n} \bar{X}_{ni}(\bar{X}_{ni}-\theta_i) I_{[|\bar{X}_{ni}|>c_i]} \right|$$

$$\left. \bar{X}_{n1}, \cdots, \bar{X}_{n\overline{i-1}}, \bar{X}_{n\overline{i+1}}, \cdots, \bar{X}_{np}\right],$$

Now using integration by parts,

$$\int_{c_i}^{\infty} \frac{(N(\bar{x}_n)-2)^+\tau(m_n)}{m_n} \bar{x}_{ni}(\bar{x}_{ni}-\theta_i) \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x}_{ni}-\theta_i)^2\right) d\bar{x}_{ni} \tag{2.6}$$

$$=\left(N\left(1+\sum_{\substack{j=1\\ j\neq i}}^{p} I_{[|\bar{x}_{nj}|>c_j]}\right)-2\right)^+ \int_{c_i}^{\infty} \frac{\tau(m_n)}{m_n} \bar{x}_{ni}(\bar{x}_{ni}-\theta_i) \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}}$$

$$\cdot \exp\left(-\frac{n}{2\sigma^2}(\bar{x}_{ni}-\theta_i)^2\right) d\bar{x}_{ni}$$

$$=\left(N\left(1+\sum_{\substack{j=1\\ j\neq i}} I_{[|\bar{x}_{nj}|>c_j]}\right)-2\right)^+$$

$$\cdot \frac{\sigma^2}{n} \frac{\tau\left(c_i^2+\sum_{\substack{j=1\\ j\neq i}}^{p} \bar{x}_{nj}^2\right)}{c_i^2+\sum_{\substack{j=1\\ j\neq i}}^{p} \bar{x}_{nj}^2} c_i \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n}{2\sigma^2}(c_i-\theta_i)^2\right)$$

$$+\frac{\sigma^2}{n} \int_{c_i}^{\infty} \frac{\partial}{\partial \bar{x}_{ni}}\left(\frac{\tau(m_n)}{m_n} \bar{x}_{ni}\right) \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x}_{ni}-\theta_i)^2\right) d\bar{x}_{ni}$$

$$\geqq (\sigma^2/n) \int_{c_j}^{\infty} (N(\bar{x}_n)-2)^+ \frac{\partial}{\partial \bar{x}_{ni}}\left(\frac{\tau(m_n)}{m_n} \bar{x}_{ni}\right) \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}}$$

$$\cdot \exp\left(-\frac{n}{2\sigma^2}(\bar{x}_{ni}-\theta_i)^2\right) d\bar{x}_{ni}\,.$$

A similar lower bound exists for

$$\int_{-\infty}^{-c_i} (N(\bar{\boldsymbol{x}}_n)-2)^+(\tau(m_n)/m_n)\bar{x}_{ni}(\bar{x}_{ni}-\theta_i)\frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{n}{2\sigma^2}(\bar{x}_{ni}-\theta_i)^2\right)d\bar{x}_{ni}$$

and one gets

$$\text{(2.7)}\qquad \mathrm{E}\left[\frac{(N(\bar{\boldsymbol{X}}_n)-2)^+\tau(M_n)}{M_n}\bar{X}_{ni}(\bar{X}_{ni}-\theta_i)I_{[|X_{ni}|\geqq c_i]}\right]$$
$$\geqq(\sigma^2/n)\,\mathrm{E}\left[(N(\bar{\boldsymbol{X}}_n)-2)^+\left\{\frac{\partial}{\partial\bar{X}_{ni}}\left(\frac{\tau(M_n)}{M_n}\bar{X}_{ni}\right)\right\}I_{[|\bar{X}_{ni}|>c_i]}\right].$$

But, using (ii),

$$\text{(2.8)}\qquad \frac{\partial}{\partial\bar{X}_{ni}}\left(\frac{\tau(M_n)}{M_n}\bar{X}_{ni}\right)=\frac{\tau(M_n)}{M_n}+\frac{\tau'(M_n)}{M_n}(2\bar{X}_{ni}^2)-\frac{\tau(M_n)}{M_n^2}(2\bar{X}_{ni}^2)$$
$$\geqq\frac{\tau(M_n)}{M_n}-\frac{\tau(M_n)}{M_n^2}(2\bar{X}_{ni}^2).$$

From (2.3), (2.4), (2.7) and (2.8), one gets after some algebraic simplifications that

$$\text{(2.9)}\qquad R(\boldsymbol{\theta},\boldsymbol{\delta}^B)-R(\boldsymbol{\theta},\boldsymbol{\delta})\leqq-(\sigma^4/n^2)\,\mathrm{E}\,[\{N(\bar{\boldsymbol{X}}_n-2)^+\}^2$$
$$\cdot\,(\tau(M_n)/M_n)(2-\tau(M_n))]\leqq 0.$$

This completes the proof of the first part of the theorem. The assertion (2.2) follows by examining the penultimate line of (2.9).

