Ann. Inst. Statist. Math.
38 (1986), Part A, 245-262

RATES OF UNIFORM CONVERGENCE OF EXTREME ORDER STATISTICS

M. FALK

(Received May 20, 1985; revised Oct. 1, 1985)

Summary

Bounds for the convergence uniformly over all Borel sets of the largest order statistic as well as of the joint distribution of extremes are established which reveal in which way these rates are determined by the distance of the underlying density from the density of the corresponding generalized Pareto distribution.

The results are highlighted by several examples among which there is a bound for the rate at which the joint distribution of the k largest order statistics from a normal distribution converges uniformly to its limit.

1. Introduction

Let P be a probability measure on the real line with distribution function ($\equiv df$) F and denote by P^n the n-fold independent product of P. Moreover, $Z_{i:n}: R^n \rightarrow R$ denotes the i-th order statistic for the sample size n and $P^n * g$ the distribution induced by P^n and a measurable function g.

If the distribution $P^n * (a_n^{-1}(Z_{n:n}-b_n))$ tends to some nondegenerate limit G (in the sense of weak convergence) for some choice of constants $a_n>0$, b_n, $n \in N$, then we know from Gnedenko [5] that this limit G must be one of the following types with $\alpha>0$

$$
\begin{aligned}
&G_{1,\alpha}(x) := \exp(-x^{-\alpha}), \qquad x>0; \\
(1.1) \qquad &G_{2,\alpha}(x) := \exp(-(-x)^{\alpha}), \qquad x \leqq 0; \\
&G_3(x) := \exp(-e^{-x}), \qquad x \in R.
\end{aligned}
$$

Moreover, Gnedenko [5] gave necessary and sufficient conditions for P to belong to the domain of attraction ($\equiv \mathcal{D}(G)$) of each of the above limits.

Key words and phrases: Generalized Pareto distribution, uniform distance, joint distribution of extremes, normal extremes.

While the rate of convergence (uniformly over all Borel-sets) of central order statistics to a normal distribution is of order $O(n^{-1/2})$ which is well-known (see Reiss [16] and, for results on asymptotic expansions, Reiss [17]), the situation changes completely for extreme order statistics.

Here the rate of convergence of $\sup_{x\in R}|F^n(a_nx+b_n)-G(x)|$ to zero may be of order $O(1/\log(n))$ in the normal case (see Hall [6]) as well as of order $O(n^{-1})$ in the exponential case (see Hall and Wellner [7]). Therefore, results on the rate of (weak) convergence have only been sporadic and a unifying approach was given only by Cohen [1] for the case G_3 and by Smith [22] for the case G_1. A nonuniform bound for the rate of (weak) convergence is given by Theorem 2.10.1 of Galambos [4].

These results were mainly derived by the concept of regularly varying functions, and although this approach has proved quite successful we will choose a different way to obtain bounds for the uniform convergence of extremes. We will explain this way in the following.

Taylor-expansion of $\log(1+x)$ immediately yields

$$(1.2)\qquad \begin{aligned} &F^n(a_nx+b_n)\underset{n\in N}{\longrightarrow}G(x)\,, \qquad x\in R\,,\\ &\iff [1-F(a_nx+b_n)]/[1-(1+\log(G(x)^{1/n}))]\underset{n\in N}{\longrightarrow}1\,,\\ &\qquad\qquad x\in G^{-1}(0,1)\,. \end{aligned}$$

Now, denote by W_i a generalized Pareto distribution ($\equiv gPd$), i.e.

$$(1.3)\qquad \begin{aligned} &W_{1,\alpha}(x)=1-x^{-\alpha}\,, \qquad x\geqq 1,\ \alpha>0\,,\\ &W_{2,\alpha}(x)=1-(-x)^{\alpha}\,, \qquad x\in[-1,0],\ \alpha>0\,,\\ &W_3(x)=1-\exp(-x)\,, \qquad x\geqq 0\,, \end{aligned}$$

then,

$$(1.4)\qquad \begin{aligned} &1+\log(G_{1,\alpha}(x)^{1/n})=W_{1,\alpha}(n^{1/\alpha}x)=:\ W_{1,\alpha,(n)}(x)\,, \qquad x\geqq n^{-1/\alpha}\,,\\ &1+\log(G_{2,\alpha}(x)^{1/n})=W_{2,\alpha}(n^{-1/\alpha}x)=:\ W_{2,\alpha,(n)}(x)\,, \qquad -n^{1/\alpha}\leqq x\leqq 0\,,\\ &1+\log(G_3(x)^{1/n})=W_3(x+\log(n))=:\ W_{3,(n)}(x)\,, \qquad x\geqq -\log(n)\,. \end{aligned}$$

Therefore, by (1.2) and (1.4) we have

$$(1.5)\qquad \begin{aligned} &F^n(a_nx+b_n)\underset{n\in N}{\longrightarrow}G_i(x)\,, \qquad x\in R\,,\\ &\iff [1-F(a_nx+b_n)]/[1-W_{i,(n)}(x)]\underset{n\in N}{\longrightarrow}1\,, \qquad x\in G_i^{-1}(0,1)\,. \end{aligned}$$

This result states that a probability measure lies in the domain of attraction of an extreme value distribution if and only if it can be approximated in an appropriate way by a shifted *gPd*.

The importance of the *gPd*'s was first observed by Pickands [15] who showed that, roughly speaking, $F\in\mathcal{D}(G)$ if and only if the conditional distribution $[F(u+y)-F(u)]/[1-F(u)]$ is approximately given by an appropriately shifted *gPd* if u is large. We remark that this result can also easily be motivated by (1.5).

Now, suppose that F has a derivative f near the right endpoint of its support. Then, if $[1-F(a_nx+b_n)]/[1-W_{(n)}(x)] \underset{n\in N}{\longrightarrow} 1$, $x \in G^{-1}(0,1)$, a suitable version of the mean value theorem implies for any $x, y \in G^{-1}(0,1)$, $x<y$,

$$[F(a_nx+b_n)-F(a_ny+b_n)]/[W_{(n)}(x)-W_{(n)}(y)]$$
$$=a_nf(a_n\theta_n+b_n)/W'_{(n)}(\theta_n) \underset{n\in N}{\longrightarrow} 1\,, \qquad \theta_n \in (x, y)\,,$$

and hence, one might expect that if f satisfies certain regularity conditions we have

$$na_nf(a_nx+b_n)/w(x) \underset{n\in N}{\longrightarrow} 1\,, \qquad x \in G^{-1}(0,1)\,, \tag{1.6}$$

where with

$$g(x):=G'(x)\,, \qquad x\in G^{-1}(0,1)\,, \tag{1.7}$$

and

$$w(x):=[\log(G(x))]'=g(x)/G(x)\,, \qquad x\in G^{-1}(0,1)\,, \tag{1.8}$$

$n^{-1}w(x)$ is for $x \in W_{(n)}^{-1}(0,1)$ the Lebesgue density of the shifted *gPd* $W_{(n)}$, i.e.

