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FACTORIAL ORTHOGONALITY IN THE PRESENCE OF COVARIATES

RAHUL MUKERJEE AND HARUO YANAI

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Summary

The present paper obtains necessary and sufficient conditions for factorial orthogonality in the presence of covariates. In particular, when interactions are absent, combinatorial characterizations of the conditions, as natural generalizations of the well-known equal and proportional frequency criteria, have been derived.

1. Introduction

In a multifactor set up, the problem of orthogonality has been considered by a number of researchers. For a factorial experiments in a completely randomized design, Seber [13] derived conditions for orthogonality. In the contexts of factorial block designs and fractional plans, conditions for orthogonality have been obtained among others by Chakravarti [2], Kurkjian and Zelen [7], Addelman [1], Kshirsagar [6], Cotter, John and Smith [4], Lewis and John [9], Kuwada and Nishii [8], Mukerjee [10], [11]. Recently, the problem has been considered by Rao and Yanai [12] and also Takeuchi, Yanai and Mukerjee [14] using projection operators.

The present paper considers the problem of factorial orthogonality in the presence of covariates. The problem is of practical importance and for real-life examples of factorial experimentation incorporating covariates, from the fields of industry and biometry, one may see Wishart [15], Cochran and Cox ([3], 176–180) and Cox ([5], 259–260) among others. Under a situation where the level combinations are arranged in a completely randomized design and there are p covariates, two cases are considered in this paper. First the regression coefficients are allowed to vary over the level combinations and a combinatorial characterization for factorial orthogonality is derived. Next, the situation where the regression coefficients are the same for all level combinations is

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also taken care of.

2. Notations and preliminaries

Suppose that there are m factors $F_1, \cdots, F_m$ at $s_1, \cdots, s_m$ levels respectively. Let Y denote the study variate and suppose that there are p nonstochastic covariates $x^{(1)}, \cdots, x^{(p)}$. There are $v=\prod_{j=1}^{m} s_j$ level combinations of the m factors and a typical level combination will be denoted by $i=(i_1, \cdots, i_m)$, $0 \leqq i_j \leqq s_j-1$; $1 \leqq j \leqq m$. Hereafter, the level combinations will be assumed to be lexicographically ordered (cf. Kurkjian and Zelen [7]).

With the level combinations arranged in a completely randomized design, suppose that the i-th level combination is applied n_i times and denote the corresponding observational vector by $\boldsymbol{Y}_i=(Y_{i1}, \cdots, Y_{in_i})$. Further, let the corresponding values of the w-th covariate be, say, $\boldsymbol{x}_i^{(w)}=(x_{i1}^{(w)}, \cdots, x_{in_i}^{(w)})'$, $(1 \leqq w \leqq p)$. Then, allowing regression coefficients to vary over level combinations, one may take the linear model,

$$E(Y_{ij})=\mu+\sum_{w=1}^{p} \gamma_{wi} x_{ij}^{(w)}+\tau_i\,, \qquad 1 \leqq j \leqq n_i \text{ and } 1 \leqq i \leqq v\,, \tag{2.1}$$

where τ_i is the effect due to i-th level combination, and the random variables Y_{ij} are homoscedastic and uncorrelated. Here, γ_{wi} $(1 \leqq w \leqq p$, $1 \leqq i \leqq v)$ is the regression coefficient of Y on $x^{(w)}$ in the presence of the i-th level combination. Define the covariate matrix as $\boldsymbol{X}_i^{(n_i \times p)}=(\boldsymbol{x}_i^{(1)}, \cdots, \boldsymbol{x}_i^{(p)})$, a matrix of order $n_i \times p$, for $i=1, \cdots, v$. Further, we define

$$\boldsymbol{D}_X^{(n\times pv)}=\begin{pmatrix} \boldsymbol{X}_1 & & & 0 \\ & \boldsymbol{X}_2 & & \\ & & \ddots & \\ 0 & & & \boldsymbol{X}_v \end{pmatrix} \quad \text{and} \quad \boldsymbol{G}^{(n\times v)}=\begin{pmatrix} \mathbf{1}_{n_1} & & & 0 \\ & \mathbf{1}_{n_2} & & \\ & & \ddots & \\ 0 & & & \mathbf{1}_{n_v} \end{pmatrix}.$$

