Ann. Inst. Statist. Math.
38 (1986), Part A, 297-309

ON AN AUTOREGRESSIVE MODEL WITH TIME-DEPENDENT COEFFICIENTS

GEA HWA KWOUN AND YOSHIHIRO YAJIMA

(Received Sep. 6, 1984; revised Mar. 5, 1985)

Summary

As one of the non-stationary time series model, we consider a first-order autoregressive model in which the autoregressive coefficient is assumed to be a function, $f_t(\theta)$, of time t. We establish several assumptions on $f_t(\theta)$, not on the terms in the Taylor expansion of log-likelihood function, and show that the estimators of unknown parameters involved in $f_t(\theta)$ have strong consistency and asymptotic normality under these assumptions when sample size tends to infinity.

1. Introduction

The usual assumptions that are made in the time series analysis are that the time series are stationary and the structure of the series can be described by a linear model which has constant coefficients such as an autoregressive (AR) model. In general, these models deal with the case when the structure of the series does not change over time. But, too often, we actually encounter time series whose structure varies over time.

For this reason, it seems natural to generalize this AR model by considering the case when the coefficients are themselves slowly moving through time as the structure of the series changes. In recent works about such models, Ozaki [6] has investigated an AR model whose coefficients are functions of an observation $X(t)$ at time t and Nicholls and Quinn [5] have introduced a random coefficient autoregressive (RCA) model, in which the coefficients are subject to random perturbations. Priestley [7] pointed out that these state-dependent models can provide not only better fits to the data, but more importantly, they can reveal interesting behaviour patterns (such as limit cycles) which can never be captured by classical linear models.

Key words: Time-dependent coefficients, strong consistency, asymptotic normality.

In this paper we consider an autoregressive time series in which the coefficients are assumed to be a function of time t. Quinn and Nicholls derived conditions for the stationarity of RCA model and under these conditions they considered the properties of estimates. But obviously our case is non-stationary. We can see the variance of the process changes over time and depends on t. We show that under certain conditions the least squares estimators of unknown parameters are strongly consistent and asymptotically normally distributed.

The model we consider is

$$X_t = f_t(\theta) X_{t-1} + e_t \,, \tag{1.1}$$

where $\{e_t, t=1, 2, \cdots\}$ is a sequence of independent identically distributed random variables with means zero, variances σ^2 and Ee_t^4 exist for any t, and θ is a $p \times 1$ vector of unknown parameters. $f_t(\theta)$ is a real valued function defined on a compact set Θ which contains the true value vector θ_0 as an inner point and is a subset of R^p. The values of θ_0 and σ^2 are unknown, so we have to estimate θ_0 and σ^2.

We assume the following conditions;

(1) There is a constant α (>0) such that

$$\sum_{j=1}^{t} \left\{ \prod_{i=0}^{j-1} f_{t-i}^2(\theta) \right\} \leqq \alpha$$

for any $t \in T = \{1, 2, \cdots\}$ and $\theta \in \Theta$.

(2) The derivatives $f'_{t,i}(\theta) = \dfrac{\partial f_t(\theta)}{\partial \theta_i}$, $f''_{t,ij}(\theta) = \dfrac{\partial^2 f_t(\theta)}{\partial \theta_i \partial \theta_j}$ and $f'''_{t,ijk}(\theta) = \dfrac{\partial^3 f_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k}$ $(i, j, k = 1, 2, \cdots, p)$ exist and are bounded for any $t \in T$ and $\theta \in \Theta$.

(3) For each θ ($\neq \theta_0$), there exists a set A ($\subset T$), which may depend on θ, and a positive ε_θ such that

$$|f_t(\theta_0) - f_t(\theta)| \geqq \varepsilon_\theta \,, \qquad \text{for any } t \in A$$

and

$$\liminf_{N \to \infty} \frac{\#(A_N)}{N} > 0 \,,$$

where $\#(A_N)$ denotes the number of elements in $A_N = A \cap \{1, 2, \cdots, N\}$.

(4) There exists a limit of $N^{-1} \sum_{t=2}^{N} \{f_t(\theta_0) - f_t(\theta)\}^2 \mathrm{E}\, X_{t-1}^2$.

(5) We suppose that for $i, j = 1, 2, \cdots, p$,

$$\lim_{N \to \infty} N^{-1} \sum_{t=2}^{N} f'_{t,i}(\theta) f'_{t,j}(\theta) \mathrm{E}\, X_{t-1}^2$$

exists, and if we denote it by $a_{i,j}(\theta)$ the matrix $A(\theta)=[a_{i,j}(\theta)]$ is non-singular.

The proof of the strong consistency requires the condition (1), (3), (4), and the condition that $f'_{t,i}(\theta)$ exists and is bounded for any $t \in T$ and $\theta \in \Theta$. To establish the asymptotic normality the remainder conditions are added.

