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OPTIMAL PARTIALLY BALANCED FRACTIONAL $2^{m_1+m_2}$ FACTORIAL DESIGNS OF RESOLUTION IV*

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Summary

This paper investigates some partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution IV derived from partially balanced arrays, which permit estimation of the general mean, all main effects, all two-factor interactions within each set of the m_k factors ($k=1, 2$) and some linear combinations of the two-factor interactions between the sets of the m_k ones. In addition, optimal designs with respect to the generalized trace criterion defined by Shirakura (1976, *Ann. Statist.*, 4, 723–735) are presented for each pair (m_1, m_2) with $2 \leqq m_1 \leqq m_2$ and $m_1+m_2 \leqq 6$, and for values of N (the number of observations) in a reasonable range.

1. Introduction

As a special case of an asymmetrical balanced array of type 2 defined by Nishii [7], a partially balanced array (PB-array) has been studied by Kuwada [3]. Necessary and sufficient conditions for the existence of a PB-array have been obtained by Kuwada and Kuriki [4]. A-optimal partially balanced fractional $2^{m_1+m_2}$ factorial ($2^{m_1+m_2}$-PBFF) designs of resolution V derived from PB-arrays have been obtained by Kuwada [3].

In this paper, we consider the situation in which the three-factor and higher order interactions are assumed to be negligible, the set of the factors is divided into two disjoint sets (m_1 factors and m_2 ones, say), and furthermore the two-factor interactions between the sets of the m_k factors ($k=1, 2$) are not of immediate interest from the point of view of estimation, etc., but they are possibly not negligible. In this situation, we study a fractional $2^{m_1+m_2}$ factorial design derived from a PB-array such that the general mean, all main effects, all two-factor

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interactions within each set of the m_k factors ($k=1, 2$) and some linear combinations of the two-factor interactions between the sets of the m_k factors are estimable, and that the covariance matrix of these estimates is invariant under any permutation on the m_k factors for each k. Such a design is called a $2^{m_1+m_2}$-PBFF design of resolution IV.

For earlier works on a design of even resolution, see for example, Kuwada [2], Margolin [5], [6], Shirakura [8]–[11], Srivastava and/or Anderson [1], [13] and Webb [14]. Especially, by use of the properties of the triangular multidimensional partially balanced (TMDPB) association algebra, it was shown in [8] that under some conditions, a balanced array with $\mu_l=0$ yields a balanced fractional 2^m factorial design of resolution $2l$ such that all effects up to the $(l-1)$-factor interactions and some linear combinations of the l-factor interactions are estimable.

For the reader's convenience, we shall recall the definition of a PB-array here: A (0, 1) matrix $[T^{(1)}; T^{(2)}]$ of size $N\times(m_1+m_2)$, in which $T^{(k)}$ ($k=1, 2$) are of size $N\times m_k$, is called a PB-array of strength t_1+t_2, size N, m_1+m_2 constraints, 2 levels and index set $\{\mu(i_1, i_2)\,|\,0\leqq i_k\leqq t_k\}$, written PBA $(N, m_1+m_2, 2, t_1+t_2, \{\mu(i_1, i_2)\})$ for brevity, if for fixed values of t_k, every submatrix $[T_0^{(1)}; T_0^{(2)}]$ of size $N\times(t_1+t_2)$ is such that every (0, 1) vector with weight i_k in $T_0^{(k)}$ occurs exactly $\mu(i_1, i_2)$ times as a row of $[T_0^{(1)}; T_0^{(2)}]$, where $T_0^{(k)}$ are of size $N\times t_k$ and are composed of t_k columns of $T^{(k)}$, and the weight of a (0, 1) vector means the number of ones in the vector.

2. Preliminaries

Consider a factorial experiment with m_1+m_2 factors each at two levels (0 and 1, say), where $m_k\geqq 2$ for $k=1, 2$. Further consider the situation in which the three-factor and higher order interactions are assumed to be negligible. The vector of unknown effects is then given by $(\{\theta(0;0)\}; \{\theta(u;0)\}; \{\theta(0;v)\}; \{\theta(u_1u_2;0)\}; \{\theta(0;v_1v_2)\}; \{\theta(u;v)\})$ $(=\Theta'$, say), where $1\leqq u\leqq m_1$, $1\leqq v\leqq m_2$, $1\leqq u_1<u_2\leqq m_1$ and $1\leqq v_1<v_2\leqq m_2$. Here A' denotes the transpose of a matrix A. Let $[T^{(1)}; T^{(2)}]$ $(=T$, say) be a fraction with N assemblies, then T can be expressed as a (0, 1) matrix of size $N\times(m_1+m_2)$ whose rows denote the assemblies. The vector of N observations based on T can be expressed as

