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EFFECTS OF TRANSFORMATIONS IN HIGHER ORDER ASYMPTOTIC EXPANSIONS*

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Summary

Approximate formulae using a large number of terms of Edgeworth type asymptotic expansions for the distributions of statistics often produce spurious oscillations and give poor fits to the exact distribution functions in parts of the tails. A general method for suppressing these oscillations and evoking more accurate approximations is introduced here.

1. Introduction

Intensive investigations have been made concerning the distributions of statistics in multivariate analysis. Work has been done on the derivation of both exact and approximate distributions, and thrown light on distributions of a large number of multivariate statistics (see Muirhead [6] and Siotani, Hayakawa and Fujikoshi [11]). It should be noticed that the problem of actually tabulating percentile points and values of probability integrals by use of the results still remains to be solved: exact distributions based on hypergeometric functions of matrix arguments are too complicated to handle numerically, whereas approximate distributions are less accurate.

Some work of approximation is concerned with the derivation of asymptotic expansions for the distributions of statistics, in which the errors of approximation approach to zero as some parameter n, typically a sample size, tends to infinity. In order to improve the approximation, even for small n, further terms in an asymptotic expansion may be required. However, Wallace [12] called our attention that Edgeworth type asymptotic expansions for distributions of statistics usually are not convergent infinite series for any fixed n, and that the addi-

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tion of the next term does not always improve the approximation.

Niki and Konishi [8] gave an asymptotic expansion up to terms of order n^{-4} for the distribution of the sample correlation coefficient, r, in a bivariate normal sample, which guarantees accuracy to five decimal places even when the sample size $n+1$ is as small as 11. It is worth pointing out that the approximants to the distribution function based on a higher order asymptotic expansion for the distribution of r itself, when n is small, produced spurious oscillations and gave extremely poor fits in one tail. This is just the *tail difficulty* discussed by Wallace [12]. To overcome this difficulty, we obtained an asymptotic expansion for the distribution of the transformed statistic

$$z(r)=\frac{1}{2}\log\frac{1+r}{1-r}$$

instead of r itself.

In general, finite sums of asymptotic expansions suffering from the tail difficulty contain notable oscillations around the exact distribution functions in parts of one or both tails. Suppressing these oscillations in some manner may induce convergence for divergent asymptotic expansions and must be useful for obtaining highly accurate approximations.

The main purpose of this paper is to introduce a procedure which restrains the finite sums of the Edgeworth type asymptotic expansions from oscillating and eliminates the *tail difficulty*. In Section 2, we discuss the attribution of the oscillations. A class of transformations for suppressing these oscillations and evoking better approximations is introduced in Section 3. In Section 4, the efficacy of the transformations is illustrated through the examples of the sample correlation coefficient and a χ^2 variate.

Konishi ([3], [4]) has introduced a procedure for finding normalizing transformations to obtain simple and accurate approximations to the distributions of statistics. The theoretical approach discussed there is further developed to the case of asymptotic expansions.

The larger part of formulae in this paper is obtained with the help of a computer algebra (formula manipulation by computer) system, REDUCE (Hearn [5]).

2. Order of Hermite polynomials in asymptotic expansion

Let T_n be a statistic of which distribution depends on parameters n and θ. Assume that there exists $\mu=\mu(\theta)$ and $\sigma=\sigma(\theta)$ such that the standardized quantity

$$X_n = \frac{\sqrt{n}\{T_n - \mu(\theta)\}}{\sigma(\theta)}$$

has a limiting normal distribution with mean 0 and variance 1 as n tends to infinity. Let κ_j $(j=1, 2, \cdots)$ be the j-th cumulant of the distribution of X_n, and assume that κ_j's are of the form

$$(2.1) \qquad \kappa_1 \sim \sum_{k=1}^{\infty} \kappa_{1,2k-1} n^{-(2k-1)/2} , \qquad \kappa_2 \sim 1 + \sum_{k=1}^{\infty} \kappa_{2,2k} n^{-k}$$

and for $i>1$

$$(2.2) \qquad \kappa_{2i-1} \sim \sum_{k=i-1}^{\infty} \kappa_{2i-1,2k-1} n^{-(2k-1)/2} , \qquad \kappa_{2i} \sim \sum_{k=i-1}^{\infty} \kappa_{2i,2k} n^{-k}$$

where the coefficients $\kappa_{i,j}$ depend on the cumulants of the population distribution.