Where $\mathbf{1}_{n_i}=(1, 1, \cdots, 1)'$ is the n_i dimensional vector with unit elements, and $n=n_1+\cdots+n_v$. Then the linear model (2.1) can be expressed in matrix notation as

$$E(\boldsymbol{Y})=\mu \mathbf{1}_n+\boldsymbol{D}_X \boldsymbol{\gamma}+\boldsymbol{G}\boldsymbol{\tau}=\boldsymbol{Z}\boldsymbol{\beta} \tag{2.2}$$

where $\boldsymbol{Z}=(\mathbf{1}_n, \boldsymbol{D}_X, \boldsymbol{G})$ is the design matrix and $\boldsymbol{\beta}=(\mu, \boldsymbol{\gamma}', \boldsymbol{\tau}')'$ is the parameters, and μ is the general mean, $\boldsymbol{\tau}=(\tau_1, \cdots, \tau_v)'$, $\boldsymbol{\gamma}_i^{(p\times 1)}=(\gamma_{1i}, \cdots, \gamma_{pi})'$ and $\boldsymbol{\gamma}^{(vp\times 1)}=(\boldsymbol{\gamma}_1', \cdots, \boldsymbol{\gamma}_v')'$.

Further, we may assume $V(\boldsymbol{Y})=\sigma^2 \boldsymbol{I}_n$ where $\boldsymbol{I}_n$ is the identity matrix of order n.

Thus, the coefficient matrix of normal equation for β follows as

$$Z'Z=\begin{pmatrix} n & 1_n'D_X & n' \\ D_X'1_n & D_X'D_X & W' \\ n & W & D_n \end{pmatrix}$$

where $D_n=\begin{bmatrix} n_1 & & & 0 \\ & n_2 & & \\ & & \ddots & \\ 0 & & & n_v \end{bmatrix}$, $n=\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_v \end{bmatrix}$ and $W^{(v\times pv)}=\begin{bmatrix} 1_{n_1}'X_1 & & 0 \\ & \ddots & \\ 0 & & 1_{n_v}'X_v \end{bmatrix}$.

It is reasonable to assume that the design is such that for each i, writing $\alpha_i=\mu+\tau_i$, from $Y_i=(Y_{i1},\cdots,Y_{in_i})'$ alone α_i should be estimable. This will happen when

$$\text{rank}\,(X_i,\,1_n)=\text{rank}\,(X_i)+1 \tag{2.3}$$

which implies that $X_id_i\neq 1_n$ for any vector d_i. It should be noted, however, that (2.3) does not imply that the vectors of covariate $x_i^{(1)}$, $\cdots$, $x_i^{(p)}$ are linearly independent.

Let $V=(1_n,\,D_X)'(1_n,\,D_X)$. Then the reduced normal equations for τ are seen to have the coefficient matrix

$$\begin{aligned} C&=G'G-G'(1_n,\,D_X)[(1_n,\,D_X)'(1_n,\,D_X)]^-(1_n,\,D_X)'G \\ &=D_n-(n,\,W)V^-(n,\,W)' \end{aligned} \tag{2.4}$$

where V^- is any generalized inverse matrix of V. Put $H=(1_n,\,D_X)$. Then using the Theorem 5 of Rao and Yanai [12], we have

$$H(H'H)^-H'=P_M+P_{D_X/M} \tag{2.5}$$

where $P_M=1_n1_n'/n$, $Q_M=I_n-P_M$ and $P_{D_X/M}=Q_MD_X(D_X'Q_MD_X)^-D_X'Q_M$ are the orthogonal projectors onto the subspaces $S(1_n)$, $S(1_n)^\perp$ and $S(Q_MD_X)$ respectively (see Takeuchi, Yanai and Mukherjee [14]).

Substituting (2.5) into (2.4), we have

$$\begin{aligned} C&=G'G-G'(P_M+P_{D_X/M})G=G'Q_MG-G'P_{D_X/M}G \\ &=G'Q_M(I_n-P_{D_X/M})Q_MG \end{aligned}$$

in view of the result $P_{D_X/M}Q_M=Q_MP_{D_X/M}=P_{D_X/M}$. Because of (2.3), $S(Q_MD_X)$ and $S(Q_MG)$ are disjoint. Thus, we have

$$\text{rank}\,(C)=\text{rank}\,(Q_MG)=\text{rank}\,(G)-1=v-1\,. \tag{2.6}$$

The result shows that all contrasts of the level combination effects are estimable, and it is easy to see that the matrix C have all row and column sum zero, since