$$y(T)=E_T\Theta+e_T\ ,$$

where E_T is the $N\times\nu$ design matrix whose elements are either 1 or -1 and e_T is the error vector whose components are assumed to be uncorrelated each having mean zero and same variance σ^2. Here $\nu=1+(m_1+m_2)+\binom{m_1+m_2}{2}$ and σ^2 is a constant which may or may not be known, where $\binom{p}{q}$ denotes the binomial coefficient. As a special case, $\binom{p}{q}=0$ if and only if $p<q$ or $q<0$. The normal equation for estimating $\boldsymbol{\Theta}$ can be expressed as

$$M_T\hat{\boldsymbol{\Theta}}=E_T'y(T)\ , \tag{2.1}$$

where $M_T=E_T'E_T$ $(\nu\times\nu)$ being called the information matrix.

Now let $S(a_1a_2)=\{\theta(u_1\cdots u_{a_1};v_1\cdots v_{a_2})\,|\,1\leqq u_1<\cdots<u_{a_1}\leqq m_1,\ 1\leqq v_1<\cdots<v_{a_2}\leqq m_2\}$. Then $|S(a_1a_2)|=\binom{m_1}{a_1}\binom{m_2}{a_2}$, where $|S|$ denotes the cardinality of a set S. Suppose a relation of association is defined among the sets of effects in such a way that $\theta(u_1\cdots u_{a_1};v_1\cdots v_{a_2})\in S(a_1a_2)$ and $\theta(u_1'\cdots u_{b_1}';v_1'\cdots v_{b_2}')\in S(b_1b_2)$ are the $(\alpha_1\alpha_2)$-th associates if

$$|\{u_1,\cdots,u_{a_1}\}\cap\{u_1',\cdots,u_{b_1}'\}|=\min(a_1,b_1)-\alpha_1$$

and

$$|\{v_1,\cdots,v_{a_2}\}\cap\{v_1',\cdots,v_{b_2}'\}|=\min(a_2,b_2)-\alpha_2\ ,$$

where $\min(a,b)$ denotes the minimum value of integers a and b. The scheme thus defined is called an extended TMDPB (ETMDPB) association scheme (see [3]), and it can be regarded as a generalization of the TMDPB association scheme (e.g., Yamamoto, Shirakura and Kuwada [15], [16]). For the ETMDPB association scheme, we shall use the same matrix notations, $D_{\alpha_1\alpha_2}^{(a_1a_2,b_1b_2)}$, $A_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)}$ and $D_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)}$ as in [3], where $0\leqq a_1+a_2,\ b_1+b_2\leqq 2$. The reader, therefore, is referred to the paper mentioned above for the properties of these matrices used here.

Let T be a PBA $(N,\ m_1+m_2,\ 2,\ t_1+t_2,\ \{\mu(i_1,i_2)\})$, where

$$t_k=\begin{cases} m_k & \text{if } m_k=2,3\ ,\\ 4 & \text{if } m_k\geqq 4\ .\end{cases} \tag{2.2}$$

In this paper, we shall consider only a PB-array with (2.2). Then the information matrix M_T can be expressed as

$$M_T=\sum_{a_1a_2}\sum_{b_1b_2}\sum_{\alpha_1\alpha_2}\gamma_{|a_1-b_1|+2\alpha_1,\,|a_2-b_2|+2\alpha_2}D_{\alpha_1\alpha_2}^{(a_1a_2,b_1b_2)}\ ,$$

where $\sum_{a_1a_2}$ and $\sum_{b_1b_2}$ stand for the summations over $a_1a_2,\ b_1b_2=00,\ 10,\ 01,\ 20$ (if $m_1\geqq 4$), 02 (if $m_2\geqq 4$), 11, and a connection between γ_{j_1,j_2} and indices $\mu(i_1,i_2)$ of a PB-array is given by

$$\gamma_{j_1,j_2}=\sum_{i_1=0}^{t_1}\sum_{i_2=0}^{t_2}\left[\prod_{k=1}^{2}\left\{\sum_{p_k=0}^{j_k}(-1)^{p_k}\binom{j_k}{p_k}\binom{t_k-j_k}{i_k-j_k+p_k}\right\}\right]\mu(i_1,i_2)\ . \tag{2.3}$$