The X_t of (1.1) can be expressed in terms of e_t and a given initial value X_0 as

$$X_t=\prod_{i=1}^{t} f_i(\theta)X_0+\sum_{j=0}^{t-1}\left\{\prod_{i=0}^{j-1} f_{t-i}(\theta)\right\}e_{t-j}\,.$$

Hereafter, we define $\prod_{k=i}^{j} f_k(\theta)=1$ whenever $j<i$. We assume X_0 is independent of e_t and satisfies $\mathrm{E}\,X_0^4<\infty$. Moreover we assume $\mathrm{E}\,X_0=0$, which entails no loss of generality. Thus we have

$$\begin{aligned}\mathrm{E}\,X_t&=\prod_{i=1}^{t} f_i(\theta)\,\mathrm{E}\,X_0\\&=0\end{aligned}$$

and

$$\mathrm{Var}\,X_t=\prod_{i=1}^{t} f_i^2(\theta)\,\mathrm{Var}\,X_0+\sum_{j=0}^{t-1}\left\{\prod_{i=0}^{j-1} f_{t-i}^2(\theta)\right\}\sigma^2\,. \tag{1.2}$$

We call $\{X_t\}$ stable if $\mathrm{Var}\,X_t$ is bounded. Then $\{X_t\}$, which is generated by (1.1) and satisfies assumption (1), is stable.

We suppose that observations $\{X_1, X_2, \cdots, X_N\}$ are generated by the model (1.1). The least squares estimator $\hat{\theta}_N$ of θ_0 minimizes

$$\begin{aligned}Q_N(\theta)&=N^{-1}\sum_{t=2}^{N}(X_t-f_t(\theta)X_{t-1})^2\\&=N^{-1}\sum_{t=2}^{N}[\{f_t(\theta_0)-f_t(\theta)\}X_{t-1}+e_t]^2\end{aligned}$$

with respect to θ. The estimator $\hat{\sigma}_N^2$ of σ^2 is $Q_N(\hat{\theta}_N)$.

In establishing asymptotic properties of estimators, $Q_N(\theta)$ is often expanded in a third order Taylor series about θ_0 and the properties are shown under assumption that each term of Taylor series converges almost surely or in probability (see for example Klimko and Nelson [4]). The purpose of this paper can be considered to derive some conditions on $f_t(\theta)$ for the convergence of the terms and to prove that the estimators have good asymptotic properties under these conditions.

2. The strong consistency of the estimators $\hat{\theta}_N$ and $\hat{\sigma}_N^2$

To prove the main Theorems 1 and 2 we need the following lemmas.

LEMMA 1. *Under the conditon* (1),

$$N^{-1}\sum_{t=2}^{N}\{f_t(\theta_0)-f_t(\theta)\}X_{t-1}e_t$$

converges almost surely to zero as $N\to\infty$.

PROOF. Let $g_t(\theta)=f_t(\theta_0)-f_t(\theta)$ for convenience. Then there exists a constant α_1 (>0) such that

$$\begin{aligned}\mathrm{E}\left(N^{-1}\sum_{t=2}^{N}g_t(\theta)X_{t-1}e_t\right)^2&=N^{-2}\sum_{t=2}^{N}g_t^2(\theta)\,\mathrm{E}\,X_{t-1}^2\sigma^2\\&\leqq N^{-1}\alpha_1\end{aligned}$$

for any $t\in T$ and $\theta\in\Theta$. In this paper, the expectation is taken with respect to the distribution indexed by $\theta_0\in\Theta$. This lemma can be proved by making use of the same argument as in the Proof of Theorem X 6.2 of Doob [3].

LEMMA 2. *We assume the conditions* (1) *and* (4). *Then*

$$q(\theta)=\lim_{N\to\infty}N^{-1}\sum_{t=2}^{N}g_t^2(\theta)X_{t-1}^2 \tag{2.1}$$

exists almost surely. If we assume the condition (3) *furthermore then* $q(\theta)$ *is uniquely minimized at* $\theta=\theta_0$.

PROOF. If we show $\mathrm{Var}\left(N^{-1}\sum_{t=2}^{N}g_t^2(\theta)X_{t-1}^2\right)=O(N^{-1})$, it can be proved that

$$\lim_{N\to\infty}\left(N^{-1}\sum_{t=2}^{N}g_t^2(\theta)X_{t-1}^2-N^{-1}\sum_{t=2}^{N}g_t^2(\theta)\,\mathrm{E}\,X_{t-1}^2\right)=0$$

with probability 1. This result also follows from the proof of Theorem X 6.2 of Doob [3]. We therefore consider the following variance.