$$w_{(n)}(x):=n^{-1}w(x)=W'_{(n)}(x)\,, \qquad x\in W_{(n)}^{-1}(0,1)\,. \tag{1.9}$$

In particular, with $\alpha>0$

$$\begin{aligned} w_{1,\alpha}(x)&=\alpha x^{-(\alpha+1)}\,, && x>0\,,\\ w_{2,\alpha}(x)&=\alpha(-x)^{\alpha-1}\,, && x<0\,,\\ w_3(x)&=\exp(-x)\,, && x\in R\,. \end{aligned} \tag{1.10}$$

Consequently, the condition $na_nf(a_nx+b_n)/w(x) \underset{n\in N}{\longrightarrow} 1$ implies that the density of $F(a_nx+b_n)$ can be approximated by $w_{(n)}$ as follows

$$a_nf(a_nx+b_n)=w_{(n)}(x)[1+h_n(x)] \tag{1.11}$$

where $h_n(x) \underset{n\in N}{\longrightarrow} 0$, $x\in G^{-1}(0,1)$. Thus, we are led to an expression of the closeness of $F(a_nx+b_n)$ and $W_{(n)}$ in terms of their densities which is basic for a general statistical theory (see, for expample, the book by Pfanzagl [14]).

Example 1.12. Consider the standard normal distribution $N_{(0,1)}$ with *df* Φ and density φ. Choose the norming constants b_n as the solution of the equation

$$b_n = n\varphi(b_n) \tag{1.13}$$

and $a_n = 1/b_n$. Then,

$$\begin{aligned} b_n^{-1}\varphi(b_n^{-1}x+b_n) &= \exp(-(x+\log(n)))\exp(-b_n^{-2}x^2/2) \\ &= w_{3,(n)}(x)[1+h_n(x)]\ , \end{aligned} \tag{1.14}$$

where (see [3], p. 374)

$$h_n(x) = \exp(-b_n^{-2}x^2/2) - 1 = O(x^2)/\log(n)\ . \tag{1.15}$$

Notice that $O(1/\log(n))$ is the exact rate at which $N^n_{(0,1)} * (b_n(Z_{n:n} - b_n))$ tends (weakly) to G_3 (see Hall [6]). Therefore, representation (1.11) and the preceding example give rise to the idea that the rate at which h_n tends to zero determines also the rate at which the distribution of $Z_{n:n}$ under $F^n(a_n x + b_n)$ tends to G. This conjecture will be made rigorous in the next section.

Moreover, representation (1.11) immediately yields the following consequence. Assume that $F^n(a_n x+b_n) \underset{n\in N}{\longrightarrow} G(x)$, $x \in R$, and that (1.11) holds. Then,

$$\begin{aligned} dF^n(a_n x+b_n)/dx &= F^{n-1}(a_n x+b_n) n a_n f(a_n x+b_n) \underset{n\in N}{\longrightarrow} G(x)w(x) \\ &= g(x)\ , \qquad x \in G^{-1}(0,1)\ , \end{aligned} \tag{1.16}$$

and consequently, by Scheffé's Lemma we get

$$\sup_{B\in\mathcal{B}} |P^n\{a_n^{-1}(Z_{n:n}-b_n)\in B\} - G(B)| \underset{n\in N}{\longrightarrow} 0\ , \tag{1.17}$$

where $\mathcal{B}$ denotes the Borel-σ-algebra on R; i.e. condition (1.6) entails uniform convergence of the largest order statistic which is therefore the type of convergence which we will have to study.

We remark that on the other hand the convergence $na_n f(a_n x+b_n)/w(x) \underset{n\in N}{\longrightarrow} 1$, $x \in G^{-1}(0,1)$, does not necessarily entail $F^n(a_n x+b_n) \underset{n\in N}{\longrightarrow} G(x)$, $x \in R$. Put, for example, $F = W_{2,1}$, $a_n = n^{-1}$ and $b_n = -n^{-1/2}$. Then, for $x<0$, $na_n f(a_n x+b_n) = 1 = w_{2,1}(x)$ if n is large, but $n(1-F(a_n x+b_n)) = -x + n^{1/2}$.

While in [3] we proved that (1.6) holds under fairly general von Mises type conditions on F, we will establish in the present paper bounds for the rates of uniform convergence of the largest order statistic as well as for the joint distribution of extremes which will in particular reveal the influence of the remainder term h_n in the representation $a_n f(a_n x+b_n) = w_{(n)}(x)[1+h_n(x)]$ (see formula (1.11)) on the rate of uniform convergence of extreme order statistics.

Imposing further conditions on F, we can specify this influence of

h_n to be the upper bound $\left(\int h_n^2 dG\right)^{1/2}$ and $\left(\int h_n^2 d\left(\sum_{i=1}^{k} G_{(i)}\right)\right)^{1/2}$ in the multivariate case, where $G_{(i)}/R$ denotes the i-th marginal distribution of $G^{(k)}/R^k$ which is the (weak) limit of $P^n * ((a_n^{-1}(Z_{n-i+1:n}-b_n))_{i=1}^k)$ (see [24] and [25]).

In particular we have that

$$g_{(k)}(x) := g(x)[-\log(G(x))]^{k-1}/(k-1)!\ , \qquad x \in G^{-1}(0,1)\ , \tag{1.18}$$

is the Lebesgue density of the probability measure $G_{(k)}$ with df

$$G_{(k)}(x) := G(x) \sum_{i=0}^{k-1} [-\log(G(x))]^i/i!\ , \qquad x \in G^{-1}(0,1)\ , \tag{1.19}$$

where $G \in \{G_1, G_2, G_3\}$ and $G^{(k)}/R^k$ has Lebesgue density

$$g^{(k)}(\boldsymbol{x}) := g(x_k) \prod_{i=1}^{k-1} w(x_i)\ , \qquad x_1 > x_2 > \cdots > x_k\ , \tag{1.20}$$

where $g \in \{g_1, g_2, g_3\}$.

Results on the rate of uniform convergence of extremes were derived, as far as we know, only in several earlier mentioned articles by Reiss [18], [19] and, together with Kohne, [11], where bounds for the uniform distance for the special cases of the uniform and the exponential distribution were established. In particular in formula (4.4) in Reiss [19] he indicates how the probability transformation theorem can be applied to derive rates of uniform convergence of extremes for general distributions which will be basic also for our results.

2. Main results

At first we state some auxiliary results.