Let $\mathcal{A}_{\beta_1\beta_2}=[D_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)}\,|\,0\leqq a_1+a_2\leqq 2,\ 0\leqq b_1+b_2\leqq 2]$ for $\beta_1\beta_2=00,\ 10,\ 01,$

20 (if $m_1 \geqq 4$), 02 (if $m_2 \geqq 4$), 11. Then $\mathcal{A}_{\beta_1\beta_2}$ are two-sided ideals of the ETMDPB association algebra $\mathcal{A}$ generated by the linear closure of all $D_{\alpha_1\alpha_2}^{(a_1a_2,b_1b_2)}$ and also generated by the linear closure of all $D_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)}$. For T being a PB-array, let $K_{\beta_1\beta_2}$ be the irreducible matrix representations of M_T with respect to $\mathcal{A}_{\beta_1\beta_2}$, where $K_{\beta_1\beta_2}=\|\kappa_{\beta_1\beta_2}^{a_1a_2,b_1b_2}\|$. In this case, we denote them by $M_T \sim K_{\beta_1\beta_2}$ for $\beta_1\beta_2=00, 10, 01, 20$ (if $m_1 \geqq 4$), 02 (if $m_2 \geqq 4$), 11. A connection between the elements $\kappa_{b_1b_2}^{a_1a_2,b_1b_2}$ of $K_{\beta_1\beta_2}$ and the values γ_{j_1,j_2} is given by

$$\kappa_{\beta_1\beta_2}^{a_1a_2,b_1b_2}=\sum_{\alpha_1\alpha_2}\left[\prod_{k=1}^{2}\{z_{\beta_k\alpha_k}^{*(a_k,b_k)}\}\right]\gamma_{|a_1-b_1|+2\alpha_1,|a_2-b_2|+2\alpha_2}, \tag{2.4}$$

where for $0 \leqq a \leqq b \leqq m$, and $0 \leqq \alpha, \beta \leqq \min\{\min(a, m-a), \min(b, m-b)\}$,

$$z_{\beta\alpha}^{*(a,b)}=\begin{cases} \text{vanish} \quad \text{if (I) } m=2 \text{ (i) } (a,b)=(1,2) \\ \qquad (1)\ \alpha=0,\ \beta=1, \\ \qquad (2)\ \alpha=1,\ \beta=0,1, \\ \quad \text{(ii) } (a,b)=(2,2) \\ \qquad (1)\ \alpha=0,\ \beta=1,2, \\ \qquad (2)\ \alpha=1,2,\ \beta=0,1,2, \\ \text{(II) } m=3,\ (a,b)=(2,2) \\ \qquad (1)\ \alpha=0,1,\ \beta=2, \\ \qquad (2)\ \alpha=2,\ \beta=0,1,2, \\ \displaystyle\sum_{p=0}^{\alpha}(-1)^{\alpha-p}\frac{\binom{a-\beta}{p}\binom{a-p}{a-\alpha}\binom{m-a-\beta+p}{p}}{\binom{b-a+p}{p}}\left\{\binom{m-a-\beta}{b-a}\binom{b-\beta}{b-a}\right\}^{1/2} \\ \qquad \text{otherwise}. \end{cases}$$

3. $2^{m_1+m_2}$-PBFF designs of resolution IV

A fraction T is called a $2^{m_1+m_2}$-PBFF design of resolution IV when the vector of unknown effects $(\{\theta(u;0)\};\{\theta(0;v)\})$ $(=\Theta_0'$, say) is estimable and the covariance matrix, $\mathrm{Var}[\hat{\Theta}_0]$, say, of the BLUE $\hat{\Theta}_0$ of Θ_0 is invariant under any permutation on the m_k factors for each $k=1,2$. Consider the vector Θ' of unknown effects partitioned into $(\Theta_1';\Theta_2')$, where $\Theta_1'=(\{\theta(0;0)\};\{\theta(u;0)\};\{\theta(0;v)\};\{\theta(u_1u_2;0)\};\{\theta(0;v_1v_2)\})$ and $\Theta_2'=(\{\theta(u;v)\})$. Then if Θ_1 is estimable and $\mathrm{Var}[\hat{\Theta}_1]$ is invariant under any permutation on the m_k factors, T is of course a $2^{m_1+m_2}$-PBFF design of resolution IV, which is treated here.