$$Q_MG1_v=(I_n-1_n1_n'/n)G1_v=1_n-1_n(1_n'G1_v)/n=0\,.$$

To simplify the expression (2.4) for C, further note that

$$H(H'H)^{-}H' = P_{D_X} + P_{1_n/D_X} \tag{2.7}$$

where $P_{1_n/D_X} = Q_{D_X}1_n(1_n'Q_{D_X}1_n)^{-1}1_n'Q_{D_X}$, $Q_{D_X} = I_n - P_{D_X}$ and

$$P_{D_X} = D_X(D_X'D_X)^{-}D_X' = \begin{bmatrix} P_{X_1} & & & 0 \\ & P_{X_2} & & \\ & & \ddots & \\ 0 & & & P_{X_v} \end{bmatrix}, \qquad (P_{X_j} = X_j(X_j'X_j)^{-}X_j') .$$

Substitution of (2.7) into (2.4) yields

$$\begin{aligned} C &= G'G - G'(P_{D_X} + P_{1_n/D_X})G \\ &= (G'G - G'P_{D_X}G) - G'Q_{D_X}1_n(1_n'Q_{D_X}1_n)^{-1}1_n'Q_{D_X}G \\ &= D_\xi - \xi\xi'/f \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} & f = n - \sum_{i=1}^{v} \phi_i , \qquad \xi = (\xi_1, \cdots, \xi_v)' , \qquad \xi_i = n_i - \phi_i , \\ & \phi_i = 1_{n_i}'P_{X_i}1_{n_i} = 1_{n_i}'X_i(X_i'X_i)^{-}X_i'1_{n_i} \end{aligned} \tag{2.9}$$

and

$$D_\xi = \begin{pmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_v \end{pmatrix} ,$$

Further, because of the relationship (2.3), we note

$$\xi_i = n_i - \phi_i = 1_{n_i}'1_{n_i} - 1_{n_i}'P_{X_i}1_{n_i} = 1_{n_i}'Q_{X_i}1_{n_i} > 0 . \tag{2.10}$$

3. Condition for orthogonality

For $1 \leqq j \leqq m$, let us define $E_j = 1_{s_j}1_{s_j}'$, and let P_j be an $(s_j - 1) \times s_j$ matrix such that $(s_j^{-1/2}1_{s_j}, P_j')$ is orthogonal. Let Ω denote the class of m component nonnull vectors with elements 0 or 1. Then for any $a = (a_1, \cdots, a_m) \in \Omega$, writing

$$P^a = P_1^{a_1} \otimes P_2^{a_2} \otimes \cdots \otimes P_m^{a_m}$$

where $\otimes$ denotes Kronecker product and for $1 \leqq j \leqq m$,

$$P_j^{a_j} = \begin{cases} s_j^{-1/2}1_{s_j}' , & \text{if } a_j = 0 \\ P_j , & \text{if } a_j = 1 , \end{cases}$$

one can check that $P^a\tau$ represents a complete set of orthonormal contrasts belonging to the factorial effect $F_1^{a_1}\cdots F_m^{a_m}$ (cf. Kurkjian and Zelen [7], Mukerjee [10]). Denoting by $\boldsymbol{P}^a\hat{\boldsymbol{\tau}}$, i.e., the best linear unbiased estimator of $\boldsymbol{P}^a\boldsymbol{\tau}$, the design will satisfy factorial orthogonality if for every $a, b \in \Omega$ $(a\neq b)$,

$$\mathrm{Cov}\,(\boldsymbol{P}^a\hat{\boldsymbol{\tau}}, \boldsymbol{P}^b\hat{\boldsymbol{\tau}})=0 .$$

Following Mukerjee ([10], [11]), one gets a necessary and sufficient condition for the above stated below.

THEOREM 3.1. *The design satisfies factorial orthogonality if and only if for each* $a, b \in \Omega$, $a\neq b$, $\boldsymbol{P}^a\boldsymbol{C}(\boldsymbol{P}^b)'=\boldsymbol{0}$.

It is possible to give a combinatorial characterization of this necessary and sufficient condition. Thus, following precisely the steps in Mukerjee [11], if one interprets

$$\boldsymbol{P}^a\boldsymbol{C}(\boldsymbol{P}^b)'=\boldsymbol{0}$$

element by element for each $a, b\,(a\neq b) \in \Omega$, then one derives an equivalent form of Theorem 3.1 as stated below.