LEMMA 2.1. *Let P_1, P_2 be probability measures on a measurable space $(X, \mathcal{A})$ which are dominated by a σ-finite measure μ. Denote by f_1 and f_2 the respective μ-densities of P_1 and P_2. Then the following inequality holds, where $M := \{f_1 > 0\}$.*

$$\sup_{A \in \mathcal{A}} |P_1(A) - P_2(A)| \leq \left\{1 - P_2(M)^2 \exp\left\langle \int_M \log(f_1/f_2) dP_2/P_2(M) \right\rangle\right\}^{1/2}.$$

PROOF. From [19] we know (see also Section 2 in [13]) that

$$\begin{aligned}\sup_{A \in \mathcal{A}} |P_1(A) - P_2(A)| &\leq \left\{1 - \left(\int (f_1 f_2)^{1/2} d\mu\right)^2\right\}^{1/2} \\ &= \left\{1 - \left(\int (f_1/f_2)^{1/2} dP_2\right)^2\right\}^{1/2}\end{aligned}$$

$$=\left\{1-\left(\int_M (f_1/f_2)^{1/2}dP_2\right)^2\right\}^{1/2}$$
$$=\left\{1-P_2(M)^2\left(\int_M (f_1/f_2)^{1/2}dP_2/P_2(M)\right)^2\right\}^{1/2}$$
$$\leqq\left\{1-P_2(M)^2\exp\left\langle\int_M \log (f_1/f_2)dP_2/P_2(M)\right\rangle\right\}^{1/2}$$

by Jensen's inequality. For a further investigation of the uniform distance and the Kullback-Leibler mean information we refer to Ikeda [8] (see also Ikeda [9] for further results).

The next auxiliary result which is basic, is suggested by a result by Ikeda and Matsunawa [10]. It follows from Theorem 2.6 and Theorem 3.2 by Reiss [19].

LEMMA 2.2. *There exists a universal constant* $C>0$ *such that for any* $n\in N$ *and* $k\in\{1,\cdots,n\}$

$$\sup_{B\in\mathcal{B}^k}|Q^n\{(n(Z_{n-i+1:n}-1))_{i=1}^k\in B\}-G_{2,1}^{(k)}(B)|\leqq Ck/n\ ,$$

where Q *denotes the uniform distribution on* $(0,1)$.

We now turn to the first main result of this chapter. First we consider the case of the largest order statistic. Throughout the rest of this section we assume that F is absolutely continuous with density f, by C we denote the universal constant occuring in Lemma 2.2 and by E the standard exponential distribution, i.e. we put $E=W_3$.

THEOREM 2.3. *Let* $G\in\{G_1,G_2,G_3\}$ and $a_n>0$, $b_n\in R$, $n\in N$. *Then the following bound holds*

$$\sup_{B\in\mathcal{B}}|P^n\{a_n^{-1}(Z_{n:n}-b_n)\in B\}-G(B)|$$
$$\leqq(C+2)/n+\{1-G(M_n)^2\exp(-B_n/G(M_n))\}^{1/2}$$

where

$$(2.4)\qquad B_n=\int_{M_n} na_nf(a_nx+b_n)/w(x)-1-\log\{na_nf(a_nx+b_n)/w(x)\}G(dx)$$
$$+\int_{M_n}1+\log(G(x))G(dx)\qquad \textit{if } G\in\{G_1,G_3\}\ ,$$

$$(2.5)\qquad B_n=\int_{M_n} na_nf(a_nx+b_n)/w(x)-1-\log\{na_nf(a_nx+b_n)/w(x)\}G(dx)$$
$$+\int_{M_n}1+\log(G(x))+n(1-F(b_n))G(dx)\qquad \textit{if } G=G_2$$

and

$$M_n = \{x \in R:\ f(a_n x + b_n) > 0\} . \tag{2.6}$$

If $G(M_n)=1$, the above result takes the following simpler form which is immediate from integration by parts and the inequality $1-\exp(x) \leqq -x$, $x \in R$.

COROLLARY 2.7. *Let $G \in \{G_1, G_2, G_3\}$, $a_n > 0$ and $b_n \in R$, $n \in N$. Then, if $G(M_n)=1$ we have the bound*

$$\sup_{B \in \mathcal{B}} |P^n\{a_n^{-1}(Z_{n:n}-b_n) \in B\} - G(B)| \leqq (C+2)/n + \tilde{B}_n^{1/2}$$

where

$$\tilde{B}_n = \int n a_n f(a_n x + b_n)/w(x) - 1 - \log\{n a_n f(a_n x + b_n)/w(x)\} G(dx) \quad \text{if } G \in \{G_1, G_3\} \tag{2.8}$$

and

$$\tilde{B}_n = \int n a_n f(a_n x + b_n)/w(x) - 1 - \log\{n a_n f(a_n x + b_n)/w(x)\} G(dx) + n(1-F(b_n)) \quad \text{if } G = G_2 . \tag{2.9}$$

Before we discuss these results we prove Theorem 2.3.

PROOF. The probability transformation theorem implies $P^n * Z_{n:n} = Q^n * (F^{-1}(Z_{n:n}))$ where $F^{-1}(t) := \inf\{x \in R:\ F(x) \geqq t\}$, $t \in (0, 1]$, $(\inf \phi := \infty)$ denotes the generalized inverse of F. Thus, by Lemma 2.2

$$\sup_{B \in \mathcal{B}} |P^n\{a_n^{-1}(Z_{n:n}-b_n) \in B\} - E\{a_n^{-1}(\tilde{F}^{-1}(1-n^{-1}\pi_1)-b_n) \in B\}| \leqq C/n \tag{2.10}$$

where $\tilde{F}^{-1}(x) := F^{-1}(x) 1_{(0,1)}(x)$ and $\pi_1(x) = x$, $x \in R$.

Furthermore, for any $B \in \mathcal{B}$ with $E = W_3$

$$\begin{aligned} &|E\{a_n^{-1}(\tilde{F}^{-1}(1-n^{-1}\pi_1)-b_n) \in B\} \\ &\quad - E\{a_n^{-1}(F^{-1}(1-n^{-1}\pi_1)-b_n) \in B,\ 0<\pi_1<n\}| \leqq \exp(-n) . \end{aligned} \tag{2.11}$$

Moreover, the measure $\mu/\mathcal{B}$, defined by

$$\mu(B) := E\{a_n^{-1}(F^{-1}(1-n^{-1}\pi_1)-b_n) \in B,\ 0<\pi_1<n\} \tag{2.12}$$

has the following Lebesgue density for $x \in R$

$$h(x) := \exp\{-n(1-F(a_n x + b_n))\} n a_n f(a_n x + b_n) . \tag{2.13}$$

Hence, $\tilde{\mu} := \mu/(1-\exp(-n))$ is a probability measure on the real line with density $\tilde{h} := h/(1-\exp(-n))$ and thus, by Lemma 2.1,

$$\sup_{B \in \mathcal{B}} |\tilde{\mu}(B) - G(B)| \tag{2.14}$$

$$\leq \left(1-G(M_n)^2 \exp\left\{\int_{M_n} \log(\tilde{h}/g)dG/G(M_n)\right\}\right)^{1/2}$$
$$= \left(1-G(M_n)^2 \exp\left\{\int_{M_n} \log(h/g)-\log(1-\exp(-n))dG/G(M_n)\right\}\right)^{1/2}$$
$$\leq \left(1-G(M_n)^2 \exp\left\{\int_{M_n} -n(1-F(a_nx+b_n))+\log(na_nf(a_nx+b_n)) -\log(G(x)w(x))G(dx)/G(M_n)\right\}\right)^{1/2}.$$