$$\begin{aligned}&\mathrm{Var}\left(N^{-1}\sum_{t=2}^{N}g_t^2(\theta)X_{t-1}^2\right)\\&=N^{-2}\Bigg[2\sum_{t=1}^{N-1}(g_{t+1}^2\,\mathrm{E}\,X_t^2)^2+4\sum_{t=2}^{N-1}\sum_{i=1}^{t-1}g_{t+1}^2g_{t+1-i}^2\prod_{j=0}^{i-1}f_{t-j}^2(\mathrm{E}\,X_{t-i}^2)^2\\&\quad+\sum_{t=1}^{N-1}g_{t+1}^4\,\mathrm{Cum}\,(X_t,X_t,X_t,X_t)\\&\quad+2\sum_{t=2}^{N-1}\sum_{i=1}^{t-1}g_{t+1}^2g_{t+1-i}^2\,\mathrm{Cum}\,(X_t,X_t,X_{t-i},X_{t-i})\Bigg].\end{aligned} \tag{2.2}$$

It is easily seen the first and second terms of (2.2) is $O(N^{-1})$. As we have already assumed, $\{e_t\}$ are i.i.d. random variables and independent of X_0. Then, we have

$$
\begin{aligned}
(2.3)\quad & |\mathrm{Cum}\,(X_t, X_t, X_t, X_t)| \\
&= \Bigg|\mathrm{Cum}\Bigg[\sum_{j=0}^{t-1}\Bigg(\prod_{i=0}^{j-1} f_{t-i}(\theta)\Bigg)e_{t-j},\ \sum_{j=0}^{t-1}\Bigg(\prod_{i=0}^{j-1} f_{t-i}(\theta)\Bigg)e_{t-j}, \\
&\qquad\qquad \sum_{j=0}^{t-1}\Bigg(\prod_{i=0}^{j-1} f_{t-i}(\theta)\Bigg)e_{t-j},\ \sum_{j=0}^{t-1}\Bigg(\prod_{i=0}^{j-1} f_{t-i}(\theta)\Bigg)e_{t-j}\Bigg]\Bigg| \\
&\quad + \Bigg|\mathrm{Cum}\Bigg[\prod_{i=0}^{t-1} f_{t-i}X_0,\ \prod_{i=0}^{t-1} f_{t-i}X_0,\ \prod_{i=0}^{t-1} f_{t-i}X_0,\ \prod_{i=0}^{t-1} f_{t-i}X_0\Bigg]\Bigg| \\
&= \sum_{j=0}^{t-1}\Bigg(\prod_{i=0}^{j-1} f_{t-i}^4(\theta)\Bigg)|\mathrm{Cum}\,(e_{t-j}, e_{t-j}, e_{t-j}, e_{t-j})| \\
&\quad + \prod_{i=0}^{t-1} f_{t-i}^4(\theta)|\mathrm{Cum}\,(X_0, X_0, X_0, X_0)| \\
&\leqq L
\end{aligned}
$$

for a suitable constant L (>0). According to Theorem 2.3.1 of Brillinger [1], (2.3) holds. We have the last inequality by using the assumptions (1), $\mathrm{E}\,e_t^4<\infty$ and $\mathrm{E}\,X_0^4<\infty$. Iterating equation (1.1) $t-t'-1$ times, we obtain

$$X_t=\sum_{j=0}^{t-t'-1}\Bigg(\prod_{i=0}^{j-1} f_{t-i}\Bigg)e_{t-j}+\prod_{i=0}^{t-t'-1} f_{t-i}X_{t'}$$

for $t>t'$. Hence similarly we have

$$
\begin{aligned}
\Bigg|\sum_{i=1}^{t-1}\mathrm{Cum}\,(X_t, X_t, X_{t-i}, X_{t-i})\Bigg| &= \sum_{j=1}^{t-1}\Bigg(\prod_{j=0}^{i-1} f_{t-j}^2\Bigg)|\mathrm{Cum}\,(X_{t-i}, X_{t-i}, X_{t-i}, X_{t-i})| \\
&\leqq L\,.
\end{aligned}
$$

Thus the other terms of (2.2) is $O(N^{-1})$, too. From (2.1) $q(\theta)$ becomes zero when $\theta=\theta_0$. Next suppose $q(\theta)$ is to be minimized at $\theta\neq\theta_0$. Then since

$$\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}\{f_{t+1}(\theta_0)-f_{t+1}(\theta)\}^2X_t^2=0$$

almost surely, we have

$$\lim_{N\to\infty}\frac{\#(A_N)}{N}\,\frac{1}{\#(A_N)}\sum_{t\in A_N}\varepsilon_\theta^2X_t^2=0$$

almost surely. However, we assumed

$$\liminf_{N\to\infty}\frac{\#(A_N)}{N}>0$$

and we have

$$\liminf_{N\to\infty} \frac{1}{\#(A_N)} \sum_{t\in A_N} X_t^2 \geqq \lim_{N\to\infty} \frac{1}{\#(A_N)} \sum_{t\in A_N} (2f_t(\theta)X_{t-1}e_t + e_t^2) = \sigma^2$$

almost surely. Hence we have a contradiction and $q(\theta)$ is uniquely minimized at $\theta=\theta_0$.