THEOREM 3.2. *The design satisfies factorial orthogonality if and only if*

$$\xi_i=n_i-\phi_i=n_i-\boldsymbol{1}'_{n_i}\boldsymbol{X}_i(\boldsymbol{X}_i'\boldsymbol{X}_i)^-\boldsymbol{X}_i'\boldsymbol{1}_{n_i}$$

is constant over i $(1\leqq i\leqq v)$.

The above condition can be anticipated from indirect consideration as well. In the conventional set-up with no covariate, the reduced normal equations for τ has the coefficient matrix

$$\boldsymbol{D}_n^*=\boldsymbol{D}_n-\boldsymbol{n}\boldsymbol{n}'/n \tag{3.1}$$

and a necessary and sufficient condition for factorial orthogonality is given by the standard equal frequency criterion (Mukerjee [11]) where n_i is constant over i $(1\leqq i\leqq v)$. The reduced normal equation in the present set-up has the coefficient matrix $\boldsymbol{C}$, which, by (2.8), can be obtained from (3.1) by replacing n_i by ξ_i $(1\leqq i\leqq v)$. It is, therefore, only natural that the necessary and sufficient condition for factorial orthogonality will also be given by an "equal frequency-type" criterion in terms of ξ_i's. This is precisely given by Theorem 3.2.

In the conventional set-up where covariates are not taken into account, Seber [13] derives a necessary and sufficient condition for factorial orthogonality in terms of the proportional frequency criterion. Unlike our definition, his definition of contrasts belonging to factorial effects involves the replication numbers n_i's and is, therefore, different from ours.

While Theorem 3.2 generalizes the equal frequency criterion for orthogonality to the present set-up, it is also possible to generalize the proportional frequency criterion considering a particular situation described below.

Suppose prior knowledge is available regarding the absence of factorial interactions. Thus, if Ω^* be the subset of Ω comprising the multiples with at least two variates, one gets

$$P^a\boldsymbol{\tau}=\mathbf{0}\,, \qquad \text{for any } a\in\Omega^*\,. \tag{3.2}$$

Clearly, then interest lies in the estimation of only the factorial main effects; i.e., $P^a\boldsymbol{\tau}$ for $a\in(\Omega-\Omega)^*$, i.e.; $\boldsymbol{\tau}^*=P\boldsymbol{\tau}$ where $P'=(P^{(0,\cdots,0,1)'}, P^{(0,\cdots,1,0)'},\cdots, P^{(1,\cdots,0,0)'})$. Under (3.2) together with $\sum_{i=1}^{v}\tau_i=0$, if one proceeds as in Mukerjee [11] to derive the normal equations, then after some simplification, the coefficient matrix of the reduced normal equations for τ^* can be obtained simply as PCP' where C is as defined in (2.8).

The following analogue of Theorem 3.1 is evident.

THEOREM 3.3. *In the absence of factorial interactions, the design satisfies factorial orthogonality if and only if for each a and $b\in\Omega-\Omega^*$, $a\neq b$, $\boldsymbol{P}^a\boldsymbol{C}(\boldsymbol{P}^b)'=\mathbf{0}$.*

To derive a combinatorial characterization of Theorem 3.3, recall that a typical level combination is denoted by $i=(i_1,\cdots,i_m)$. Hence writing $\xi_i=\xi_{i_1\cdots i_m}$, define for $1\leqq j\leqq m$, $0\leqq i_j\leqq s_j-1$,

$$\xi_{i_j}^{(j)}=\sum_{i_1=0}^{s_1-1}\cdots\sum_{i_{j-1}=0}^{s_{j-1}-1}\sum_{i_{j+1}=0}^{s_{j+1}-1}\cdots\sum_{i_m=0}^{s_m-1}\xi_{i_1\cdots i_{j-1}i_ji_{j+1}\cdots i_m}$$

i.e. $\xi_{i_j}^{(j)}$ represents the marginal total obtained by summing $\xi_{i_1\cdots i_m}$ over all suffixes other than i_j. In a similar fashion, for $1\leqq j<k\leqq m$, $0\leqq i_l\leqq s_l-1$ $(l=j,k)$, the marginal total $\xi_{i_ji_k}^{(j,k)}$ may be defined by summing $\xi_{i_1\cdots i_m}$ over all suffixes other than i_j and i_k. For example, if $m=3$, then $\xi_{i_1i_3}^{(1,3)}=\sum_{i_2=0}^{s_2-1}\xi_{i_1i_2i_3}$ and so on. Similarly, interpreting ϕ_i and n_i as $\phi_i=\phi_{i_1\cdots i_m}$ and $n_i=n_{i_1\cdots i_m}$, define the marginal totals $\phi_{i_j}^{(j)}, n_{i_j}^{(j)}$ $(1\leqq j\leqq m$, $0\leqq i_j\leqq s_j-1)$ and $\phi_{i_ji_k}^{(j,k)}, n_{i_ji_k}^{(j,k)}$ $(1\leqq j<k\leqq m$, $0\leqq i_l\leqq s_l-1$, $l=j,k)$. Clearly it follows that