Furthermore, by Fubini's theorem

$$(2.15) \quad \int_{M_n} 1-F(a_nx+b_n)G(dx)=\int_{M_n}\int_{a_nx+b_n}^{\infty} f(y)dyG(dx)$$
$$=\int_{M_n} a_n \int_x^{\infty} f(a_ny+b_n)dyG(dx)$$
$$=a_n\int_{-\infty}^{\infty} f(a_ny+b_n)\int_{-\infty}^{y} g(x)1_{M_n}(x)dxdy\ .$$

In the case G_1 the last term on the right-hand side above is less than or equal to

$$(2.16) \quad a_n\int_0^{\infty} f(a_ny+b_n)G_1(y)dy=a_n\int_{M_n} f(a_ny+b_n)/w_1(y)G_1(dy)\ .$$

In the case G_3 it is bounded above by

$$(2.17) \quad a_n\int_{-\infty}^{\infty} f(a_ny+b_n)G_3(y)dy=a_n\int_{M_n} f(a_ny+b_n)/w_3(y)G_3(dy)\ ,$$

whereas in the case G_2 we get the estimate

$$(2.18) \quad a_n\int_{-\infty}^{0} f(a_ny+b_n)/w_2(y)G_2(dy)+a_n\int_0^{\infty} f(a_ny+b_n)G_2(M_n)dy$$
$$=a_n\int_{M_n} f(a_ny+b_n)/w_2(y)G_2(dy)+G_2(M_n)(1-F(b_n))$$
$$=\int_{M_n} a_nf(a_ny+b_n)/w_2(y)+(1-F(b_n))G_2(dy)\ .$$

Now Theorem 2.3 follows from (2.10), (2.11), (2.14)-(2.18) and the fact that $\sup_{B\in\mathcal{B}} |\tilde{\mu}(B)-\mu(B)| \leq \exp(-n)$.

Remarks 2.19. If $F\in\mathcal{D}(G_2)$ then $\omega(F)=F^{-1}(1)<\infty$ and hence, b_n will usually be chosen equal to $\omega(F)$. Thus, by doing so, the term $1-F(b_n)$ in (2.9) vanishes yielding the same bound as (2.8).

Moreover, if F satisfies one of the von Mises type conditions stated in [3], the results of that paper imply $M_n \underset{n\in N}{\longrightarrow} G^{-1}(0, 1)$, and hence,

$$(2.20) \quad \int_{M_n} 1+\log(G(x))G(dx) \underset{n\in N}{\longrightarrow} 1+\int \log(G(x))g(x)dx=0$$

which is immediate from integration by parts.

Furthermore, from the inequality $x-1-\log(x)=x-1+\log(x^{-1})\leqq x-1+x^{-1}-1=(x-1)^2/x$, $x>0$, we obtain

$$(2.21)\quad \int_{M_n} na_nf(a_nx+b_n)/w(x)-1-\log\{na_nf(a_nx+b_n)/w(x)\}G(dx)$$
$$\leqq\int_{M_n}\{na_nf(a_nx+b_n)/w(x)-1\}^2/\{na_nf(a_nx+b_n)/w(x)\}G(dx)$$

but the first bound proves to be more practicable in some cases.

Moreover, one can show that the preceding bounds are still valid if instead of the uniform distance the Hellinger distance is considered (see [20]).

Example 2.22. Let $P=N_{(0,1)}$. Then, with b_n, $n\in N$, chosen as in (1.13)

$$\sup_{B\in\mathcal{B}}|N^n_{(0,1)}\{b_n(Z_{n:n}-b_n)\in B\}-G_3(B)|$$
$$\leqq O(n^{-1})+\left\{\int_R nb_n^{-1}\varphi(b_n^{-1}x+b_n)\exp(x)-1-\log\{nb_n^{-1}\varphi(b_n^{-1}x+b_n)\cdot\exp(x)\}G_3(dx)\right\}^{1/2}$$
$$=O(n^{-1})+\left\{\int_R\exp(-b_n^{-2}x^2/2)-1+b_n^{-2}x^2/2G_3(dx)\right\}^{1/2}$$
$$=O(n^{-1})+\left\{\int_R\exp(-\theta b_n^{-2}x^2/2)b_n^{-4}x^4/8G_3(dx)\right\}^{1/2}$$
$$=O(n^{-1}+b_n^{-2})=O(1/\log(n))$$

which is the optimal rate (see [2], p. 374 and [6]).

We conclude the discussion of Theorem 2.3 by remarking that it can be shown by examples that neither the term $n(1-F(b_n))$ in formula (2.5) nor the term $\int_{M_n}1+\log(G(x))G(dx)$ in (2.4) and (2.5) can be dispensed with. They may be regarded as measures of the differences of the supports of $F(a_nx+b_n)$ and G.

Now, one might hope that those techniques which led to Theorem 2.3 can also be applied to obtain similar results on the rate of convergence of the k-th extreme or of the joint distribution of the k largest order statistics. This will be done in the following.

However, we will only investigate the joint distribution of equally standardized extremes which yields of course a bound for the rate of convergence of the distribution of the k-th largest order statistic. But, in view of Theorem 3.2 in Reiss [19], one might suppose that these rates are close to each other anyway.

The following auxiliary result is a multivariate analogue to Theorem 2.3.

LEMMA 2.23. *For any* $n\in N$, $k\in\{1,\cdots,n\}$ *and* $a_n>0$, $b_n\in R$ *we have*

$$\sup_{B\in\mathcal{B}^k}|P^n\{(a_n^{-1}(Z_{n-i+1:n}-b_n))_{i=1}^k\in B\}-G^{(k)}(B)|$$
$$\leqq (C+2)k/n+\Big(1-G^{(k)}(M_n^{(k)})^2\exp\Big\{\int_{M_n^{(k)}}-n(1-F(a_nx_k+b_n))$$
$$+\sum_{i=1}^k\log\langle na_nf(a_nx_i+b_n)/w(x_i)\rangle$$
$$-\log(G(x_k))G^{(k)}(d\boldsymbol{x})/G^{(k)}(M_n^{(k)})\Big\}\Big)^{1/2}$$

where $M_n^{(k)}:=\Big\{\boldsymbol{x}\in R^k:\ x_1>\cdots>x_k,\ \prod_{i=1}^k f(a_nx_i+b_n)>0\Big\}$ *and* $G\in\{G_1, G_2, G_3\}$.