Now, we shall consider a PB-array T satisfying the following conditions:

$$\det(K_{\beta_1\beta_2}) \neq 0 \quad \text{for } \beta_1\beta_2=00, 10, 01, 20 \text{ (if } m_1 \geqq 4\text{)}, \tag{3.1}$$

02 (if $m_2 \geqq 4$) and $\det(K_{11})=0$,

where $\det(A)$ denotes the determinant of a matrix A.

LEMMA 3.1. *For T being a PBA $(N, m_1+m_2, 2, t_1+t_2, \{\mu(i_1, i_2)\})$, $\det(K_{11})=0$, i.e., $\kappa_{11}^{11,11}=0$, yields $\mu(i_1, i_2)=0$ for $1 \leqq i_k \leqq t_k-1$ $(k=1, 2)$.*

PROOF. From (2.3), (2.4) and Vandermonde convolution formula, it follows that

$$\kappa_{11}^{11,11}=N-\gamma_{0,2}-\gamma_{2,0}+\gamma_{2,2}=16\sum_{i_1}\sum_{i_2}\binom{t_1-2}{i_1-1}\binom{t_2-2}{i_2-1}\mu(i_1, i_2) .$$

Since the values $\binom{t_1-2}{i_1-1}\binom{t_2-2}{i_2-1}$ are positive integers for $1 \leqq i_k \leqq t_k-1$ $(k=1, 2)$ and $\mu(i_1, i_2)$ are non-negative, $\kappa_{11}^{11,11}=0$ yields $\mu(i_1, i_2)=0$ for $1 \leqq i_k \leqq t_k-1$. This completes the proof.

Let C be a matrix of order ν such that

$$C=\operatorname{diag}[I_{\nu^*}; H] \in \mathcal{A} ,$$

where I_p denotes the unit matrix of order p. Here

$$\nu^*=\nu-m_1m_2=\{2+m_1+m_2+(m_1)^2+(m_2)^2\}/2$$

and

$$H=h_{00}A_{00}^{\#(11,11)}+h_{10}A_{10}^{\#(11,11)}+h_{01}A_{01}^{\#(11,11)} ,$$

where $h_{\beta_1\beta_2}$ $(\beta_1\beta_2=00, 10, 01)$ are real constants. Then it holds that $C \sim \Gamma_{\beta_1\beta_2}$ for $\beta_1\beta_2=00, 10, 01, 20$ (if $m_1 \geqq 4$), 02 (if $m_2 \geqq 4$), where

$$\Gamma_{00}=\operatorname{diag}[I_5; h_{00}] ,$$

(3.2)
$$\Gamma_{10}=\begin{cases}\operatorname{diag}[1; h_{10}] & \text{if } m_1=2 , \\ \operatorname{diag}[I_2; h_{10}] & \text{if } m_1 \geqq 3 ,\end{cases}$$

$$\Gamma_{01}=\begin{cases}\operatorname{diag}[1; h_{01}] & \text{if } m_2=2 , \\ \operatorname{diag}[I_2; h_{01}] & \text{if } m_2 \geqq 3 ,\end{cases}$$

$$\Gamma_{20}=\begin{cases}\text{vanish} & \text{if } m_1=2, 3 , \\ [1] & \text{if } m_1 \geqq 4\end{cases}$$

and

$$\Gamma_{02}=\begin{cases}\text{vanish} & \text{if } m_2=2, 3 , \\ [1] & \text{if } m_2 \geqq 4 .\end{cases}$$

LEMMA 3.2. *Let T be a PBA $(N, m_1+m_2, 2, t_1+t_2, \{\mu(i_1, i_2)\})$ satisfying the condition* (3.1). *Then there exists a matrix X of order ν such that $XM_T=C$ and $X \in \mathcal{A}$.*

PROOF. Let $\chi_{\beta_1\beta_2}=\Gamma_{\beta_1\beta_2}K_{\beta_1\beta_2}^{-1}$ and consider a matrix X such that $X\sim \chi_{\beta_1\beta_2}$ for $\beta_1\beta_2=00, 10, 01, 20$ (if $m_1\geqq 4$), 02 (if $m_2\geqq 4$), where $\Gamma_{\beta_1\beta_2}$ are given by (3.2). Then since $M_T\sim K_{\beta_1\beta_2}$, we have $XM_T=C$ and $X\in\mathcal{A}$.