$$\xi_{i_j}^{(j)}=n_{i_j}^{(j)}-\phi_{i_j}^{(j)}\,, \tag{3.3}$$

$$\xi_{i_ji_k}^{(j,k)}=n_{i_ji_k}^{(j,k)}-\phi_{i_ji_k}^{(j,k)}\,. \tag{3.4}$$

Then again as in Mukerjee [11] if one interprets $\boldsymbol{P}^a\boldsymbol{C}(\boldsymbol{P}^b)'=\mathbf{0}$ element by element for each a, b $(a\neq b)\in \Omega-\Omega^*$, the following equivalent form of Theorem 3.3 is obtained.

THEOREM 3.4. *In the absence of factorial interactions the design satisfies factorial orthogonality if and only if for every* $1\leq j<k\leq m$ *and* $0\leq i_l\leq s_l-1$ $(l=j, k)$, *the relation*

$$\xi_{i_j i_k}^{(j,k)}=\xi^{-1}\xi_{i_j}^{(j)}\xi_{i_k}^{(k)}\,, \tag{3.5}$$

where $\xi=\sum_{i_1}\cdots\sum_{i_m}\xi_{i_1\cdots i_m}$, *holds.*

The above is just a generalization of the standard proportional frequency criterion for factorial orthogonality in the conventional set-up (where covariates are not taken into account) under the absence of interactions (see, e.g., Addelman [1] and Mukerjee [11]). It may be noted, in this connection, that in the conventional setting the proportional frequency criterion is stated in terms of $n_{i_1\cdots i_m}$'s, which, in view of (2.8) and (3.1), are replaced by $\xi_{i_1\cdots i_m}$'s in the present set-up.

4. Some allied results

In the above development the regression coefficients of Y on the covariates vary over the level combinations. Suppose now such variations are not allowed and the regression coefficients, say $\gamma_1,\cdots,\gamma_p$ are the same for all level combinations. The notations, unless otherwise stated, remain the same as before.

Then defining,

$$\bar{x}^{(w)}=n^{-1}\sum_{i=1}^{v}\sum_{j=1}^{n_i}x_{ij}^{(w)},\ \boldsymbol{x}^{(w)(n\times 1)}=(\boldsymbol{x}_1^{(w)\prime},\cdots,\boldsymbol{x}_v^{(w)\prime})'$$

where $w=1,\cdots,p$,

$\boldsymbol{\gamma}^*=(\gamma_1,\cdots,\gamma_p)'$, a vector of order p,

$\boldsymbol{X}=[\boldsymbol{x}^{(1)}-\bar{x}^{(1)}\boldsymbol{1}_n,\cdots,\boldsymbol{x}^{(p)}-\bar{x}^{(p)}\boldsymbol{1}_n]$, a matrix of order $n\times p$,

one can take the linear model

$$E(\boldsymbol{Y})=\mu\boldsymbol{1}_n+\boldsymbol{X}\boldsymbol{\gamma}^*+\boldsymbol{G}\boldsymbol{\tau}=\boldsymbol{Z}^*\boldsymbol{\beta}^* \quad\text{and}\quad V(\boldsymbol{Y})=\sigma^2\boldsymbol{I}_n \tag{4.1}$$

where $\boldsymbol{Z}^*(\boldsymbol{1}_n, \boldsymbol{X}, \boldsymbol{G})$ and $\boldsymbol{\beta}^*=[\mu, (\boldsymbol{\gamma}^*)', (\boldsymbol{\tau})']'$. As before, the standard restriction $\sum_{i=1}^{v}\tau_i=0$ is there.