PROOF. We utilize again those ideas which led to Theorem 2.3. 2.2 implies with $\tilde{F}^{-1}(x)=F^{-1}(x)1_{(0,1)}(x)$, $x\in R$,

$$\text{(2.24)}\qquad \sup_{B\in\mathcal{B}^k}\Big|P^n\{(a_n^{-1}(Z_{n-i+1:n}-b_n))_{i=1}^k\in B\}$$
$$-E^k\Big\{\Big(a_n^{-1}\Big(\tilde{F}^{-1}\Big(1-n^{-1}\sum_{j=1}^i\pi_j\Big)-b_n\Big)\Big)_{i=1}^k\in B\Big\}\Big|\leqq Ck/n\ ,$$

where again E denotes the standard exponential distribution and $\pi_i(\boldsymbol{x})=x_i$ for $\boldsymbol{x}=(x_1,\cdots,x_n)\in R^n$, $i=1,\cdots,n$. Furthermore,

$$\text{(2.25)}\quad \sup_{B\in\mathcal{B}^k}\Big|E^k\Big\{\Big(a_n^{-1}\Big(\tilde{F}^{-1}\Big(1-n^{-1}\sum_{j=1}^i\pi_j\Big)-b_n\Big)\Big)_{i=1}^k\in B\Big\}$$
$$-E^k\Big\{\Big(a_n^{-1}\Big(F^{-1}\Big(1-n^{-1}\sum_{j=1}^i\pi_j\Big)-b_n\Big)\Big)_{i=1}^k\in B,\ 0<\pi_1<\sum_{j=1}^k\pi_j<n\Big\}\Big|$$
$$\leqq\exp(-n)\sum_{i=0}^{k-1}n^i/i!\ .$$

This follows easily from the fact that $E^k*\Big(-\sum_{j=1}^k\pi_j\Big)=G_{2,1,(k)}$.

Moreover, the measure $\mu^{(k)}/\mathcal{B}^k$, defined by

$$\text{(2.26)}\quad \mu^{(k)}(B):=E^k\Big\{\Big(a_n^{-1}\Big(F^{-1}\Big(1-n^{-1}\sum_{j=1}^i\pi_j\Big)-b_n\Big)\Big)_{i=1}^k\in B,\ 0<\pi_1<\sum_{j=1}^k\pi_j<n\Big\}$$

has Lebesgue density for $\boldsymbol{x}\in R^k$

$$\text{(2.27)}\qquad h^{(k)}(\boldsymbol{x}):=\exp\{-n(1-F(a_nx_k+b_n))\}\prod_{i=1}^k na_nf(a_nx_i+b_n)$$
$$\cdot\ 1_{\{\boldsymbol{y}\in R^k:y_1>\cdots>y_k\}}(\boldsymbol{x})\ .$$

Hence, Lemma 2.1 yields

$$\sup_{B\in\mathcal{B}^k}|\mu^{(k)}(B)-G^{(k)}(B)|$$
$$\leqq\left(1-G^{(k)}(M_n^{(k)})^2\exp\left\{\int_{M_n^{(k)}}\log(h^{(k)}/g^{(k)})dG^{(k)}/G^{(k)}(M_n^{(k)})\right\}\right)^{1/2}$$
$$=\left(1-G^{(k)}(M_n^{(k)})^2\exp\left\{\int_{M_n^{(k)}}-n(1-F(a_nx_k+b_n))\right.\right.$$
$$\left.\left.+\sum_{i=1}^{k}\log(na_nf(a_nx_i+b_n))-\log(g^{(k)}(x))G^{(k)}(dx)/G^{(k)}(M_n^{(k)})\right\}\right)^{1/2}$$

which together with (1.20) yields the assertion.

In the case that $G^{(k)}(M_n^{(k)})=1$, Lemma 2.23 reduces to the following result which fits completely with Corollary 2.7 by putting $k=1$.

COROLLARY 2.28. *Assume that* $G^{(k)}(M_n^{(k)})=1$ *where* $G\in\{G_1,G_2,G_3\}$. *Then we have for any* $n\in N$ *and* $k\in\{1,\cdots,n\}$, $a_n>0$, $b_n\in R$

$$\sup_{B\in\mathcal{B}^k}|P^n\{(a_n^{-1}(Z_{n-i+1:n}-b_n))_{i=1}^k\in B\}-G^{(k)}(B)|\leqq(C+2)k/n+(\tilde{B}_n^{(k)})^{1/2}$$

where

$$\text{(2.29)}\quad \tilde{B}_n^{(k)}:=\int[na_nf(a_nx+b_n)/w(x)-1-\log\{na_nf(a_nx+b_n)/w(x)\}]$$
$$\cdot\sum_{i=1}^{k}g_{(i)}(x)dx\qquad \textit{if } G\in\{G_1,G_3\}$$

and

$$\text{(2.30)}\quad \tilde{B}_n^{(k)}:=\int[na_nf(a_nx+b_n)/w(x)-1-\log\{na_nf(a_nx+b_n)/w(x)\}]$$
$$\cdot\sum_{i=1}^{k}g_{(i)}(x)dx+n(1-F(b_n))\qquad \textit{if } G=G_2\,.$$

Remarks 2.31. As mentioned in Remark 2.19, the term $1-F(b_n)$ in formula (2.30) will usually be equal to zero yielding the same bound $\tilde{B}_n^{(k)}$ for any of the three cases G_1, G_2, G_3.

Furthermore, we see that again the rate at which $na_nf(a_nx+b_n)/w(x)$ tends to one is crucial for the rate of convergence at which even the joint distribution of extremes tends to $G^{(k)}$. Roughly speaking, this rate is of order $O\left(\left\{\int[na_nf(a_nx+b_n)/w(x)-1]^2\sum_{i=1}^{k}g_{(i)}(x)dx\right\}^{1/2}\right)$.

PROOF OF COROLLARY 2.28. The equality $G^{(k)}*\pi_i=G_{(i)}$, $1\leqq i\leqq k$, together with integration by parts yields (see formula (1.19)).

$$\int_{M_n^{(k)}}1-F(a_nx_k+b_n)G^{(k)}(dx)$$

$$=\int 1-F(a_nx+b_n)G_{(k)}(dx)$$
$$=a_n\int f(a_nx+b_n)G_{(k)}(x)dx$$
$$=a_n\int f(a_nx+b_n)G(x)\sum_{i=0}^{k-1}[-\log (G(x))]^i/i!dx\ .$$

Thus, if $G\in\{G_1, G_3\}$

$$1-\exp\left\{\int_{M_n^{(k)}} -n(1-F(a_nx_k+b_n))+\sum_{i=1}^{k}\log \langle na_nf(a_nx_i+b_n)/w(x_i)\rangle -\log (G(x_k))G^{(k)}(d\boldsymbol{x})\right\}$$
$$\leqq\int [na_nf(a_nx+b_n)/w(x)-1]g(x)\sum_{i=0}^{k-1}[-\log (G(x))]^i/i!$$
$$-\log \langle na_nf(a_nx+b_n)/w(x)\rangle g(x)\sum_{i=0}^{k-1}[-\log (G(x))]^i/i!dx$$
$$+\int g(x)\sum_{i=0}^{k-1}[-\log (G(x))]^i/i!+\log (G(x))g_{(k)}(x)dx\ .$$

Now, $\int g(x)\sum_{i=0}^{k-1}[-\log (G(x))]^i/i!dx=\sum_{i=0}^{k-1}\int g_{(i)}(x)dx=k$ (see formula (1.18)) and, by integration by parts, $\int \log (G(x))g_{(k)}(x)dx=-\int [g(x)/G(x)]G_{(k)}(x)dx$ $=-\int \sum_{i=1}^{k} g_{(i)}(x)dx=-k$ (see formula (1.19)). Thus, Lemma 2.23 yields the first part of Corollary 2.28, i.e. formula (2.29).