THEOREM 3.1. *For T being an array of Lemma* 3.2, *a parametric function*

$$\Psi=C\Theta=\begin{bmatrix}\Theta_1\\ H\Theta_2\end{bmatrix} \quad \left(=\begin{bmatrix}\Theta_1\\ \Psi_2\end{bmatrix},\ say\right)$$

is an estimable function of Θ. The BLUE $\hat{\Psi}$ of Ψ is given by $\hat{\Psi}=XE_T'y(T)$, where X is the matrix given in Lemma 3.2.

PROOF. From Lemma 3.2 and $\mathrm{E}[y(T)]=E_T\Theta$, it holds that $\mathrm{E}[\hat{\Psi}]=XE_T'\,\mathrm{E}[y(T)]=XE_T'E_T\Theta=XM_T\Theta=C\Theta=\Psi$, where E stands for an expected value. Also it follows from the Gauss-Markoff Theorem that the BLUE $\hat{\Psi}$ of Ψ is uniquely given by $\hat{\Psi}=C\hat{\Theta}$, where $\hat{\Theta}$ is a solution of (2.1). It, therefore, holds that $\hat{\Psi}=XM_T\hat{\Theta}=XE_T'y(T)$.

Note that for T being an array of Lemma 3.2, $\mathrm{Var}[y(T)]=\sigma^2 I_N$ and $XM_T=C$ yield $\mathrm{Var}[\hat{\Psi}]=\sigma^2 XC$.

Let

$$\begin{aligned}(3.3)\qquad X_1&=\mathrm{diag}\,[X_{11};\, 0_{m_1m_2}]\\ &=\sum_{a_1a_2}\sum_{b_1b_2}\kappa^{00}_{a_1a_2,b_1b_2}D_{00}^{\#(a_1a_2,b_1b_2)}+\sum_{u_1u_2}\sum_{v_1v_2}\kappa^{10}_{u_1u_2,v_1v_2}D_{10}^{\#(u_1u_2,v_1,v_2)}\\ &\quad+\sum_{w_1w_2}\sum_{s_1s_2}\kappa^{01}_{w_1w_2,s_1s_2}D_{01}^{\#(w_1w_2,s_1s_2)}+\kappa^{20}_{20,20}D_{20}^{\#(20,20)}+\kappa^{02}_{02,02}D_{02}^{\#(02,02)},\end{aligned}$$

where (i) when $m_1, m_2=2, 3$, $\kappa^{20}_{20,20}$ and $\kappa^{02}_{02,02}$ vanish, (ii) when $m_1\geqq 4$ and $m_2=2, 3$, $\kappa^{02}_{02,02}$ vanishes, and (iii) when $m_1=2, 3$ and $m_2\geqq 4$, $\kappa^{20}_{20,20}$ vanishes. Here X_{11} is the $\nu^*\times\nu^*$ submatrix of X, 0_p denotes the matrix of order p whose elements are all zero, and $\kappa^{\beta_1\beta_2}_{a_1a_2,b_1b_2}$ are the (a_1a_2, b_1b_2)-th elements of $K_{\beta_1\beta_2}^{-1}$. Then we have

$$(3.4)\qquad \begin{aligned}\mathrm{Var}[\hat{\Theta}_1]&=\sigma^2 X_{11},\\ \mathrm{Var}[\hat{\Psi}_2]&=\sigma^2\{\sum_{\beta_1\beta_2}(h_{\beta_1\beta_2})^2\kappa^{\beta_1\beta_2}_{11,11}A^{\#(11,11)}_{\beta_1\beta_2}\},\end{aligned}$$

where $\sum_{\beta_1\beta_2}$ stands for $\beta_1\beta_2=00, 10, 01$. Since X_1 belongs to $\mathcal{A}$, the following is immediately established:

THEOREM 3.2. *For T being an array of Lemma* 3.2, *T is a* $2^{m_1+m_2}$-

PBFF design of resolution IV *such that* Var $[\hat{\Theta}_1]$ *is invariant under any permutation on each set of the* m_k *factors, and that* $A^{\sharp(11,11)}_{\beta_1\beta_2}\Theta_2$ $(\beta_1\beta_2=00, 10, 01)$ *are estimable.*

To illustrate the usefulness of the results in this paper, we present an example here.