Then the coefficient matrix of normal equation for β^* is easily seen to be

$$(\boldsymbol{Z}^*)'(\boldsymbol{Z}^*)=\begin{pmatrix} n & \boldsymbol{0}' & \boldsymbol{n}' \\ \boldsymbol{0} & \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{U}' \\ \boldsymbol{n} & \boldsymbol{U} & \boldsymbol{D}_n \end{pmatrix}$$

where $\boldsymbol{U}^{(v\times p)}=\boldsymbol{G}'\boldsymbol{X}=[(u_{iw})]$, $u_{iw}=n_i(\bar{x}_i^{(w)}-\bar{x}^{(w)})$ and $\bar{x}_i^{(w)}=n_i^{-1}\sum_{j=1}^{n_i} x_{ij}^{(w)}$ $(1\leq \leq i\leq v,\ 1\leq w\leq p)$.

Now, for estimating all contrasts of $\boldsymbol{\tau}$, it is necessary and sufficient that

$$\text{rank}\,(X, G)=\text{rank}\,(X)+v\,. \tag{4.3}$$

Thus, the reduced normal equations for $\boldsymbol{\tau}$ may be seen to have the coefficient matrix

$$\begin{aligned} \boldsymbol{C}^* &= \boldsymbol{G}'\boldsymbol{G}-\boldsymbol{G}'(\boldsymbol{1}_n, \boldsymbol{X})[(\boldsymbol{1}_n, \boldsymbol{X})'(\boldsymbol{1}_n, \boldsymbol{X})]^-(\boldsymbol{1}_n, \boldsymbol{X})'\boldsymbol{G} \\ &= \boldsymbol{G}'\boldsymbol{G}-\boldsymbol{G}'(\boldsymbol{P}_M+\boldsymbol{Q}_M\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{Q}_M\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Q}_M)\boldsymbol{G} \\ &= \boldsymbol{G}'\boldsymbol{Q}_M\boldsymbol{G}-\boldsymbol{G}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{G} \\ &= \boldsymbol{D}_n-n^{-1}\boldsymbol{n}\boldsymbol{n}'-\boldsymbol{U}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{U}'\,, \end{aligned}$$

in view of $\boldsymbol{Q}_M\boldsymbol{X}=\boldsymbol{X}$ and using the decomposition theorem of the orthogonal projector. Observe that the matrix $\boldsymbol{C}^*$ has rank $v-1$ and, as usual, all row and column sums zero.

Then, analogously to Theorem 3.1, in the general setting with no assumption regarding the absence of interactions it follows that the design satisfies factorial orthogonality if and only if for each $a, b \in \Omega$, $a\neq b$, $\boldsymbol{P}^a\boldsymbol{C}^*(\boldsymbol{P}^b)'=\boldsymbol{0}$. The matrix $\boldsymbol{C}^*$ is somewhat complicated and even if one follows the lines of the earlier researchers, a combinatorial characterization of this condition cannot be obtained in general. For example, if as in Mukerjee [11] one tries to interpret this condition element by element for each $a, b \in \Omega$ $(a\neq b)$, the resulting expressions will involve the covariates in a manner too complex to comprehend, thus defeating the very purpose of further reduction.

If, however, prior knowledge is available regarding the absence of factorial interactions, one can obtain a combinatorial characterization. In that case, analogously to Theorem 3.3, it is seen that the design satisfies factorial orthogonality if and only if for each $a, b \in \Omega-\Omega^*$, $a\neq b$,

$$\boldsymbol{P}^a\boldsymbol{C}^*(\boldsymbol{P}^b)'=\boldsymbol{0}\,. \tag{4.4}$$

While deriving a combinatorial characterization of (4.4), first for the sake of notational simplicity the case $m=2$ will be taken up, although the approach is quite general and can be easily extended to the case with any number of m.

With $m=2$, recall that a typical level combination is $i=(i_1, i_2)$ and it is replicated $n_i = n_{i_1 i_2}$ times. For $0 \leqq i_j \leqq s_j - 1$ $(j=1, 2)$, define

$$n_{i_1 \cdot} = \sum_{i_2=0}^{s_2-1} n_{i_1 i_2} \qquad \text{and} \qquad n_{\cdot i_2} = \sum_{i_1=0}^{s_1-1} n_{i_1 i_2}$$