If $G=G_2$ then

$$\int_{M_n^{(k)}} 1-F(a_nx_k+b_n)G_2^{(k)}(d\boldsymbol{x})$$
$$=a_n\int_{-\infty}^{0} f(a_nx+b_n)G_2(x)\sum_{i=0}^{k-1}[-\log (G_2(x))]^i/i!dx$$
$$+a_n\int_0^\infty f(a_nx+b_n)dx\ .$$

Since $a_n\int_0^\infty f(a_nx+b_n)dx=1-F(b_n)$, the assertion of Corollary 2.28 is now completed by arguments which are analogue to the preceding ones.

By means of Corollary 2.28 the following results can easily be derived. First we deal with the standard Cauchy distribution C.

Example 2.32. We have uniformly for any $n\in N$ and $k\in\{1,\cdots,n\}$

$$\sup_{B\in\mathfrak{B}^k}|C^n\{((\pi/n)Z_{n-i+1:n})_{i=1}^k\in B\}-G_{1,1}^{(k)}(B)|=O(k^{3/2}/n)\ .$$

PROOF. We have

$$\tilde{B}_n^{(k)} = \int_0^\infty \{\log\langle 1+\pi^2 n^{-2} x^{-2}\rangle + [1+\pi^2 n^{-2} x^{-2}]^{-1} - 1\} G_{1,1,(k)}(x) x^{-2} dx$$
$$\leqq \pi^2 n^{-2} \sum_{i=0}^{k-1} \int_0^\infty x^{i+2} \exp(-x) dx/i! = \pi^2 n^{-2} \sum_{i=0}^{k-1} (i+1)(i+2) .$$

Thus, the assertion is now immediate from Corollary 2.28.

According to the long history of the investigation of extremes from a normal population, our next example may be regarded as a main result.

Example 2.33. For the normal distribution we have with b_n chosen as in (1.13)

$$\sup_{B\in\mathcal{B}^k} |N_{(0,1)}^n\{((b_n(Z_{n-i+1:n}-b_n))_{i=1}^k \in B\} - G_3^{(k)}(B)|$$
$$= O\Big(\Big\{\sum_{i=1}^k \int x^4 g_{3,(i)}(x) dx\Big\}^{1/2}\Big)\Big/\log(n)$$
$$= O(k^{1/2}\log^2(k+1)/\log(n)) .$$

PROOF. We have in analogy to the proof of Example 2.22

$$\tilde{B}_n^{(k)} \leqq b_n^{-4}/8 \int x^4 \sum_{i=1}^k g_{3,(i)}(x) dx = b_n^{-4}/8 \int x^4 G_{3,(k)}(x) \exp(-x) dx .$$

Moreover, since $0 < G_{3,(k)} < 1$,

$$\int x^4 G_{3,(k)}(x)\exp(-x) dx \leqq \int_{-\infty}^0 x^4 G_{3,(k)}(x) \exp(-x) dx + 4! .$$

Now, integration by parts and substitution $y = -\log(x)$ yields

$$\int_{-\infty}^0 x^4 G_{3,(k)}(x) \exp(-x) dx$$
$$= \int_{-\infty}^0 \exp(-x)\{4x^3 G_{3,(k)}(x) + x^4 g_{3,(k)}(x)\} dx$$
$$\leqq \int_{-\infty}^0 \exp(-x) x^4 g_{3,(k)}(x) dx = \int_1^\infty \log^4(x) \exp(-x) x^k dx/(k-1)!$$
$$\leqq Lk \log^4(k+1) , \qquad L \text{ being a generic constant} .$$

The assertion is now immediate from Corollary 2.28.

Imposing further conditions on F we will study in the following by means of Lemma 2.23 the influence of h in the representation $f = w_{(n)}[1+h]$ on the rate at which the joint distribution of the k largest order statistics under P^n reaches its limit in more detail.

Therefore, define for $c > -1$ and $L \geqq 0$ a neighborhood $U_{c,L}(W_{(n)})$ of $W_{(n)}$ by

(2.34) $U_{c,L}(W_{(n)}):=\{P\colon P$ has Lebesgue density $f=w_{(n)}[1+h]$, where $c\leqq h\leqq L\}$.

Notice that we can suppose $h\equiv 0$ on $\{w_{(n)}=0\}$.

Now, by means of Lemma 2.23 we can prove the following result.

THEOREM 2.35. *For any $c>-1$ and $L\geqq 0$ there exist positive constants D_1, D_2 such that for any $K\subset U_{c,L}(W_{(n)})$ and $k\in\{1,\cdots,n\}$, $n\in\boldsymbol{N}$, the following bound holds.*

$$\sup_{P\in K}\sup_{B\in\mathcal{B}^k}|P^n\{(Z_{n-i+1:n})_{i=1}^k\in B\}-G^{(k)}(B)|$$
$$\leqq D_1k/n+D_2\sup_{P\in K}\left\{\int_{\{w_{(n)}>0\}}h^2(x)\left(\sum_{i=1}^k G_{(i)}\right)(dx)\right\}^{1/2}$$

where $G\in\{G_1, G_2, G_3\}$ is the (weak) limit of $W_{(n)}^n$, $n\in\boldsymbol{N}$.

Putting $h\equiv 0$ in the preceding result we obtain the following consequence for *gPd*'s, thus getting back Lemma 2.2.

COROLLARY 2.36. *There exists a constant $D>0$ such that for any shifted gPd $W_{(n)}$ with corresponding limiting distribution G*

$$\sup_{B\in\mathcal{B}^k}|W_{(n)}^n\{(Z_{n-i+1:n})_{i=1}^k\in B\}-G^{(k)}(B)|\leqq Dk/n\ .$$

Example 2.37. For the usual Pareto distribution $W_{1,\alpha}$ we get uniformly for all $\alpha>0$

$$\sup_{B\in\mathcal{B}^k}|W_{1,\alpha}^n\{(n^{-1/\alpha}Z_{n-i+1:n})_{i=1}^k\in B\}-G_{1,\alpha}^{(k)}(B)|\leqq Dk/n\ .$$

Example 2.38. In the case of the standard exponential distribution we obtain

$$\sup_{B\in\mathcal{B}^k}|E^n\{(Z_{n-i+1:n}-\log(n))_{i=1}^k\in B\}-G_3^{(k)}(B)|\leqq Dk/n\ .$$