Example. Let

$$T'=\left[\begin{array}{c|cc|c|cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

Then T is a PBA $(10, 2+2, 2, 2+2, \{\mu(0,0)=1, \mu(0,1)=1, \mu(0,2)=1, \mu(1,0)=1, \mu(1,1)=0, \mu(1,2)=1, \mu(2,0)=0, \mu(2,1)=1, \mu(2,2)=0\})$, and $\det(K_{00})=262144$, $\det(K_{10})=\det(K_{01})=128$ and $\det(K_{11})=0$. From Theorem 3.2, the vectors $\Theta_1'=(\theta(0;0), \theta(1;0), \theta(2;0), \theta(0;1), \theta(0;2), \theta(12;0), \theta(0;12))$, $A^{\sharp(11,11)}_{10}\Theta_2$, i.e., $\{\theta(1;1)+\theta(1;2)-\theta(2;1)-\theta(2;2)\}/4$, and $A^{\sharp(11,11)}_{01}\Theta_2$, i.e., $\{\theta(1;1)-\theta(1;2)+\theta(2;1)-\theta(2;2)\}/4$, are estimable. Note that there do not exist the matrices K_{20} and K_{02}, since $m_1=m_2=2$.

4. GT-optimal $2^{m_1+m_2}$-PBFF designs of resolution IV

Consider a $2^{m_1+m_2}$-PBFF design T of resolution IV with N assemblies. Then since for $N \geqq \nu$, there exist $2^{m_1+m_2}$-PBFF designs of resolution V (e.g., [3]), we consider the only case in which $\nu^{**} \leqq N < \nu$, where $\nu^{**}=\nu^*+1+\phi_{10}+\phi_{01}=\{3(m_1+m_2)+(m_1)^2+(m_2)^2\}/2$, where

$$\phi_{\beta_1\beta_2}=\prod_{k=1}^{2}\left\{\binom{m_k}{\beta_k}-\binom{m_k}{\beta_k-1}\right\}. \tag{4.1}$$

For choosing a design which allows the estimates of at least m_1+m_2 main effects and further maximizes the amount of the information in some sense, we shall consider the sum of the variances of the estimate $\hat{\Theta}_1$ and the $(1+\phi_{10}+\phi_{01})$ normalized independent parameters in $A^{\sharp(11,11)}_{\beta_1\beta_2}\Theta_2$ for $\beta_1\beta_2=00, 10, 01$. Let $\Psi^{\beta_1\beta_2}=\{(m_1m_2)/\phi_{\beta_1\beta_2}\}^{1/2}A^{\sharp(11,11)}_{\beta_1\beta_2}\Theta_2$ for $\beta_1\beta_2=00, 10, 01$. Then $\Psi^{\beta_1\beta_2}$ are normalized parametric functions of Θ_2. It follows from (3.4) that every element in the BLUEs $\hat{\Psi}^{\beta_1\beta_2}$ of $\Psi^{\beta_1\beta_2}$ has the same variance $\sigma^2\kappa^{\beta_1\beta_2}_{11,11}$. Thus the following yields:

THEOREM 4.1. *For* T *being a* $2^{m_1+m_2}$*-PBFF design of resolution* IV *derived from a PBA* $(N, m_1+m_2, 2, t_1+t_2, \{\mu(i_1, i_2)\})$ *satisfying* (3.1), *the sum of the variances of* ν^* *BLUEs of the effects in* Θ_1 *and normalized independent parameters in* $\Psi^{\beta_1\beta_2}$ *is given by* $\sigma^2 S_T$, *where*