Then an asymptotic expansion for the distribution of X_n, collecting terms according to the power of $n^{-1/2}$, is obtained in the form

$$(2.3) \qquad \Pr[X_n < x] \sim \Phi(x) - \varphi(x)\{n^{-1/2}a_1(x) + n^{-1}a_2(x) + n^{-3/2}a_3(x) + \cdots\}$$

where $\Phi(x)$ and $\varphi(x)$ denote the distribution function of the standard normal variate and its derivative, respectively, and the coefficients $a_1(x), a_2(x), a_3(x), \cdots$ are given by

$$a_1(x) = \frac{1}{6}\kappa_{3,1}H_2 + \kappa_{1,1} ,$$

$$a_2(x) = \frac{1}{72}\kappa_{3,1}^2 H_5 + \left(\frac{1}{6}\kappa_{1,1}\kappa_{3,1} + \frac{1}{24}\kappa_{4,2}\right)H_3 + \left(\frac{1}{2}\kappa_{1,1}^2 + \frac{1}{2}\kappa_{2,2}\right)H_1 ,$$

$$\begin{aligned} a_3(x) = & \frac{1}{1296}\kappa_{3,1}^3 H_8 + \left(\frac{1}{72}\kappa_{1,1}\kappa_{3,1}^2 + \frac{1}{144}\kappa_{3,1}\kappa_{4,2}\right)H_6 \\ & + \left(\frac{1}{12}\kappa_{1,1}^2\kappa_{3,1} + \frac{1}{24}\kappa_{1,1}\kappa_{4,2} + \frac{1}{12}\kappa_{2,2}\kappa_{3,1} + \frac{1}{120}\kappa_{5,3}\right)H_4 \\ & + \left(\frac{1}{6}\kappa_{1,1}^3 + \frac{1}{2}\kappa_{1,1}\kappa_{2,2} + \frac{1}{6}\kappa_{3,3}\right)H_2 + \kappa_{1,3} , \end{aligned}$$

$$\cdots .$$

Here $H_j = H_j(x)$ $(j=1, 2, 3, \cdots)$ is the Hermite polynomial of order j. For further coefficients $a_j(x)$ $(4 \leq j \leq 8)$ and Hermite polynomials of order 23 or less, see Niki [7]. The validity of this type of expansion has been discussed by Wallace [12], Bhattacharya and Ghosh [1] and so on.

In practice the values of the probability integral of T_n are approximated by using a finite sum of the asymptotic series (2.3), that is,

$$\begin{aligned} \Pr[T_n < t] &= \Pr[X_n < x] \\ &\fallingdotseq F_m(x) = \Phi(x) - \varphi(x)\{n^{-1/2}a_1(x) + n^{-1}a_2(x) + \cdots + n^{-m/2}a_m(x)\} , \end{aligned}$$

where x is taken as $x=\sqrt{n}\{t-\mu(\theta)\}/\sigma(\theta)$. It is of interest to note that $F_m(x)$ contains the term

$$\left(\frac{\kappa_{3,1}}{\sqrt{n}}\right)^m \cdot \frac{1}{(3!)^m m!} H_{3m-1}(x)\varphi(x) , \tag{2.4}$$

since each term in $a_m(x)$ $(m=1, 2, 3, \cdots)$ can be expressed as

$$\frac{1}{(s_1!)^{p_1}(s_2!)^{p_2}\cdots(s_q!)^{p_q}\cdot p_1!p_2!\cdots p_q!}\kappa_{s_1,t_1}^{p_1}\kappa_{s_2,t_2}^{p_2}\cdots\kappa_{s_q,t_q}^{p_q}\cdot H_{p_1s_1+p_2s_2+\cdots+p_qs_q-1}(x)$$

where $p_1t_1+p_2t_2+\cdots+p_qt_q=m$ (Petrov [9] gave the form of $a_m(x)$ when the j-th cumulant of the distribution of a statistic can be exactly expressed as κ_j). For large j, the function $h_j(x)=H_j(x)\varphi(x)$ is a highly oscillatory function with j zero points and since