Remark 2.1. For known σ^2, if instead of testing $H_0:\boldsymbol{\theta}=\mathbf{0}$ against $H_1:\boldsymbol{\theta}\neq\mathbf{0}$, one tests $H_0:\boldsymbol{\theta}=\boldsymbol{\lambda}$ against $H_1:\boldsymbol{\theta}\neq\boldsymbol{\lambda}$ where $\boldsymbol{\lambda}$ is some specified constant, then the usual preliminary test estimator should be modified as $\boldsymbol{\delta}^{\lambda}(\bar{\boldsymbol{X}}_n)=(\boldsymbol{\delta}_1^{\lambda}(\bar{\boldsymbol{X}}_n),\cdots,\boldsymbol{\delta}_p^{\lambda}(\bar{\boldsymbol{X}}_n))$, with

$$\delta_i^{\lambda}(\bar{\boldsymbol{X}}_n)=\lambda_i+(\bar{X}_{ni}-\lambda_i)I_{[|\bar{X}_{ni}-\lambda_i|>c_i]},\qquad i=1,\cdots,p.$$

Now a class of estimators (similar to the one given in (2.1)) dominating $\boldsymbol{\delta}^{\lambda}$ is given by $\boldsymbol{\delta}_0^{\lambda}(\bar{\boldsymbol{X}}_n)$ with

$$\delta_{0i}^{\lambda}(\bar{\boldsymbol{X}}_n)=\lambda_i+\left(1-\frac{\sigma^2(N_0^{\lambda}(\bar{\boldsymbol{X}}_n)-2)^+\tau(M_n^{\lambda})}{nM_n^{\lambda}}\right)(\bar{X}_{ni}-\lambda_i)I_{[|\bar{X}_{ni}-\lambda_i|>c_i]},$$

$i=1,\cdots,p$, where $N_0^{\lambda}(\bar{\boldsymbol{X}}_n)=\#\{i:\ |\bar{x}_{ni}-\lambda_i|>c_i\}$, and $M_n^{\lambda}=\|\bar{\boldsymbol{X}}_n-\boldsymbol{\lambda}\|^2$.

Next we examine the case when σ^2 is unknown. In this case σ^2 is replaced by S_n^2 in the improved estimator, and also M_n in Theorem 2.1 needs to be changed. More precisely, we have the following theorem.

THEOREM 2.2. *Let* $F_n=\|\bar{\boldsymbol{X}}_n\|^2/S_n^2$. *Let* τ *be a real-valued function satisfying* (i) *and* (ii) *of Theorem* 2.1. *Then the estimator*

$$\tilde{\boldsymbol{\delta}}^B(\bar{\boldsymbol{X}}_n, S_n^2) = (\tilde{\delta}_1^B(\bar{\boldsymbol{X}}_n, S_n^2), \cdots, \tilde{\delta}_p^B(\bar{\boldsymbol{X}}_n, S_n^2))$$

with

$$\tilde{\delta}_i^B(\bar{\boldsymbol{X}}_n, S_n^2) = \left(1 - \frac{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\tau(F_n)}{nF_n}\right)\bar{X}_{ni} I_{[|\bar{X}_{ni}|/S_n > c_i]}, \tag{2.10}$$

$i=1,\cdots,p$, *when* $N(\bar{\boldsymbol{X}}_n, S_n^2) = \#\{i: |\bar{x}_{ni}|/S_n > c_i\}$ *dominates the estimator* $\tilde{\boldsymbol{\delta}}$ *given in* (1.4). *Also, a lower bound of the risk improvement is given by*

$$(\sigma^2/n^2)\,\mathrm{E}\,[\{(N(\bar{\boldsymbol{X}}_n)-2)^+\}^2\tau(F_n)(2-\tau(F_n))/F_n]\,. \tag{2.11}$$

PROOF. First write the risk difference

$$\begin{aligned} &R(\boldsymbol{\theta}, \tilde{\boldsymbol{\delta}}^B) - R(\boldsymbol{\theta}, \tilde{\boldsymbol{\delta}}) \\ &= -2n^{-1}\sum_{i=1}^{p}\mathrm{E}\left[\frac{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\tau(F_n)}{F_n}\bar{X}_{ni}(\bar{X}_{ni}-\theta_i)I_{[|\bar{X}_{ni}|/S_n>c_i]}\right] \\ &\quad + n^{-2}\,\mathrm{E}\left[\frac{\{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\}^2\tau^2(F_n)}{F_n^2}\sum_{i=1}^{p}\bar{X}_{ni}^2 I_{[|\bar{X}_{ni}|/S_n>c_i]}\right]. \end{aligned} \tag{2.12}$$