Consequently, it is proved that $Q_N(\theta)$ converges almost surely to $q(\theta)+\sigma^2$ as $N\to\infty$ by Lemmas 1 and 2. We are now to prove the strong consistency of estimators.

THEOREM 1. *Let $\hat{\theta}_N$ and $\hat{\sigma}_N^2$ be the least squares estimators of θ_0 and σ^2. Under the conditions* (1)–(4), *$\hat{\theta}_N$ and $\hat{\sigma}_N^2$ are strongly consistent estimators of θ_0 and σ^2.*

PROOF. Let $\Theta_0=\{\theta_i,\ i=1, 2, \cdots\}$ be a countable dense set of Θ, and we put

$$\Omega_{\theta_i}=\{\omega\,;\lim_{N\to\infty} Q_N(\theta_i)=q(\theta_i)+\sigma^2\}\,,$$

$$\Omega_\alpha=\left\{\omega\,;\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}(X_t^2-\mathrm{E}\,X_t^2)=0\right\},$$

$$\Omega_\beta=\left\{\omega\,;\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}e_t^2=\sigma^2\right\},$$

and

$$\Omega_t=\{\omega\,;|e_t|<\infty\}\,.$$

We proved that $\Pr(\Omega_{\theta_i})$, $\Pr(\Omega_\alpha)$ and $\Pr(\Omega_\beta)$ are 1 respectively in Lemmas 1 and 2. Furthermore $\mathrm{E}\,e_t<\infty$, $t=1, 2, \cdots$, imply $\Pr\left(\bigcap_{t=1}^{\infty}\Omega_t\right)=1$. Thus if we put

$$\Gamma=\bigcap_{\theta_i\in\Theta_0}\Omega_{\theta_i}\cap\Omega_\alpha\cap\Omega_\beta\cap\left(\bigcap_{t=1}^{\infty}\Omega_t\right),$$

we have

$$\Pr(\Gamma)=1\,.$$

Hereafter we consider for a fixed sample $\omega\in\Gamma$. First we need to show that $\{Q_N(\theta)\}$ uniformly converges to $q(\theta)+\sigma^2$ on Θ for any fixed $\omega\in\Gamma$. That is, we need to show that for any subsequence $\{Q_{N'}(\theta)\}$ of $\{Q_N(\theta)\}$ there exists a subsequence $\{Q_{N''}(\theta)\}$ of $\{Q_{N'}(\theta)\}$ such that $\lim_{N''\to\infty} Q_{N''}(\theta)=q(\theta)+\sigma^2$ uniformly on Θ. The proof consists of two stages. For any continuous function $\psi(\theta)$ on Θ, $\psi(\theta)\in\{Q_N(\theta)\}$, we define $\|\psi(\theta)\| =\sup_{\theta\in\Theta}|\psi(\theta)|$. First it is shown that $\{Q_N(\theta)\}$ is relative compact to the topology induced by this norm. Then any subsequence $\{Q_{N'}(\theta)\}$ of $\{Q_N(\theta)\}$ contains a uniformly convergent subsequence $\{Q_{N''}(\theta)\}$ of $\{Q_{N'}(\theta)\}$ and let $\tilde{Q}(\theta)$ be a uniform limit point of $\{Q_{N''}(\theta)\}$. Next we show that $\tilde{Q}(\theta)$ is always equal to $q(\theta)+\sigma^2$ for any $\theta\in\Theta$. By following Ascoli-Arzelà theorem, if $\{Q_N(\theta)\}$ is equicontinuous on Θ and uniformly bounded, then $\{Q_N(\theta)\}$ is relative compact. In our case it is easily seen that $\{Q_N(\theta)\}$ is uniformly bounded for any $\theta\in\Theta$ and all N. Thus we need only to show $\{Q_N(\theta)\}$ is equicontinuous on Θ. Since $Q_N(\theta)$ is differentiable on Θ, it holds that

$$Q_N(\theta_\alpha)-Q_N(\theta_\beta)=(\theta_\alpha-\theta_\beta)^T\begin{bmatrix}\partial Q_N(\theta_\lambda^1)/\partial\theta_1\\ \vdots\\ \partial Q_N(\theta_\lambda^p)/\partial\theta_p\end{bmatrix}$$

where for all i, $\|\theta_\lambda^i-\theta_\beta\|\leqq\|\theta_\alpha-\theta_\beta\|$. The transpose of vector θ is denoted by θ^T. Here $\sup_N\sup_\theta|\partial Q_N(\theta)/\partial\theta|$ is finite, since