$$|h_{2k}(0)|=\frac{1}{\sqrt{2\pi}}\frac{(2k)!}{k!}$$

and

$$|h_{2k+1}(\zeta_{2k+2})|>\frac{1}{\zeta_{2k}}\int_0^{\zeta_{2k}}|h_{2k+1}(x)|dx=\frac{1}{\zeta_{2k}}|h_{2k}(0)|\geqq\frac{1}{\sqrt{2\pi}}\frac{(2k)!}{k!} ,$$

the magnitude of oscillations may be said to be of order $\Gamma(j)/\Gamma(j/2)$, where ζ_{2k} is the smallest positive zero point of $h_{2k}(x)$. Hence the magnitude of oscillations of the function

$$g_m(x)=\frac{1}{(3!)^m m!}h_{3m-1}(x)=\frac{1}{(3!)^m m!}H_{3m-1}(x)\varphi(x)$$

gets greater with *hyper exponential order* as m becomes larger (see Fig. 1). The maximum and minimum values of $g_m(x)$ for $2\leqq x\leqq 3$ and $3\leqq x\leqq 4$ are given in Table 1.

This table suggests that, if the value of $|\kappa_{3,1}|$ is nearly equal to

Table 1. The maximum and minimum values of $g_m(x)$ in the tail

	$\max/\min\ g_m(x)$ for $2\leqq x\leqq 3$							
m	2	3	4	5	6	7	8	9
max	0.001	0.011	0.002	0.013	0.033	0.027	0.108	0.222
min	−0.013	−0.002	−0.011	−0.007	−0.018	−0.064	−0.046	−0.197
	$\max/\min\ g_m(x)$ for $3\leqq x\leqq 4$							
m	2	3	4	5	6	7	8	9
max	0.002	0.000	0.003	0.000	0.005	0.014	0.015	0.084
min	0.001	−0.002	−0.000	−0.004	−0.003	−0.009	−0.041	−0.029

$\sqrt{n}$ or larger (this condition is often realized for relatively small n), the absolute value of (2.4) for large m may exceed the value of $\Pr[X_n<x]$ or $\Pr[X_n\geqq x]=1-\Pr[X_n<x]$ in parts of one or both tails. In such a case, noting that $H_{3m-1}(x)$ is the highest order Hermite polynomial in $F_m(x)$, it may be seen that the approximation error

$$C_m(x)=F_m(x)-\Pr[X_n<x]$$

is mainly caused by the oscillations in the tail parts of $g_m(x)$. As seen in Section 4, the error curve of $C_m(x)$ may also have notable oscillations, which violate both monotonicity and the 0-1 range property. This seems to be the main source of the *tail difficulty* which Wallace