PROOF OF THEOREM 2.35. Choose $P\in U_{c,L}(W_{(n)})$. Then,

$$\begin{aligned}(2.39)\qquad 1-G^{(k)}(M_n^{(k)})&=1-G^{(k)}\{\boldsymbol{x}\in(G^{-1}(0,1))^k\colon w_{(n)}(x_k)>0\}\\ &=1-G_{(k)}\{x\in G^{-1}(0,1)\colon w_{(n)}(x)>0\}\\ &=\exp(-n)\sum_{i=0}^{k-1}n^i/i!=O(k^2/n^2)\end{aligned}$$

which is trivial. Moreover, denoting the *df* of P by F we have

$$\begin{aligned}(2.40)\qquad \int_{M_n^{(k)}}1-F(x_k)G^{(k)}(d\boldsymbol{x})&=\int_{\{w_{(n)}>0\}}1-F(x)G_{(k)}(dx)\\ &=\int_{\{w_{(n)}>0\}}f(y)G_{(k)}(y)dy-G_{(k)}\{w_{(n)}\leqq 0\}\ .\end{aligned}$$

Furthermore, integration by parts yields

$$(2.41)\qquad \int_{M_n^{(k)}} \log(G(x_k))G^{(k)}(d\boldsymbol{x}) = \int_{\{w_{(n)}>0\}} \log(G(x))G_{(k)}(dx) = n\exp(-n)\sum_{i=0}^{k-1} n^i/i! - \int_{\{w_{(n)}>0\}} \sum_{i=1}^{k} g_{(i)}(x)dx$$

where g, $g_{(i)}$ denote the densities of G and $G_{(i)}$.

Finally, we can estimate by means of (2.39) for $i=1,\cdots,k$

$$(2.42)\qquad \int_{M_n^{(k)}} \log\{f(x_i)/w_{(n)}(x_i)\}G^{(k)}(d\boldsymbol{x}) = \int_{M_n^{(k)}} \log\{1+h(x_i)\}G^{(k)}(d\boldsymbol{x})$$
$$= \int_{G^{-1}(0)}^{\infty} \log\{1+h(x)\}G_{(i)}(dx) - \int_{(M_n^{(k)})^c} \log\{1+h(x_i)\}G^{(k)}(d\boldsymbol{x})$$
$$\geqq \int_{\{w_{(n)}>0\}} \log\{1+h(x)\}G_{(i)}(dx) - D\exp(-n)\sum_{i=0}^{k-1} n^i/i!$$

since $\log(1+h)\leqq\log(1+L)$, where D denotes a generic constant.

Now, putting together (2.39)–(2.42) we obtain from Lemma 2.23 the following bound for $P\in U_{c,L}(W_{(n)})$

$$\sup_{B\in\mathcal{B}^k} |P^n\{(Z_{n-i+1:n})_{i=1}^k \in B\} - G^{(k)}(B)|$$
$$\leqq \Big[1-G^{(k)}(M_n^{(k)})^2\exp\Big\{\Big\langle\int_{\{w_{(n)}>0\}} -nf(y)G_{(k)}(y)+\sum_{i=1}^{k}\log\{1+h(y)\} \cdot g_{(i)}(y)dy - Dk\exp(-n)\sum_{i=0}^{k-1} n^i/i!\Big\rangle\Big/G^{(k)}(M_n^{(k)})\Big\}\Big]^{1/2} + (C+2)k/n$$
$$= \Big[1-G^{(k)}(M_n^{(k)})^2\exp\Big\{\Big\langle\int_{\{w_{(n)}>0\}} (\log\{1+h(y)\}-h(y))\sum_{i=1}^{k} g_{(i)}(y)dy - Dk\exp(-n)\sum_{i=0}^{k-1} n^i/i!\Big\rangle\Big/G^{(k)}(M_n^{(k)})\Big\}\Big]^{1/2} + (C+2)k/n$$
$$\leqq \Big[1-G^{(k)}(M_n^{(k)})^2\exp\Big\{-\Big\langle D\int_{\{w_{(n)}>0\}} h^2(y)\sum_{i=1}^{k} g_{(i)}(y)dy + Dk\exp(-n)\sum_{i=0}^{k-1} n^i/i!\Big\rangle\Big/G^{(k)}(M_n^{(k)})\Big\}\Big]^{1/2} + (C+2)k/n$$

since $nG_{(k)}=\sum_{i=1}^{k} g_{(i)}/(n^{-1}w)=\sum_{i=1}^{k} g_{(i)}/w_{(n)}$ if $w_{(n)}>0$. Hence, the assertion is immediate from (2.39) and elementary computations.

The preceding considerations can be specified as follows. Define for $c>-1$ and $L\geqq 0$ a neighborhood $V_{c,L}(W)$ of a *gPd* W by

$$(2.43)\quad V_{c,L}(W) := \{P\colon P \text{ has Lebesgue density } f = w_{(1)}[1+h], \text{ where } c \leq h \leq L\},$$

where again $w_{(1)}$ denotes the density of $W = W_{(1)}$. The following result is now immediate from Theorem 2.35.

PROPOSITION 2.44. *For any $c > -1$, $L \geq 0$ there exist positive constants D_1 and D_2 such that for any $K \subset V_{c,L}(W)$*

$$(2.45)\quad \sup_{P \in K} \sup_{B \in \mathcal{B}^k} |P^n\{(n^{-1/\alpha} Z_{n-i+1/n})_{i=1}^k \in B\} - G_{1,\alpha}^{(k)}(B)| \leq D_1 k/n + D_2 \sup_{P \in K} \left\{\int_{n^{-1/\alpha}}^{\infty} h(n^{1/\alpha}x)^2 \left(\sum_{i=1}^{k} G_{1,\alpha,(i)}\right)(dx)\right\}^{1/2} \quad \text{if } W = W_{1,\alpha},$$

$$(2.46)\quad \sup_{P \in K} \sup_{B \in \mathcal{B}^k} |P^n\{(n^{1/\alpha} Z_{n-i+1:n})_{i=1}^k \in B\} - G_{2,\alpha}^{(k)}(B)| \leq D_1 k/n + D_2 \sup_{P \in K} \left\{\int_{-n^{1/\alpha}}^{0} h(n^{-1/\alpha}x)^2 \left(\sum_{i=1}^{k} G_{2,\alpha,(i)}\right)(dx)\right\}^{1/2} \quad \text{if } W = W_{2,\alpha},$$

$$(2.47)\quad \sup_{P \in K} \sup_{B \in \mathcal{B}^k} |P^n\{(Z_{n-i+1:n} - \log(n))_{i=1}^k \in B\} - G_3^{(k)}(B)| \leq D_1 k/n + D_2 \sup_{P \in K} \left\{\int_{-\log(n)}^{\infty} h(x + \log(n))^2 \left(\sum_{i=1}^{k} G_{3,(i)}\right)(dx)\right\}^{1/2} \quad \text{if } W = W_3.$$

Consequently, imposing further growth-conditions on h, we are led to the following result.

COROLLARY 2.48. *For any $c > -1$ and δ, $L_1, L_2 > 0$ there exist positive constants D_1 and D_2 such that the following holds.*

(2.49) *Put* $V_{c,L_1,\delta,L_2}(W_{1,\alpha}) := \{P \in V_{c,L_1}(W_{1,\alpha})\colon |h(x)| \leq L_2 x^{-\delta\alpha}\}$.