Again, calculations analogous to (2.5)–(2.7) give

$$\begin{aligned} &\sum_{i=1}^{p}\mathrm{E}\left[\frac{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\tau(F_n)}{F_n}\bar{X}_{ni}(\bar{X}_{ni}-\theta_i)I_{[|\bar{X}_{ni}|/S_n>c_i]}\right] \\ &\quad \geqq (\sigma^2/n)\,\mathrm{E}\,[\{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\}^2\tau(F_n)/F_n]\,. \end{aligned} \tag{2.13}$$

Again,

$$\begin{aligned} &\mathrm{E}\left[\frac{\{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\}^2\tau^2(F_n)}{F_n^2}\sum_{i=1}^{p}\bar{X}_{ni}^2 I_{[|\bar{X}_{ni}|/S_n>c_i]}\right] \\ &\quad \leqq \mathrm{EE}\left[\frac{\{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\}^2\tau^2(F_n)}{F_n}S_n^2\,|\,\bar{\boldsymbol{X}}_n\right]. \end{aligned} \tag{2.14}$$

Note that conditional on $\bar{\boldsymbol{X}}_n = \bar{\boldsymbol{x}}_n$, if $0 = z_{n0} \leqq z_{n1} \leqq \cdots \leqq z_{np}$ denote the ordered $\bar{x}_{ni}^2/c_i^2$, then $(N(\bar{\boldsymbol{x}}_n, s_n^2)-2)^+$ assumes the value $p-2-i$ for $z_{ni} \leqq s_n^2 \leqq z_{n\overline{i+1}}$ $(i=0,1,\cdots,p-3)$, and $(N(\bar{\boldsymbol{x}}_n, s_n^2)-2)^+ = 0$ for $s_n^2 > z_{n\overline{p-2}}$. Write $b = \sigma^2((n-1)p+2)^{-1}$, $k=(n-1)p$. Noting that $S_n^2 \sim b\chi_k^2$, and using the independence of $\bar{\boldsymbol{X}}_n$ and S_n^2, one gets,

$$\begin{aligned} &\mathrm{E}\,[\{(N(\bar{\boldsymbol{X}}_n, S_n^2)-2)^+\}^2(\tau^2(F_n)/F_n)S_n^2\,|\,\boldsymbol{X}_n = \bar{\boldsymbol{x}}_n] \\ &= \sum_{i=0}^{p-3}(p-2-i)^2\int_{z_{ni}}^{z_{n\overline{i+1}}}\frac{\tau^2(\|\bar{\boldsymbol{x}}_n\|^2/y)}{(\|\bar{\boldsymbol{x}}_n\|^2/y)}\,y\exp(-y/2b)\frac{(y/2b)^{k/2-1}}{\Gamma(k/2)}\frac{dy}{2b}\,. \end{aligned} \tag{2.15}$$

Now, using integration by parts, and the fact that $\tau(m) \uparrow$ in m, one gets, rhs of (2.15)

$$(2.16)\quad \le \frac{(2b)^2}{\|\bar{x}_n\|^2\Gamma(k/2)} \sum_{i=0}^{p-3} (p-2-i)^2\tau^2(\|\bar{x}_n\|^2/z_{n\overline{i+1}})[\exp(-z_{ni}/2b)(z_{ni}/2b)^{k/2+1} - \exp(-z_{n\overline{i+1}}/2b)(z_{n\overline{i+1}}/2b)^{k/2+1}] + \frac{(2b)^2}{\|\bar{x}_n\|^2} \sum_{i=0}^{p-3} (p-2-i)^2 \int_{z_{ni}/2b}^{z_{n\overline{i+1}}/2b} \exp(-u)\frac{\partial}{\partial u} \cdot \{\tau^2(\|\bar{x}_n\|^2/2bu)u^{k/2+1}\}\frac{du}{\Gamma(k/2)} .$$

