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MODIFIED INFORMATION CRITERIA FOR A UNIFORM APPROXIMATE EQUIVALENCE OF PROBABILITY DISTRIBUTIONS

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Summary

Information criteria for two-sided uniform ϕ-equivalence, which is a newly introduced strong approximate equivalence of probability distributions, are proposed. The criteria resort to some modified K-L informations defined on suitable approximate main domains and are presented in the form of systems with double inequalities. They present systematic implements to handle many statistical approximation problems and are useful to evaluate related approximation errors quantitatively. Criteria for asymptotic cases are also derived from the presented inequalities. As applications, necessary and sufficient conditions and error evaluations are given for approximate and/or asymptotic equivalences of the probability distributions on sampling with and without replacement from a finite population and on quasi-extreme order statistics from a continuous distribution.

1. Introduction

Let X and Y be random variables defined on a measurable space $(R, \boldsymbol{B})$, where R is any abstract space and $\boldsymbol{B}$ stands for a σ-field of subsets of R. Denote the corresponding probability distributions of X and Y by P^X and P^Y, respectively. Suppose that we are interested in some measurable set $A(\in \boldsymbol{B})$ where it is convenient or natural for us to consider approximation problems between the probability distributions. On the set A, let $f^*>0$ and $g^*>0$ be the respective Radon-Nikodym derivatives of P^X and P^Y with respect to a σ-finite measure μ defined on the space $(R, \boldsymbol{B})$. Then, for any measurable set E in the sub-σ-field of $\boldsymbol{B}$ generated by the set A

$$P^X(E)=\int_E f^* d\mu \qquad \text{and} \qquad P^Y(E)=\int_E g^* d\mu .$$

Key words: Modified information criteria, K-L information, uniform ϕ-equivalence, sampling from finite population, quasi-extreme order statistics.

In this paper when we use the character A we always consider the situation that X and Y are absolutely continuous with respect to μ over some broad region containing the set A as its subset. Thus the set A is not necessarily identical with the whole space R. Further X and Y may or may not be dominated by μ outside A. In practical situations the set A may be called *an approximate main domain* or *a domain of our interest* in which a domain of attraction exists for an underlying approximation problem.

In the same set-up as above the following type approximate equivalence of X and Y is of interest: Let ε be a small postive non-negative number and let $\delta^*=\delta^*(X, Y; A)$ be a certain measure of discrepancy defined on A between the two probability distributions. The random variables X and Y are said to be *uniformly ϕ-equivalent with respect to δ^* in the sense of type $(\boldsymbol{B})_d$* and the equivalent notion is denoted as

$$X \overset{\phi}{\sim} Y , \qquad (\boldsymbol{B})_d , \tag{1.1}$$

if the following conditions are satisfied: (i) $\delta^*(X, Y; A) \leqq \varepsilon$ and (ii) there exists a non-negative function $\phi(u; \delta^*)$ of $u\,(>0)$ such that

$$D(X, Y; \boldsymbol{B}) = \sup_{E \in \boldsymbol{B}} |P^X(E) - P^Y(E)| \leqq \phi(\varepsilon; \delta^*) , \tag{1.2}$$

where $\phi(u; \delta^*) \to 0$ as $u \to 0$. To the above ϕ-equivalence Matsunawa [5] gave some criteria based on K-L information and on related measures of discrepancy restricted over the measurable subset A in $\boldsymbol{B}$, as δ^*. However, the results were not necessarily satisfactory, because those were derived from some double inequalities based on a modified affinity and so the lower and upper bounds of $D(X; Y; \boldsymbol{B})$ did not have the same order of magnitude. In this paper several improved lower and upper bounds of almost the same order of magnitude are given in the following section without resorting to the affinity. Those bounds have another important characteristic which contains explicity the information about the approximate main domain for the relevant approximation problems. Therefore, they lead us to a two-sided uniform ϕ-equivalence which is an improvement of (1.1). By the bounds the approximation errors between sampling with and without replacement from a finite population are investigated in the same section. Asymptotic theory of two sequences of random variables is discussed in Section 3 where some modified information criteria are given for the uniform asymptotic equivalence of the sequences in the sense of type $(\boldsymbol{B})_d$. As applications a few new results on asymptotic distributions of quasi-extreme order statistics are given.

2. Criteria for two-sided $(\boldsymbol{B})_a$ ϕ-equivalence

Let us begin to define a two-sided ϕ-equivalence between two probability distributions P^X and P^Y:

DEFINITION 2.1. Let ε be a non-negative small number. The random variables X and Y are said to be *two-sided uniformly ϕ-equivalent with respect to δ^* in the sense of type* $(\boldsymbol{B})_a$ and denoted as

$$X \sim Y \qquad [(\boldsymbol{B})_a; \underline{\phi}, \bar{\phi}] , \tag{2.1}$$

if the following two conditions are satisfied: (i) There exist functions $\underline{\eta}(u)$ and $\bar{\eta}(u)$ such that $\underline{\eta}(\varepsilon) \leqq \delta^*(X, Y; A) \leqq \bar{\eta}(\varepsilon)$, where $\underline{\eta}(u) \leqq \bar{\eta}(u)$ for all $u \geqq 0$. (ii) There exist non-negative functions $\underline{\phi}(u; \delta^*)$ and $\bar{\phi}(u; \delta^*)$ of u defined on $u \geqq 0$ such that

$$0 \leqq \underline{\phi}(\underline{\eta}(\varepsilon); \delta^*) \leqq D(X, Y; \boldsymbol{B}) \leqq \bar{\phi}(\bar{\eta}(\varepsilon); \delta^*) , \tag{2.2}$$

where $\underline{\phi}(u; \delta^*) \to 0$ and $\bar{\phi}(u; \delta^*) \to 0$, as $u \to 0$.