$$\begin{aligned}\sup_\theta|\partial Q_N(\theta)/\partial\theta|&=\sup_\theta\left|2N^{-1}\sum_{t=2}^N(X_t-f_t(\theta)X_{t-1})f_t'(\theta)X_{t-1}\right|\\&=\sup_\theta\left|2N^{-1}\sum_{t=2}^N(e_t+g_t(\theta)X_{t-1})f_t'(\theta)X_{t-1}\right|\\&\leqq\frac{M}{N}\sum_{t=2}^N|e_tX_{t-1}|+\frac{M}{N}\sum_{t=2}^N X_{t-1}^2\\&\leqq M'\end{aligned}$$

for suitable positive constants M and M'. Hence for every ε (>0), there exists a number δ (>0), depending only on ε, such that $|\theta_\alpha-\theta_\beta|<\delta$ implies $|Q_N(\theta_\alpha)-Q_N(\theta_\beta)|<\varepsilon$ for all θ_α, $\theta_\beta\in\Theta$ and all N. Now we shall show that $\tilde{Q}(\theta)=q(\theta)+\sigma^2$ for any $\theta\in\Theta$. We see from Lemma 2 that

$$q(\theta)=\lim_{N\to\infty}N^{-1}\sum_{t=2}^N g_t^2(\theta)\,\mathrm{E}\,X_{t-1}^2$$

for any $\theta\in\Theta$. By the same argument, $\left\{N^{-1}\sum_{t=2}^N g_t^2(\theta)\,\mathrm{E}\,X_{t-1}^2\right\}$ is uniformly bounded and equicontinuous on Θ, and, hence, $\left\{N^{-1}\sum_{t=2}^N g_t^2(\theta)\,\mathrm{E}\,X_{t-1}^2\right\}$ converges uniformly to $q(\theta)$ as $N\to\infty$. Thus $q(\theta)$ is continuous on Θ. $\tilde{Q}(\theta)$ coincides with $q(\theta)+\sigma^2$ in the countable dense set Θ_0. Because $q(\theta)+\sigma^2$ and $\tilde{Q}(\theta)$ are continuous, they coincide for any θ of the parameter space Θ. Finally we prove that $\hat{\theta}_N$ converges to θ_0 as $N\to\infty$. Now, suppose that $\hat{\theta}_N$ does not converge to θ_0. Since Θ is a compact set, there exists a convergent subsequence $\{\hat{\theta}_{N'}\}$ of $\{\hat{\theta}_N\}$ and θ^* ($\neq\theta_0$)

such that $\lim_{N'\to\infty}\hat{\theta}_{N'}=\theta^*$. Since $q(\theta)$ is continuous and $Q_N(\theta)$ converges uniformly to $q(\theta)+\sigma^2$, $Q_{N'}(\hat{\theta}_{N'})$ converges to $q(\theta^*)+\sigma^2$ as $N'\to\infty$. However since $\hat{\theta}_{N'}$ is the least squares estimator, we have

$$Q_{N'}(\hat{\theta}_{N'})\leqq Q_{N'}(\theta_0)=\frac{1}{N'}\sum_{t=1}^{N'}e_t^2 .$$

It follows by letting $N'\to\infty$ that

$$q(\theta^*)+\sigma^2\leqq\sigma^2 .$$

Hence $q(\theta^*)$ is zero. Since θ_0 uniquely attains the minimum value of $q(\theta)$, we have $\theta^*=\theta_0$. This contradiction implies that $\hat{\theta}_N$ converges almost surely to θ_0. Now

$$\begin{aligned}\hat{\sigma}_N^2&=N^{-1}\sum_{t=2}^{N}\{X_t-f_t(\hat{\theta}_N)X_{t-1}\}^2\\&=Q_N(\hat{\theta}_N)\end{aligned}$$

and $Q_N(\hat{\theta}_N)$ uniformly converges almost surely to $q(\theta_0)+\sigma^2=\sigma^2$ as $N\to\infty$. This completes the proof.

3. Asymptotic distribution of the estimators

In this section we shall show that the least squares estimators are asymptotically normally distributed.

THEOREM 2. *Let the conditions* (1)-(5) *hold. Then* $N^{1/2}(\hat{\theta}_N-\theta_0)$ *has an asymptotically normal distribution with mean zero and covariance matrix* $1/4\cdot A^{-1}(\theta_0)\sigma^2$ *defined by assumption* (5).