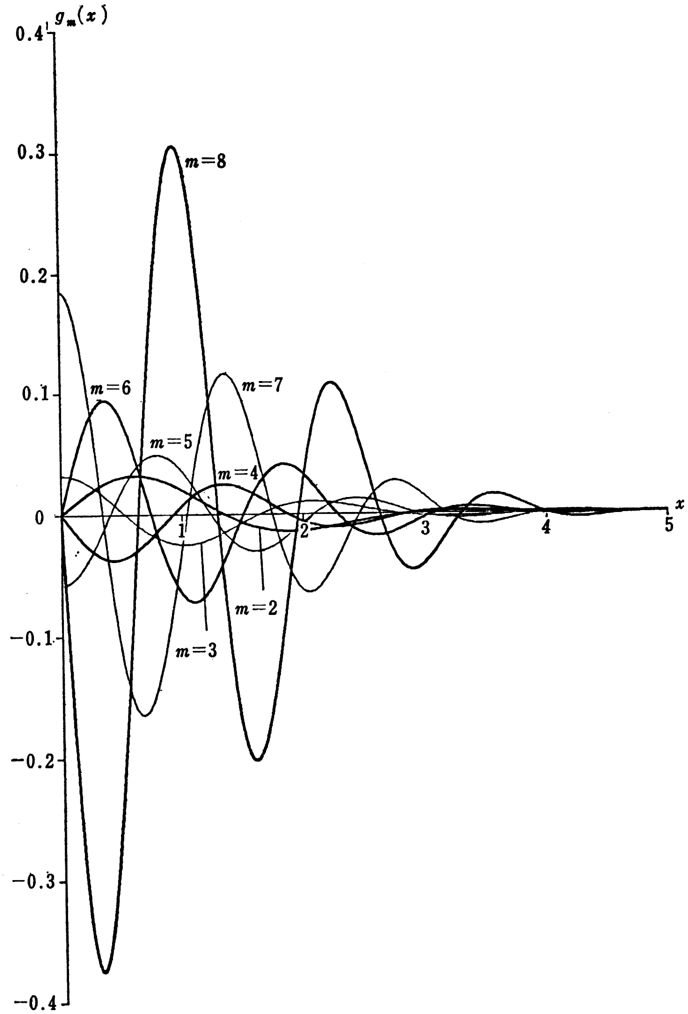


Fig. 1. Graphs of $g_m(x)=\dfrac{1}{(3!)^m m!}H_{3m-1}(x)\varphi(x)$ $(m=2, 3,\cdots, 8)$.

[12] pointed out.

The above discussion suggests an approach for suppressing the troublesome oscillations in the tails; that is, if we could eliminate the term (2.4) from $F_m(x)$, the oscillations would fade from the tails. The key of this approach to the *tail difficulty* lies in correction of the asymptotic skewness. If we take $\kappa_{3,1}$ to be zero, the highest order term in $F_m(x)$ $(m\geqq 2)$ can be reduced to the following form; when m is even,

$$\left(\frac{\kappa_{4,2}}{n}\right)^{m/2}\cdot\frac{1}{(4!)^{m/2}(m/2)!}H_{2m-1}(x)\varphi(x)$$

and when m is odd,

$$\frac{\kappa_{4,2}^{(m-3)/2}}{n^{m/2}}\left\{\frac{2\kappa_{1,1}\kappa_{4,2}}{m-1}+\frac{\kappa_{5,3}}{5}\right\}\cdot\frac{1}{(4!)^{(m-1)/2}((m-3)/2)!}H_{2(m-1)}(x)\varphi(x)\ .$$

Table 2 compares the highest orders of the Hermite polynomials in $a_m(x)$ under the assumption that $\kappa_{3,1}=0$ or $\kappa_{3,1}=\kappa_{4,2}=0$. The values in the lowest row of the table can be easily obtained in the same manner.

Table 2. The highest order of Hermite polynomials in $a_m(x)$

m	1	2	3	4	5	6	7	8	9	10
general	2	5	8	11	14	17	20	23	26	29
if $\kappa_{3,1}=0$	0	3	4	7	8	11	12	15	16	19
if $\kappa_{3,1}=\kappa_{4,2}=0$	0	1*	4	5	6	9	10	11	14	15

* may be reduced to zero by a linear transformation.

The requirement that the asymptotic skewness $\kappa_{3,1}$ is reduced to zero may be achieved by a transformation of statistic. In the next section we will give a general procedure for finding such a tranformation.

3. Transformations of statistics

Consider a one-to-one function $f(T_n)$ independent of n. We assume that the derivatives of $f(T_n)$ of requisite order are continuous in a neighbourhood of $T_n=\mu(\theta)$. It is well-known (Rao [10]) that the limiting distribution of

$$Y_n=\frac{\sqrt{n}\{f(T_n)-f(\mu(\theta))\}}{\sigma(\theta)f'(\mu(\theta))}$$

is normal with zero mean and unit variance.

The j-th cumulants λ_j of Y_n are of the same form as in (2.1) and (2.2);

$$\lambda_1 \sim n^{-1/2}\lambda_{1,1} + n^{-3/2}\lambda_{1,3} + \cdots$$
$$\lambda_2 \sim 1 + n^{-1}\lambda_{2,2} + n^{-2}\lambda_{2,4} + \cdots$$
$$\lambda_3 \sim n^{-1/2}\lambda_{3,1} + n^{-3/2}\lambda_{3,3} + \cdots$$
$$\lambda_4 \sim n^{-1}\lambda_{4,2} + n^{-2}\lambda_{4,4} + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$