(note that with the notations of the preceeding section, $n_{i_1 \cdot}$ and $n_{\cdot i_2}$ are the same as $n_{i_1}^{(1)}$ and $n_{i_2}^{(2)}$ respectively). Also we can write $x_{ij}^{(w)}$ as $x_{(i_1, i_2) j}^{(w)}$ $(1 \leqq j \leqq n_{i_1 i_2})$ for each w, since $i=(i_1, i_2)$. Similarly, $\bar{x}_i^{(w)}$ may be written as $\bar{x}_{i_1 i_2}^{(w)}$. Define for $0 \leqq i_1 \leqq s_1 - 1$

$$\bar{x}_{i_1 \cdot}^{(w)} = \sum_{i_2=0}^{s_2-1} n_{i_1 i_2} \bar{x}_{i_1 i_2}^{(w)} / n_{i_1 \cdot} , \qquad 1 \leqq w \leqq p$$

as the mean of $x^{(w)}$ from all experimental units receiving the level i_1 of the first factor. Similarly, define for $0 \leqq i_2 \leqq s_2 - 1$,

$$\bar{x}_{i_2 \cdot}^{(w)} = \sum_{i_1=0}^{s_1-1} n_{i_1 i_2} \bar{x}_{i_1 i_2}^{(w)} / n_{\cdot i_2} , \qquad 1 \leqq w \leqq p .$$

Now, the condition (4.4) with $m=2$, $a=(1, 0)$ and $b=(0, 1)$ yields

$$(4.5) \qquad (\boldsymbol{P}_1 \otimes \boldsymbol{1}'_{s_2}) \boldsymbol{C}^* (\boldsymbol{1}_{s_1} \otimes \boldsymbol{P}'_2) = \boldsymbol{0} .$$

By definition, it follows that

$$\boldsymbol{P}'_j \boldsymbol{P}_j = \boldsymbol{I}_{s_j} - s_j^{-1} \boldsymbol{E}_j \qquad (j=1, 2) .$$

On pre and postmultiplication of (4.5) by $\boldsymbol{P}'_1$ and $\boldsymbol{P}_2$ respectively, one gets

$$[(\boldsymbol{I}_{s_1} - s_1^{-1} \boldsymbol{E}_1) \otimes \boldsymbol{1}'_{s_2}] \boldsymbol{C}^* [\boldsymbol{1}_{s_1} \otimes (\boldsymbol{I}_{s_2} - s_2^{-1} \boldsymbol{E}_2)] = \boldsymbol{0} ,$$

whence, recalling that $\boldsymbol{C}^*$ has all row and column sums zero,

$$(\boldsymbol{I}_{s_1} \otimes \boldsymbol{1}'_{s_2}) \boldsymbol{C}^* (\boldsymbol{1}_{s_1} \otimes \boldsymbol{I}_{s_2}) = \boldsymbol{0} ,$$

i.e., by (4.2),

$$(4.6) \qquad \begin{aligned} &(\boldsymbol{I}_{s_1} \otimes \boldsymbol{1}'_{s_2}) \boldsymbol{D}_n (\boldsymbol{1}_{s_1} \otimes \boldsymbol{I}_{s_2}) \\ &\quad = (\boldsymbol{I}_{s_1} \otimes \boldsymbol{1}'_{s_2}) [n^{-1} \boldsymbol{n}\boldsymbol{n}' + \boldsymbol{U} (\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{U}'] (\boldsymbol{1}_{s_1} \otimes \boldsymbol{I}_{s_2}) . \end{aligned}$$

Each side of (4.6) is an $s_1 \times s_2$ matrix. For $0 \leqq i_j \leqq s_j - 1$ $(j=1, 2)$, the (i_1, i_2)-th element of the left hand side of (4.6) can be seen to be $n_{i_1 i_2}$. Similarly, the (i_1, i_2)-th element of

$$(\boldsymbol{I}_{s_1} \otimes \boldsymbol{1}'_{s_2}) (n^{-1} \boldsymbol{n}\boldsymbol{n}') (\boldsymbol{1}_{s_1} \otimes \boldsymbol{I}_{s_2})$$

is $n^{-1} n_{i_1 \cdot} n_{\cdot i_2}$. Also observe that

$$(\boldsymbol{I}_{s_1} \otimes \boldsymbol{1}'_{s_2}) \boldsymbol{U} = (\boldsymbol{I}_{s_1} \otimes \boldsymbol{1}'_{s_2}) \boldsymbol{G}' \boldsymbol{X}$$