Then,
$$\sup_{P \in V_{c,L_1,\delta,L_2}(W_{1,\alpha})} \sup_{B \in \mathcal{B}^k} |P^n\{(n^{-1/\alpha} Z_{n-i+1:n})_{i=1}^k \in B\} - G_{1,\alpha}^{(k)}(B)| \leq D_1 k/n + D_2 (k/n)^\delta k^{1/2}.$$

(2.50) *Put* $V_{c,L_1,\delta,L_2}(W_{2,\alpha}) := \{P \in V_{c,L_1}(W_{2,\alpha})\colon |h(x)| \leq L_2 |x|^{\delta\alpha}\}$.

Then,
$$\sup_{P \in V_{c,L_1,\delta,L_2}(W_{2,\alpha})} \sup_{B \in \mathcal{B}^k} |P^n\{(n^{1/\alpha} Z_{n-i+1:n})_{i=1}^k \in B\} - G_{2,\alpha}^{(k)}(B)| \leq D_1 k/n + D_2 (k/n)^\delta k^{1/2}.$$

(2.51) *Put* $V_{c,L_1,\delta,L_2}(W_3) := \{P \in V_{c,L_1}(W_3)\colon |h(x)| \leq L_2 \exp(-\delta x)\}$.

Then,
$$\sup_{P \in V_{c,L_1,\delta,L_2}(W_3)} \sup_{B \in \mathcal{B}^k} |P^n\{(Z_{n-i+1:n} - \log(n))_{i=1}^k \in B\} - G_3^{(k)}(B)|$$

$$\leqq D_1 k/n + D_2(k/n)^\delta k^{1/2}\ .$$

Notice that the upper bound in the preceding result tends to zero as n increases iff

$$k = k(n) = o(n^{2\delta/(2\delta+1)}) \tag{2.52}$$

and examples show that this condition is sharp. Moreover, (2.50) yields essentially the result on uniform convergence of extremes by Weiss [23].

PROOF OF COROLLARY 2.48. We have with D denoting a generic constant

$$\begin{aligned}
&\sup_{P\in V_{c,L_1,\delta,L_2}(W_{1,\alpha})} \left\{\int_{n^{-1/\alpha}}^{\infty} h(n^{1/\alpha}x)^2 \sum_{i=1}^{k} g_{1,\alpha,(i)}(x)dx\right\}^{1/2} \\
&\leqq Dn^{-\delta}\left\{\sum_{i=1}^{k}\int_0^\infty x^{-2\delta\alpha} g_{1,\alpha,(i)}(x)dx\right\}^{1/2} \\
&= Dn^{-\delta}\left\{\sum_{i=1}^{k}\int_0^\infty x^{2\delta+i-1}\exp(-x)dx/(i-1)!\right\}^{1/2} \\
&= Dn^{-\delta}\left\{\sum_{i=1}^{k}\Gamma(2\delta+i)/\Gamma(i)\right\}^{1/2}
\end{aligned}$$

where $\Gamma(t) := \int_0^\infty x^{t-1}\exp(-x)dx,\ t>0$, denotes the Gamma-function.

In an analoguous way one shows that this bound is also valid in the other cases. Finally, observe that

$$\sum_{i=1}^{k}\Gamma(2\delta+i)/\Gamma(i) \leqq D\sum_{i=1}^{k} i^{2\delta} \leqq Dk^{2\delta+1}$$

which completes the proof.

Acknowledgment

The author would like to thank R.-D. Reiss for many stimulating discussions and the referee for his careful reading of the manuscript.

UNIVERSITY OF SIEGEN

REFERENCES

[1] Cohen, J. P. (1982). Convergence rates for the ultimate and penultimate approximation in extreme-value theory, *Adv. Appl. Prob.*, 14, 833-854.

[2] Cramér, H. (1946). *Mathematical Methods of Statistics*, Princeton University Press, Princeton N.J.

[3] Falk, M. (1984). *Uniform Convergence of Extreme Order Statistics* I, Preprint 122, University of Siegen.

[4] Galambos, J. (1978). *The Asymptotic Theory of Extreme Order Statistics*, Wiley, New York.

[5] Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une série aléatoire, *Ann. Math.*, **44**, 423-453.

[6] Hall, P. (1979). On the rate of convergence of normal extremes, *J. Appl. Prob.*, **16**, 433-439.

[7] Hall, W. J. and Wellner, J. A. (1979). The rate of convergence in law of the maximum of an exponential sample, *Statist. Neerlandica*, **33**, 151-154.

[8] Ikeda, S. (1963). Asymptotic equivalence of probability distributions with applications to some problems of asymptotic independence, *Ann. Inst. Statist. Math.*, **15**, 87-116.

[9] Ikeda, S. (1968). Asymptotic equivalence of real probability distributions, *Ann. Inst. Statist. Math.*, **20**, 339-362.

[10] Ikeda, S. and Matsunawa, T. (1976). Uniform asymptotic distribution of extremes, in *Essays in Probability and Statistics* (eds. S. Ikeda et al.), Shinko Tsusho, Tokyo, 419-432.

[11] Kohne, W. and Reiss, R.-D. (1983). A note on uniform approximation to distributions of extreme order statistics, *Ann. Inst. Statist. Math.*, **35**, 343-345.

[12] Kraft, C. (1955). Some conditions for consistency and uniform consistency of statistical procedures, *University of California Publications in Statistics*, **2**, 125-142.

[13] LeCam, L. (1970). On the assumptions used to prove asymptotic normality of the maximum likelihood estimates, *Ann. Math. Statist.*, **41**, 802-828.

[14] Pfanzagl, J. (1982). *Contributions to a General Asymptotic Statistical Theory*, Springer, Berlin.

[15] Pickands, J. III (1975). Statistical inference using extreme order statistics, *Ann. Statist.*, **3**, 119-131.

[16] Reiss, R.-D. (1974). On the accuracy of the normal approximation for quantiles, *Ann. Prob.*, **2**, 741-744.

[17] Reiss, R.-D. (1976). Asymptotic expansions for sample quantiles, *Ann. Prob.*, **4**, 249-258.

[18] Reiss, R.-D. (1977). Asymptotic theory for order statistics, *Lecture notes*, University of Freiburg.

[19] Reiss, R.-D. (1981). Uniform approximation to distributions of extreme order statistics, *Adv. Appl. Prob.*, **13**, 533-547.

[20] Reiss, R.-D. (1984). *Statistical Inference Using Approximate Extreme Value Models*, Preprint 124, University of Siegen.

[21] Resnick, S. I. (1971). Tail equivalence and applications, J. *Appl. Prob.*, **8**, 136-156.

[22] Smith, R. L. (1982). Uniform rates of convergence in extreme-value theory, *Adv. Appl. Prob.*, **14**, 600-622.

[23] Weiss, L. (1971). Asymptotic inference about a density function at an end of its range, *Naval Res. Logist. Quaterly*, **18**, 111-114.

[24] Weissman, I. (1975). Multivariate extremal processes generated by independent non-identically distributed random variables, *J. Appl. Prob.*, **12**, 477-487.

[25] Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations, *J. Amer. Statist. Ass.*, **73**, 812-815.