where $\lambda_{j,l}$ are given by

$$\lambda_{1,1} = \frac{\sigma f''(\mu) + 2\kappa_{1,1} f'(\mu)}{2f'(\mu)}$$

$$\lambda_{2,2} = \frac{2\sigma^2 f'''(\mu) f'(\mu) + \sigma^2 f''(\mu)^2 + 2(\kappa_{3,1} + 2\kappa_{1,1})\sigma f''(\mu) f'(\mu) + 2\kappa_{2,2} f'(\mu)^2}{2f'(\mu)^2}$$

$$\lambda_{3,1} = \frac{3\sigma f''(\mu) + \kappa_{3,1} f'(\mu)}{f'(\mu)}$$

$$\lambda_{4,2} = \frac{4\sigma^2 f'''(\mu) f'(\mu) + 12\sigma^2 f''(\mu)^2 + 12\sigma\kappa_{3,1} f''(\mu) f'(\mu) + \kappa_{4,2} f'(\mu)^2}{f'(\mu)^2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

As discussed in the previous section, a transformation which reduces $\lambda_{3,1}$ to zero, if exists, may be effective to suppressing oscillations and may give an accurate approximation. The condition $\lambda_{3,1}=0$ can be realized by finding a function which satisfies the differential equation of the second order

$$3\sigma f''(\mu) + \kappa_{3,1} f'(\mu) = 0 \ . \tag{3.1}$$

Formally the solution can be written as

$$f(\mu) = c_1 \int \exp\left(\int -\frac{\kappa_{3,1}}{3\sigma} d\mu\right) d\mu + c_2$$

with arbitrary constants c_1 and c_2 (irrelevant to the expression of Y_n). However, versatility in determining $\mu=\mu(\theta)$ and $\sigma=\sigma(\theta)$ and a problem in obtaining $\kappa_{3,1}(\mu)$ and $\sigma(\mu)$ as a function of μ remain to be fixed.

4. Illustrations

This section illustrates the efficacy of transformations through two examples concerning the χ^2 variate and the sample correlation coefficient.

Example 1. Let χ_n^2 be the χ^2 variate with n degrees of freedom.

The standardized variate

$$X_n=\frac{\sqrt{n}\,(\chi_n^2-n)}{\sqrt{2}\,n}=\sqrt{\frac{n}{2}}\left(\frac{\chi_n^2}{n}-1\right)$$

is asymptotically normally distributed with zero mean and unit variance. It follows from (3.1) and $\kappa_{3,1}=2\sqrt{2}$, that the transformation $f(x)$ must satisfy

$$3nf''(n)+2f'(n)=0\ .$$

The solution of the above differential equation with respect to n is found to be $f(n)=n^{1/3}$, which is the cubic root transformation due to Wilson and Hilferty [13] for a χ^2 variate. Hence the transforme variate is given by

$$Y_n=\frac{\sqrt{n}\,\{(\chi_n^2)^{1/3}-n^{1/3}\}}{\sqrt{2}\,n\cdot(1/3)n^{-2/3}}=\sqrt{\frac{9n}{2}}\left\{\left(\frac{\chi_n^2}{n}\right)^{1/3}-1\right\}\ .$$

The distributions of the standardized χ^2 variate X_n and the transformed variate Y_n can be, respectively, expressed by the following Edgeworth type asymptotic formulae:

$$\begin{aligned}(4.1)\quad \Pr[X_n<x]\sim\Phi(x)-\varphi(x)\Big\{&n^{-1/2}\left(\frac{\sqrt{2}}{3}H_2\right)+n^{-1}\left(\frac{1}{9}H_5+\frac{1}{2}H_3\right)\\&+n^{-3/2}\sqrt{2}\left(\frac{1}{81}H_8+\frac{1}{6}H_6+\frac{2}{5}H_4\right)+n^{-2}\left(\frac{1}{486}H_{11}+\frac{1}{18}H_9\right.\\&\left.+\frac{47}{120}H_7+\frac{2}{3}H_5\right)+n^{-5/2}\sqrt{2}\left(\frac{1}{7290}H_{14}+\frac{1}{162}H_{12}+\frac{31}{360}H_{10}\right.\\&\left.+\frac{19}{45}H_8+\frac{4}{7}H_6\right)+n^{-3}\left(\frac{1}{65610}H_{17}+\frac{1}{972}H_{15}+\frac{77}{3240}H_{13}\right.\\&\left.+\frac{493}{2160}H_{11}+\frac{153}{175}H_9+H_7\right)+n^{-7/2}\sqrt{2}\left(\frac{1}{1377810}H_{20}\right.\\&+\frac{1}{14580}H_{18}+\frac{23}{9720}H_{16}+\frac{727}{19440}H_{14}+\frac{1751}{6300}H_{12}+\frac{31}{35}H_{10}\\&\left.+\frac{8}{9}H_8\right)+n^{-4}\left(\frac{1}{16533720}H_{23}+\frac{1}{131220}H_{21}+\frac{107}{291600}H_{19}\right.\\&\left.+\frac{503}{58320}H_{17}+\frac{190261}{1814400}H_{15}+\frac{4049}{6300}H_{13}+\frac{3349}{1890}H_{11}+\frac{8}{5}H_9\right)\\&+O(n^{-9/2})\Big\}\end{aligned}$$

and

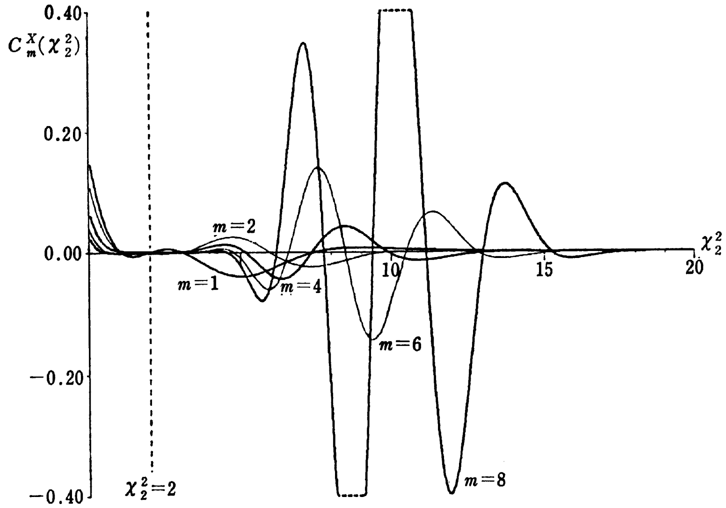
$$(4.2)\quad \Pr[Y_n<x]\sim\Phi(x)-\varphi(x)\Big\{n^{-1/2}\Big(-\frac{\sqrt{2}}{3}\Big)+n^{-1}\Big(-\frac{1}{54}H_3+\frac{1}{9}H_1\Big)$$
$$+n^{-3/2}\sqrt{2}\Big(\frac{1}{90}H_4+\frac{1}{27}H_2\Big)+n^{-2}\Big(\frac{1}{5832}H_7-\frac{13}{2430}H_5$$
$$-\frac{31}{486}H_3-\frac{26}{243}H_1\Big)+n^{-5/2}\sqrt{2}\Big(-\frac{13}{87480}H_8-\frac{181}{153090}H_6$$
$$+\frac{19}{1458}H_4+\frac{58}{729}H_2+\frac{40}{729}\Big)+n^{-3}\Big(-\frac{1}{944784}H_{11}$$
$$+\frac{137}{1312200}H_9+\frac{2761}{918540}H_7+\frac{215}{13122}H_5-\frac{22}{2187}H_3$$
$$-\frac{200}{2187}H_1\Big)+n^{-7/2}\sqrt{2}\Big(\frac{17}{14171760}H_{12}+\frac{613}{82668600}H_{10}$$
$$-\frac{851}{918540}H_8-\frac{2789}{196830}H_6-\frac{362}{6561}H_4-\frac{80}{2187}H_2+\frac{88}{6561}\Big)$$
$$+n^{-4}\Big(\frac{1}{204073344}H_{15}-\frac{241}{212576400}H_{13}-\frac{25961}{496011600}H_{11}$$
$$-\frac{14171}{37200870}H_9+\frac{13871}{2361960}H_7+\frac{1409}{19683}H_5+\frac{11578}{59049}H_3$$
$$+\frac{2240}{19683}H_1\Big)+O(n^{-9/2})\Big\}\ .$$

The finite sums of (4.1) up to terms of order $n^{-m/2}$ are referred to $F_m^X(x)$ ($m=1, 2,\cdots, 8$) in this example. The notation $F_m^Y(x)$ are similarly used as the finite sums of (4.2). In comparison with (4.1), it should be noticed that the Hermite polynomials in (4.2) have coefficients smaller in magnitude with them in addition of being of lower order.