Since, $\exp(-\nu)\nu^{k/2+1}$ is a concave function of ν with a maximum at $\nu=(k/2)+1>z_{n\overline{p-2}}$, then every term in the first expression of (2.16) is negative. If, on the other hand, $z_{nj}<(k/2)+1\le z_{n\overline{j+1}}$ for some j $(0\le j\le p-3)$, then $(p-2-i)^2\tau^2(\|\bar{x}_n\|^2/z_{n\overline{i+1}})$ being nonincreasing in i, it follows that first term in the rhs of (2.16)

$$(2.17)\quad \le \frac{(2b)^2}{\|\bar{x}_n\|^2\Gamma(k/2)}(p-2-j)^2\tau^2(\|\bar{x}_n\|^2/z_{n\overline{j+1}}) \cdot \sum_{i=1}^{p-3} [\exp(-z_{ni}/2b)(z_{ni}/2b)^{k/2+1} - \exp(-z_{n\overline{i+1}}/2b)(z_{n\overline{i+1}}/2b)^{k/2+1}]$$
$$= \frac{(2b)^2}{\|\bar{x}_n\|^2\Gamma(k/2)}(p-2-j)^2\tau^2(\|\bar{x}_n\|^2/z_{n\overline{j+1}})[\exp(-z_{n0}/2b)(z_{n0}/2b)^{k/2+1} - \exp(-z_{n\overline{p-2}}/2b)(z_{n\overline{p-2}}/2b)^{k/2+1}]<0 ,$$

since $z_{n0}=0<z_{n\overline{p-2}}$. Now, using $\tau(m)\uparrow$ in m

$$(2.18)\quad \frac{\partial}{\partial u}\{\tau^2(\|\bar{x}_n\|^2/2bu)u^{k/2+1}\} \le \left(\frac{k}{2}+1\right)\tau^2(\|\bar{x}_n\|^2/2bu)u^{k/2} .$$

Now from (2.14)-(2.18), it follows that

$$(2.19)\quad \mathrm{EE}\,[\{(N(\bar{X}_n, S_n^2)-2)^+\}^2(\tau^2(F_n)/F_n)S_n^2 \mid \bar{X}_n]$$
$$\le \sigma^2\,\mathrm{E}\left[\sum_{i=0}^{p-3}(p-2-i)^2\int_{z_{ni}}^{z_{n\overline{i+1}}} \frac{\tau^2(\|\bar{X}_n\|^2/y)}{(\|\bar{X}_n\|^2/y)}e^{-y/2b}\frac{(y/2b)^{k/2-1}}{\Gamma(k/2)}\frac{dy}{2b}\right]$$
$$=\sigma^2\,\mathrm{E}\,[\{(N(\bar{X}_n, S_n^2)-2)^+\}^2\tau^2(F_n)/F_n] .$$

From (2.12)-(2.14) and (2.19), it follows using $0<\tau(\cdot)<2$ that

$$(2.20)\quad R(\boldsymbol{\theta}, \tilde{\boldsymbol{\delta}}^B)-R(\boldsymbol{\theta}, \tilde{\boldsymbol{\delta}})\le -(\sigma^2/n^2)\,\mathrm{E}\,[\{(N(\bar{X}_n, S_n^2)-2)^+\}^2 \cdot \tau(F_n)(2-\tau(F_n))/F_n]<0 .$$

Also, (2.11) follows from the penultimate step of (2.20).

Remark 2.2. For unknown σ^2, if instead we test $H_0:\boldsymbol{\theta}=\boldsymbol{\lambda}$ against $H_1:\boldsymbol{\theta}\ne\boldsymbol{\lambda}$, then the preliminary test estimator should be modified as

$$\tilde{\boldsymbol{\delta}}^{\lambda}(\bar{X}_n, S_n^2)=\tilde{\delta}_1^{\lambda}(\bar{X}_n, S_n^2),\cdots, \tilde{\delta}_p^{\lambda}(\bar{X}_n, S_n^2))$$

with

$$\tilde{\delta}_i^{\lambda}(\bar{X}_n, S_n^2) = \lambda_i + (\bar{X}_{ni} - \lambda_i) I_{[|\bar{x}_{ni} - \lambda_i|/s_n > c_i]} , \qquad i = 1, \cdots, p .$$

The corresponding class of estimators dominating $\tilde{\boldsymbol{\delta}}^{\lambda}$ is now given by $\tilde{\boldsymbol{\delta}}_0^{\lambda}(\bar{X}_n, S_n^2)$ with its i-th component equal to

$$\tilde{\boldsymbol{\delta}}_{0i}^{\lambda}(\bar{X}_n, S_n^2) = \lambda_i + \left(1 - \frac{S_n^2 (N_0^{\lambda}(\bar{X}_n, S_n^2) - 2)^+ \tau(F_n^{\lambda})}{n F_n^{\lambda}}\right)(\bar{X}_{ni} - \lambda_i) ,$$

with $N_0^{\lambda}(X_n, S_n^2) = \#\{i: |\bar{x}_{ni} - \lambda_i|/s_n > c_i\}$ and $F_n^{\lambda} = \|\bar{X}_n - \boldsymbol{\lambda}\|^2 / S_n^2$.

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