PROOF. First we note

$$\frac{\partial Q_N(\hat{\theta}_N)}{\partial\theta_i}=\frac{\partial Q_N(\theta_0)}{\partial\theta_i}+\left[\frac{\partial^2 Q_N(\tilde{\theta}_{N,i})}{\partial\theta_i\partial\theta^T}\right](\hat{\theta}_N-\theta_0)$$

where $\|\tilde{\theta}_{N,i}-\theta_0\|\leqq\|\hat{\theta}_N-\theta_0\|$ for all $i=1, 2, \cdots, p$. Similar to the Lemma 2 we find that

$$\lim_{N\to\infty}N^{-1}\sum_{t=2}^{N}f'_{t,j}(\theta_0)f'_{t,i}(\theta_0)X_{t-1}^2$$

exists almost surely for $i, j=1, 2, \cdots, p$. Now, it can be proved by using the same method as one in the proof of Theorem 1 that $\{\partial^2 Q_N(\tilde{\theta}_{N,i})/\partial\theta\partial\theta^T\}$ converges to $\lim_{N\to\infty}\{\partial^2 Q_N(\theta_0)/\partial\theta\partial\theta^T\}\overset{\text{say}}{=}J(\theta_0)$ almost surely, since $\hat{\theta}_N$ is strongly consistent estimator of θ_0. The (i, j) element of matrix $J(\theta_0)$ is

$$\lim_{N\to\infty} N^{-1}\sum_{t=2}^{N}\{f''_{t,ij}(\theta_0)X_tX_{t-1}-f'_{t,i}(\theta_0)f'_{t,j}(\theta_0)X_{t-1}^2-f_t(\theta_0)f''_{t,ij}(\theta_0)X_{t-1}^2\}$$
$$=-\lim_{N\to\infty} N^{-1}\sum_{t=2}^{N}f'_{t,j}(\theta_0)f'_{t,i}(\theta_0)X_{t-1}^2$$

by Lemma 1. Thus we have $J(\theta_0)=-A(\theta_0)$. For this reason, we can see $\sqrt{N}(\hat{\theta}_N-\theta_0)$ and $A^{-1}(\theta_0)\sqrt{N}\,\partial Q(\theta_0)/\partial\theta$ have the same asymptotic distribution. Next, note that for any constant vector β such that β is $p\times 1$ we have

$$-\beta^T\partial Q_N(\theta_0)/\partial\theta=\frac{2}{N}\sum_{t=2}^{N}(X_t-f_t(\theta_0)X_{t-1})\beta^TF'_t(\theta_0)X_{t-1}$$
$$=\frac{2}{N}\sum_{t=2}^{N}e_tX_{t-1}\beta^TF'_t(\theta_0)$$
$$=\frac{2}{N}\sum_{t=2}^{N}\zeta_t$$

if we put $F'_t(\theta_0)=(f'_{t,1}(\theta_0),\cdots,f'_{t,p}(\theta_0))$ and $\zeta_t=e_tX_{t-1}\beta^TF'_t(\theta_0)$. Define $U_N=\sum_{t=2}^{N}\zeta_t$ and let $\mathcal{F}_N$ be the σ-field generated by $\{X_0, e_t, t\leqq N\}$. Then we find $\{U_N, \mathcal{F}_N, N=1, 2, \cdots\}$ is a martingale. Let

$$V_N^2=\sum_{t=2}^{N}\mathrm{E}\,(\zeta_t^2/\mathcal{F}_{t-1})$$
$$=\sigma^2\sum_{t=2}^{N}X_{t-1}^2(\beta^TF'_t(\theta_0))^2$$

and

$$S_N^2=\mathrm{E}\,V_N^2=\sigma^2\sum_{t=2}^{N}\mathrm{E}\,X_{t-1}^2(\beta^TF'_t(\theta_0))^2\,.$$

If we can show

(i) $V_N^2/S_N^2 \xrightarrow{p} 1$

and

(ii) $\sum_{t=2}^{N}\mathrm{E}\,\zeta_t^2I(|\zeta_t|\geqq\varepsilon S_N)/S_N^2 \longrightarrow 0$

as $N\to\infty$ for any $\varepsilon>0$, where $I(\cdot)$ being the indicator function, then we can obtain the fact that $S_N^{-1}\sum_{t=2}^{N}\zeta_t$ converges in distribution to $N(0, 1)$ by the martingale central limit Theorem 2 of Brown [2]. The condition (i) is obviously held by Lemma 2, so we need only to consider condition (ii) which is called Lindeberg condition.