$$=\begin{pmatrix} n_{0\cdot}(\bar{x}_{0\cdot}^{(1)}-\bar{x}^{(1)}) & n_{0\cdot}(\bar{x}_{0\cdot}^{(2)}-\bar{x}^{(2)}) & \cdots & n_{0\cdot}(\bar{x}_{0\cdot}^{(p)}-\bar{x}^{(p)}) \\ n_{1\cdot}(\bar{x}_{1\cdot}^{(1)}-\bar{x}^{(1)}) & n_{1\cdot}(\bar{x}_{1\cdot}^{(2)}-\bar{x}^{(2)}) & \cdots & n_{1\cdot}(\bar{x}_{1\cdot}^{(p)}-\bar{x}^{(p)}) \\ \cdots & \cdots & & \cdots \\ n_{t\cdot}(\bar{x}_{t\cdot}^{(1)}-\bar{x}^{(1)}) & n_{t\cdot}(\bar{x}_{t\cdot}^{(2)}-\bar{x}^{(2)}) & \cdots & n_{t\cdot}(\bar{x}_{t\cdot}^{(p)}-\bar{x}^{(p)}) \end{pmatrix}$$

where $t=s_1-1$ and a similar expression may be obtained for $U'(1_{s_1}\otimes I_{s_2})$. Hence, if one writes $T=n^{-1}X'X$ and $T^{-}=[(t^{wq})]$ for any generalized inverse matrix of T, one gets, after some simplification, that (i_1, i_2)-th element of

$$(I_{s_1}\otimes 1'_{s_2})[U(X'X)^{-}U'](1_{s_1}\otimes I_{s_2})$$

as

$$n^{-1}n_{i_1\cdot}n_{\cdot i_2}\sum_{w=1}^{p}\sum_{q=1}^{p}t^{wq}(\bar{x}_{i_1\cdot}^{(w)}-\bar{x}^{(w)})(\bar{x}_{\cdot i_2}^{(q)}-\bar{x}^{(q)}) .$$

Thus (4.6) yields

$$n_{i_1i_2}=n^{-1}n_{i_1\cdot}n_{\cdot i_2}\left[1+\sum_{w=1}^{p}\sum_{q=1}^{p}t^{wq}(\bar{x}_{i_1\cdot}^{(w)}-\bar{x}^{(w)})(\bar{x}_{\cdot i_2}^{(q)}-\bar{x}^{(q)})\right] \tag{4.7}$$

where $0\leqq i_j\leqq(s_j-1)$, $j=1, 2$. Since the above steps are reversible, (4.7) is not only a necessary but also a sufficient condition for orthogonality for $m=2$ under the absence of interactions. Observe that $n_{i_1i_2}=n^{-1}n_{i_1\cdot}n_{\cdot i_2}$ follows if $\bar{x}_{i_1\cdot}^{(w)}=\bar{x}^{(w)}$ holds for $w=1, 2, \cdots, p$.

To extend the above result to general m, define $n_{i_j}^{(j)}$ ($1\leqq j\leqq m$, $0\leqq i_j\leqq s_j-1$) and $n_{i_ji_k}^{(j,k)}$ ($1\leqq j<k\leqq m$, $0\leqq i_l\leqq s_l-1$, $l=j, k$) as in Section 3. Also generalizing notations $\bar{x}_{i_1\cdot}^{(w)}$ and $\bar{x}_{\cdot i_2}^{(w)}$ introduced earlier, denote by $\bar{x}_{i_j}^{(j/w)}$ the mean of $x^{(w)}$ from all experimental units where the level i_j of the j-th factor has been applied ($0\leqq i_j\leqq s_j-1$, $1\leqq j\leqq m$ and $1\leqq w\leqq p$). Then one gets the following generalization of (4.7).

THEOREM 4.1. *In the absence of factorial interactions, design satisfies factorial orthogonality if and only if for every* $1\leqq j<k\leqq m$ *and* $0\leqq i_l\leqq s_l-1$ $(l=j, k)$, *the condition*

$$n_{i_ji_k}^{(j,k)}=n^{-1}n_{i_j}^{(j)}n_{i_k}^{(k)}\left[1+\sum_{w=1}^{p}\sum_{q=1}^{p}t^{wq}(\bar{x}_{i_j}^{(j/w)}-\bar{x}^{(w)})(\bar{x}_{i_k}^{(k/q)}-\bar{x}^{(q)})\right] \tag{4.8}$$

holds.

The above is also some sort of an extension of the proportional frequency criteria to the present set-up.

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INDIAN STATISTICAL INSTITUTE
NATIONAL CENTER OF ENTRANCE EXAMINATION FOR UNIVERSITY

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