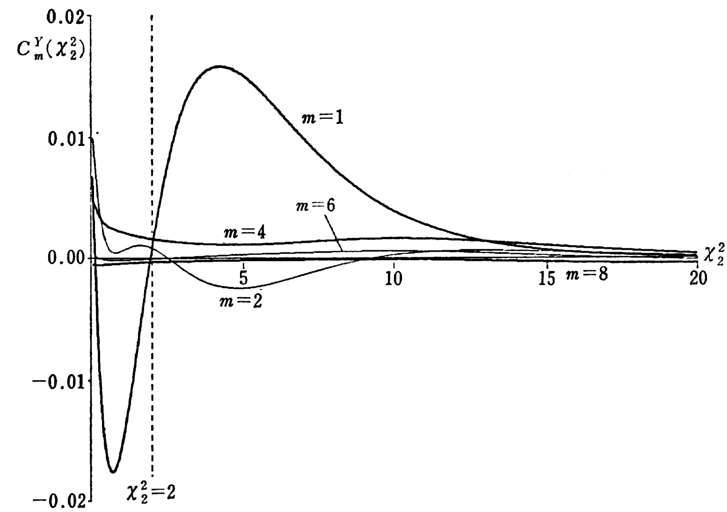
Even for $n=2$, the formula based on $F_8^Y(x)$ (formula (4.2) omitted the terms of order $n^{-9/2}$ or higher) closely approximates the exact distribution function

$$\Pr[\chi_2^2<x]=F^E(x)=1-e^{-x/2}$$

having errors less than 0.001 for $x\geqq 0.01$. The approximation errors $C_m^X(\chi_2^2)=F_m^X(X_2)-F^E(\chi_2^2)$ and $C_m^Y(\chi_2^2)=F_m^Y(Y_2)-F^E(\chi_2^2)$ for $\chi_2^2\leqq 10$ and $m=1$, 2, 4, 6, 8 are shown in Fig. 2 (a) and Fig. 2 (b), respectively (Note that the vertical scales differ each other). We can see from Fig. 2 (a) that the formulae for larger m give better fits in the neighbourhood of $\mu=2$, however, suffer from oscillation phenomena and give much poorer fits in the right tail (note that $\kappa_{3,1}/\sqrt{n}=2$ in this case). It should be also noted that the shape of oscillations resembles the right half of the corresponding oscillations in Fig. 1, which may endorse our inference about the source of the *tail difficulty*. In contrast, free from oscillation, the



(a) Standardized variate : $C^X_m(\chi^2_2)$ $(m=1,2,4,6,8)$.



(b) Transformed variate : $C^Y_m(\chi^2_2)$ $(m=1,2,4,6,8)$

Fig. 2. Errors of approximations to the χ^2_2 distribution.

approximate formulae based on transformed variate (see Fig. 2 (b)) become more accurate as the order m increases.

Example 2. Let r be the sample correlation coefficient based on a sample of size $n+1$ drawn from a bivariate normal distribution with population correlation ρ. It is known that the limiting distribution of