First, we note $\mathrm{E}\,(\zeta_t^4)=(\beta^TF'_t(\theta_0))^4\,\mathrm{E}\,X_{t-1}^4\,\mathrm{E}\,e_t^4$ is bounded for any $t\in T$ and $\theta\in\Theta$. Now

$$\begin{aligned}\mathrm{E}\,\zeta_t^2 I(|\zeta_t|\geqq \varepsilon S_N)&=\int_{|\zeta_t|\geqq\varepsilon S_N}\zeta_t^2 dF(\zeta_t)\\&\leqq\int_{|\zeta_t|\geqq\varepsilon S_N}\frac{\zeta_t^4}{(\varepsilon S_N)^2}dF(\zeta_t)\\&\leqq\frac{1}{(\varepsilon S_N)^2}\mathrm{E}\,\zeta_t^4\end{aligned}$$

and hence

$$\sum_{t=2}^{N}\mathrm{E}\,\zeta_t^2 I(|\zeta_t|\geqq\varepsilon S_N)/S_N^2\leqq\frac{1}{\varepsilon^2 S_N^4}\sum_{t=2}^{N}\mathrm{E}\,\zeta_t^4 \longrightarrow 0$$

by letting $N\to\infty$ as required. Hence $S_N^{-1}\sum_{t=2}^{N}\zeta_t$ converges in distribution to $N(0,1)$. That is, $-N^{1/2}\beta^T\partial Q_N(\theta_0)/\partial\theta=2N^{-1/2}\sum_{t=2}^{N}\zeta_t$ is asymptotically distributed as normal with mean zero and covariance $\frac{1}{4}\lim_{N\to\infty}N^{-1}S_N^2$. Hence $N^{1/2}\partial Q_N(\theta_0)/\partial\theta$ converges in distribution to $N(0, 1/4\cdot A(\theta_0)\sigma^2)$. This completes the proof of the Theorem 2.

4. Remarks

For the stability of time series and obtaining good asymptotic properties of the estimators, a number of conditions for the function $f_t(\theta)$ were required. Now we shall show an example. Let $f_t(\theta)=\theta_1\sin(\alpha t+\theta_2)$. Thus, from (1.1), we consider the following process

$$X_t=\theta_1\sin(\alpha t+\theta_2)X_{t-1}+e_t \tag{4.1}$$

where α is a known constant, θ_1 and θ_2 are unknown parameters such that $\theta_1\in\Theta_1=[\varepsilon, 1-\varepsilon]$ and $\theta_2\in\Theta_2=[0, 2\pi-\varepsilon]$ for $\varepsilon>0$. Here if we take $\alpha=\pi/2$ for convenience, then the period of $f_t(\theta)$ is 4. It is not hard to see that this example satisfies the conditions (1)-(3). Since the period of $\sin^2\left(\frac{\pi}{2}t+\theta_2\right)$ is 2, we consider the subsequences $\{\mathrm{E}\,X_{1+2t}^2\}$ and $\{\mathrm{E}\,X_{2+2t}^2\}$ of $\{\mathrm{E}\,X_t^2\}$. Then we obtain

$$\begin{aligned}\mathrm{E}\,X_{k+2t}^2&=\theta_1^{2(k+2t)}\prod_{i=1}^{k+2t}\sin^2\left(\frac{\pi}{2}i+\theta_2\right)\mathrm{Var}\,X_0\\&+\sum_{j=0}^{k+2t-1}\theta_1^{2j}\prod_{i=0}^{j-1}\sin^2\left(\frac{\pi}{2}(k+2t-i)+\theta_2\right)\sigma^2\,.\end{aligned} \tag{4.2}$$

For a fixed k, $k=1, 2$, we can see $\mathrm{E}\,X_{k+2t}^2$ converges to a finite value as $t\to\infty$, since the 2nd term on the right of (4.2) is a monotone increasing sequence and bounded for any $t\in T$ and $(\theta_1, \theta_2)\in\Theta_1\times\Theta_2$. Hence

$N^{-1}\sum_{t=1}^{N} \mathrm{E}\, X_t^2$ converges to $1/2\sum_{k=1}^{2} V_k$, where V_k denotes the limit point of $\{\mathrm{E}\, X_{k+2t}^2\}$. Using this result, it can easily be shown that the condition (4) holds. This means the estimators have strong consistency. Next in order to check the condition (5), we prove firstly $A(\theta)$ exists. We have

$$A(\theta)=\lim_{N\to\infty} N^{-1}\sum_{t=2}^{N}\begin{bmatrix} \sin^2\left(\frac{\pi}{2}t+\theta_2\right), & \frac{\theta_1}{2}\sin(\pi t+2\theta_2) \\ \frac{\theta_1}{2}\sin(\pi t+2\theta_2), & \theta_1^2\cos^2\left(\frac{\pi}{2}t+\theta_2\right) \end{bmatrix}\mathrm{E}\, X_{t-1}^2$$

$$=\frac{1}{2}\sum_{k=1}^{2}\begin{bmatrix} \sin^2\left(\frac{\pi}{2}k+\theta_2\right), & \frac{\theta_1}{2}\sin(\pi k+2\theta_2) \\ \frac{\theta_1}{2}\sin(\pi k+2\theta_2), & \theta_1^2\cos^2\left(\frac{\pi}{2}k+\theta_2\right) \end{bmatrix} V_k\,.$$

Moreover the matrix $A(\theta)$ is nonsingular, because the determinant of $A(\theta)$ is

$$\frac{\theta_1^2}{8}V_1V_2\left\{2\sin^2\left(\frac{1}{2}\pi+2\theta_2\right)+\cos(\pi+4\theta_2)+3\right\}>0\,.$$

Hence this example satisfies the conditions for establishing central limit theorem as well as strong consistency.