$$S_T=\begin{cases} \mathrm{tr}\,(K_{00}^{-1})+\phi_{10}\,\mathrm{tr}\,(K_{10}^{-1})+\phi_{01}\,\mathrm{tr}\,(K_{01}^{-1}) & \text{if } m_1, m_2=2, 3\,, \\ \mathrm{tr}\,(K_{00}^{-1})+\phi_{10}\,\mathrm{tr}\,(K_{10}^{-1})+\phi_{01}\,\mathrm{tr}\,(K_{01}^{-1})+\phi_{20}\,\mathrm{tr}\,(K_{20}^{-1}) & \\ & \text{if } m_1\geqq 4,\ m_2=2, 3\,, \\ \mathrm{tr}\,(K_{00}^{-1})+\phi_{10}\,\mathrm{tr}\,(K_{10}^{-1})+\phi_{01}\,\mathrm{tr}\,(K_{01}^{-1})+\phi_{02}\,\mathrm{tr}\,(K_{02}^{-1}) & \\ & \text{if } m_1=2, 3,\ m_2\geqq 4\,, \\ \mathrm{tr}\,(K_{00}^{-1})+\phi_{10}\,\mathrm{tr}\,(K_{10}^{-1})+\phi_{01}\,\mathrm{tr}\,(K_{01}^{-1})+\phi_{20}\,\mathrm{tr}\,(K_{20}^{-1})+\phi_{02}\,\mathrm{tr}\,(K_{02}^{-1}) & \\ & \text{if } m_1, m_2\geqq 4\,, \end{cases}$$

where $\mathrm{tr}\,(A)$ *denotes the trace of a matrix* A *and* $\phi_{\beta_1\beta_2}$ *are given by* (4.1).

For T_1 and T_2 being two $2^{m_1+m_2}$-PBFF designs of resolution IV derived respectively from a $\mathrm{PBA}\,(N, m_1+m_2, 2, t_1+t_2, \{\mu(i_1, i_2)\})$ and a $\mathrm{PBA}\,(N, m_1+m_2, 2, t_1+t_2, \{\mu^*(i_1, i_2)\})$ satisfying (3.1), T_1 is said to be better than T_2 if $S_{T_1}<S_{T_2}$. Such a criterion is called the generalized trace (GT) criterion, which was defined by Shirakura [8].

Table 1. GT-optimal 2^{2+2}-PBFF designs $(10\leqq N<11)$

N	μ	S_T
10	111101010	1.43750
	110101110	

Table 2. GT-optimal 2^{2+3}-PBFF designs $(14\leqq N<16)$

N	μ	S_T
14	non-exist	
15	110110010110	1.57060

Table 3. GT-optimal 2^{2+4}-PBFF designs $(19\leqq N<22)$

N	μ	S_T
19	010101000110100	2.12500
20	101011000101010	1.25000
21	201011000101010	1.19167
	101011000111010	

Table 4. GT-optimal 2^{3+3}-PBFF designs $(18\leqq N<22)$

N	μ	S_T
18	non-exist	
19	0110000110011100	2.42411
20	1111100010010100	1.81826
	1110100110001010	
21	1111100010011100	1.67239
	1111100110001010	

For $[T^{(1)}; T^{(2)}]$ $(=T$, say) being a $\mathrm{PBA}\,(N, m_1+m_2, 2, t_1+t_2, \{\mu(i_1, i_2)\})$, let $\bar{T}_1=[\bar{T}^{(1)}; T^{(2)}]$, $\bar{T}_2=[T^{(1)}; \bar{T}^{(2)}]$ and $\bar{T}=[\bar{T}^{(1)}; \bar{T}^{(2)}]$, where $\bar{T}^{(k)}=G_{N\times m_k}-T^{(k)}$ and $G_{p\times q}$ denotes the $p\times q$ matrix with unit elements everywhere. Then $\bar{T}_1$, $\bar{T}_2$ and $\bar{T}$ are also PB-arrays with index sets $\{\mu(t_1-i_1, i_2)\}$, $\{\mu(i_1, t_2-i_2)\}$ and $\{\mu(t_1-i_1, t_2-i_2)\}$, respectively. If T is a PB-array satisfying (3.1), then it holds that $S_T=S_{\bar{T}_1}=S_{\bar{T}_2}=S_{\bar{T}}$ (e.g., Shirakura and Kuwada [12]).

In Tables 1, 2, 3 and 4, GT-optimal 2^{2+2}-, 2^{2+3}-, 2^{2+4}- and 2^{3+3}-PBFF designs of resolution IV derived from PB-arrays satisfying (3.1) are respectively given together with the values of S_T and the indices $\mu(i_1, i_2)$ of a PB-array. Note that in each table, $\mu=(\mu(0, 0)\mu(0, 1)\cdots\mu(0, t_2)\mu(1, 0)\cdots\mu(t_1, t_2))$, where t_k are given by (2.2).

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