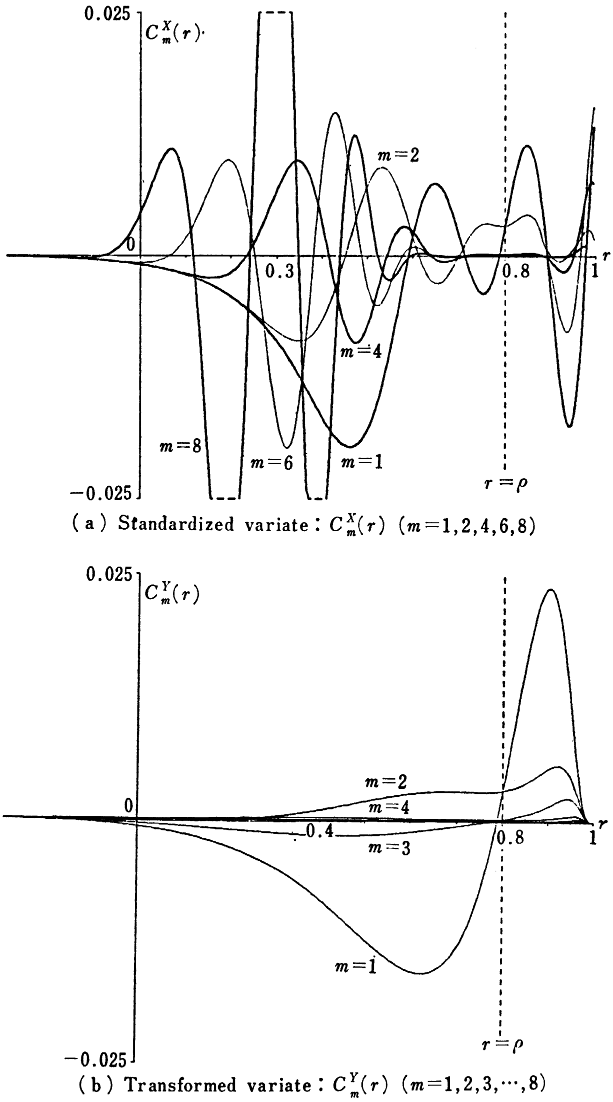
$$X_n=\frac{\sqrt{n}(r-\rho)}{1-\rho^2}$$

is the standard normal distribution. Since the term $\kappa_{3,1}$ is -6ρ, it follows from (3.1) that the transformation is required to satisfy the equation

$$(1-\rho^2)f''(\rho)-2\rho f'(\rho)=0 \ .$$

The solution of the above differential equation with respect to ρ is

$$f(\rho)=z(\rho)=\frac{1}{2}\log\frac{1+\rho}{1-\rho}$$



(a) Standardized variate: $C^X_m(r)$ $(m=1,2,4,6,8)$

(b) Transformed variate: $C^Y_m(r)$ $(m=1,2,3,\cdots,8)$

Fig. 3. Errors of approximations to the distribution of the sample correlation coefficient (sample size $n+1=11$ and population correlation coefficient $\rho=0.8$).

the z-transformation due to Fisher [2]. Since $f'(x)=1/(1-x^2)$, the variate transformed in order to make $\lambda_{3,1}=0$ is given by

$$Y_n=\frac{\sqrt{n}\,\{f(r)-f(\rho)\}}{(1-\rho^2)f'(\rho)}=\sqrt{n}\,\{z(r)-z(\rho)\}\ .$$

Concerning the distribution of the sample correlation coefficient r, an asymptotic expansion up to the terms of order n^{-4} for the distribution of the transformed variate Y_n was given in Niki and Konishi [8]. The corresponding expansion for the distribution of X_n is much more complicated to be presented and omitted here. The similar notation $F_m^X(x)$ and $F_m^Y(x)$ $(m=1, 2, \cdots, 8)$ are used for the finite sums of the above expansions as in Example 1, and the exact distribution function of r is referred to $F^E(r)$.

The approximation errors $C_m^X(r)=F_m^X(X_n)-F^E(r)$ are shown in Fig. 3 (a) for $n=10$ (sample size$=11$) and $\rho=0.8$ $(\kappa_{3,1}/\sqrt{n}=-4.8/\sqrt{10}\doteqdot-1.52)$. The oscillation phenomena are remarkable also in this case. Note that the shape of the oscillations for each m $(m=4, 6, 8)$ is very similar to the corresponding oscillations in Fig. 2 (a). On the other hand graphs of the approximation errors $C_m^Y(r)=F_m^Y(Y_{10})-F^E(r)$ in Fig. 3 (b) are much simpler than those in Fig. 3 (a). The highest order formula $F_8^Y(Y_{10})$ yields highly accurate values with

$$\max_{r=-0.995(0.005)0.995} C_8^Y(r)<8\times10^{-6}\ .$$

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