To prove strong consistency, Klimko and Nelson [4] have assumed in their theorem 2.1 three conditions, of which two conditions are concerned with 2nd partial derivatives of $Q_N(\theta)$, whereas we have required only first partial derivatives. And it is not easy to examine their condition (i):

(i) $\limsup_{N\to\infty\ \delta\to\infty} (|T_N(\theta^*)_{ij}|/N\delta)<\infty$ a.e. $i\leqq p,\ j\leqq p$

where $T_N(\theta^*)_{ij}=\dfrac{\partial^2 Q_N(\theta^*)}{\partial\theta_i\partial\theta_j}-\dfrac{\partial^2 Q_N(\theta_0)}{\partial\theta_i\partial\theta_j}$ and $\|\theta^*-\theta_0\|\leqq\|\theta-\theta_0\|$. Furthermore, they considered local minimum solutions of $\partial Q_N(\theta)/\partial\theta_i=0$, $i=1, 2, \cdots, p$, while under our conditions the solutions are to be the grobal minimum in the parameter space.

Some simulations are performed with this example. Figure 1 shows the time series of length 250 generated by the model (4.1) from the sequence $\{e_t\}$ of normally and independently distributed random variables with zero means and variances 1, where $\theta_1=0.5$ and $\theta_2=2.0$. For values of $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\sigma}^2$, and for each sample size N, the experiment was replicated 100 times. These results are summarized in Table 1, in which lines (a) gives the average of the parameter estimates, lines (b) the sample standard deviations of the corresponding estimates for

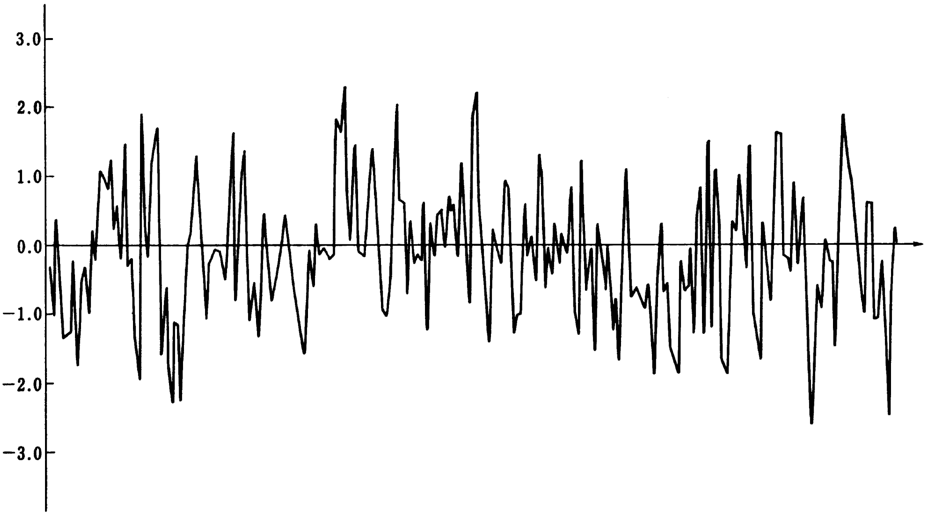

Figure 1. $X_t = 0.5 \sin\left(\frac{\pi}{2}t + 2.0\right)X_{t-1} + e_t$

Table 1. Simulation results for true values $\theta_1 = 0.5$, $\theta_2 = 2.0$ and $\sigma^2 = 1.0$

Sample size		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\sigma}^2$
50	(a)	0.5674	1.9308	0.9485
	(b)	0.2007	0.4354	0.1811
100	(a)	0.5411	1.9365	0.9892
	(b)	0.1334	0.2869	0.1345
200	(a)	0.5196	1.9413	0.9976
	(b)	0.0917	0.1830	0.1020
500	(a)	0.4977	2.0019	1.0041
	(b)	0.0574	0.1618	0.0678

the 100 replications. As the sample size becomes larger, the average of the estimates are closer to their true value in accordance with statistical theory.

We can easily construct other examples of $f_t(\theta)$ ensuring strong consistency, if we choose functions from either the family of function $\phi_t(\theta)$ which converges to $\phi(\theta)$ as $t \to \infty$ where $|\phi(\theta)| < 1 - \varepsilon$ for $\varepsilon > 0$ and $\phi(\theta)$ is a one-to-one function on Θ, or the family of periodic function whose amplitude is less than one in absolute value.

5. Acknowledgement

We would like to thank professor M. Huzii for his useful guidance and many helpful suggestions.

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