Remark 2.1. The last inequality of (2.2) is formally equivalent to (1.1). However, for concrete examples treated in this paper, ϕ's of (2.2) are constructed more accurately and conveniently than ϕ's given in Matsunawa [5]. The above definition can be extended by considering the bounds $\underline{\delta}^*$ and $\bar{\delta}^*$ such that $\underline{\eta}(\varepsilon) \leqq \underline{\delta}^* \leqq \delta^* \leqq \bar{\delta}^* \leqq \bar{\eta}(\varepsilon)$. We will say the corresponding equivalence a two-sided uniform ϕ-equivalence with respect to $(\underline{\delta}^*, \bar{\delta}^*)$, which will be considered later in Corollary 2.1.

Now, as $\delta^* = \delta^*(X, Y; A)$ let us consider the following modified quantities of the K-L information number:

$$I^*(X, Y; A) = \int_A f^* \ln (f^*/g^*) d\mu , \tag{2.3}$$

$$I^*(Y, X; A) = \int_A g^* \ln (g^*/f^*) d\mu , \tag{2.4}$$

$$I_a^*(X, Y; A) = \int_A f^* |\ln f^*/g^*| d\mu , \tag{2.5}$$

$$I_a^*(Y, X; A) = \int_A g^* |\ln g^*/f^*| d\mu \tag{2.6}$$

and consider the corresponding variation as

$$V^*(X, Y; A) = \int_A |f^* - g^*| d\mu . \tag{2.7}$$

If the set A is taken to be the whole space R, we shall delete all asterisks from the above quantities.

We can state the following fundamental inequalities whose proofs will be postponed until some preparatory lemmas are established.

THEOREM 2.1. *Let A be an approximate main domain. Then the following inequalities hold:*

$$D(X, Y; \boldsymbol{B}) \geqq \underline{V}^*/2 + |P^X(A) - P^Y(A)|/2 \ (\geqq 0)\ , \tag{2.8}$$

$$D(X, Y; \boldsymbol{B}) \leqq \bar{V}^*/2 + 1 - \{P^X(A) + P^Y(A)\}/2\ , \tag{2.9}$$

where

$$\underline{V}^* = \max\left\{\begin{array}{l} -I^*(X, Y; A)\,, \qquad -I^*(Y, X; A)\,, \\ 2l(a^+)\cdot I^*(X, Y; A^+) - (P^X(A) - P^Y(A))\ , \\ -2l(a^-)\cdot I^*(X, Y; A^-) + (P^X(A) - P^Y(A))\ , \\ -2l(b^+)\cdot I^*(Y, X; A^+) - (P^X(A) - P^Y(A))\ , \\ 2l(b^-)\cdot I^*(Y, X; A^-) + (P^X(A) - P^Y(A))\ , \\ l(a)\cdot I_a^*(X, Y; A)\,, \qquad l(b)\cdot I_a^*(Y, X; A)\ , \\ [l(a)|I^*(X, Y; A)|, \qquad l(b)|I^*(Y, X; A)|] \end{array}\right\},$$

$$\bar{V}^* = \min\left\{\begin{array}{l} 2u(c^+)\cdot I^*(X, Y; A^+) - (P^X(A) - P^Y(A))\,, \\ -2u(c^-)\cdot I^*(X, Y; A^-) + (P^X(A) - P^Y(A))\,, \\ -2u(d^+)\cdot I^*(Y, X; A^+) - (P^X(A) - P^Y(A))\,, \\ 2u(d^-)\cdot I^*(Y, X; A^-) + (P^X(A) - P^Y(A))\,, \\ u(c^-)|I^*(X, Y; A)|\,,\ u(d^-)|I^*(Y, X; A)|\,, \\ [u(c)\cdot I_a^*(X, Y; A), \qquad u(d) I_a^*(Y, X; A)] \end{array}\right\},$$

$$A^+ = \{x\,;\, f^*(x) \geqq g^*(x),\ x \in A\}\ , \qquad A^- = A - A^+\ ,$$

$$a = \inf_A (g^*/f^*)\,, \qquad a^+ = \inf_{A^+} (g^*/f^*)\,, \qquad a^- = \inf_{A^-} (g^*/f^*)\,,$$

$$b = \inf_A (f^*/g^*)\,, \qquad b^+ = \inf_{A^+} (f^*/g^*)\,, \qquad b^- = \inf_{A^-} (f^*/g^*)\,,$$

$$c = \sup_A (g^*/f^*)\,, \qquad c^+ = \sup_{A^+} (g^*/f^*)\,, \qquad c^- = \sup_{A^-} (g^*/f^*)\,,$$

$$d = \sup_A (f^*/g^*)\,, \qquad d^+ = \sup_{A^+} (f^*/g^*)\,, \qquad d^- = \sup_{A^-} (f^*/g^*)\,,$$

and for $t > 0$

$$l(t) = \frac{10t^{1/3}(1+t^{1/3})(1+t^{1/3}+t^{2/3})}{3+68t^{1/3}+98t^{2/3}+68t+3t^{4/3}}\ , \tag{2.10}$$

$$u(t) = \frac{(1+t^{1/3})(1+t^{1/3}+t^{2/3})(1+8t^{1/3}+t^{2/3})}{11+38t^{1/3}+11t^{2/3}}\ . \tag{2.11}$$

Remark 2.2. In the above theorem the quantities a, a^+, a^- and so on are tacitly assumed to exist finitely, otherwise the bounds (2.8) and (2.9) have no practical meaning. The second terms in those bounds can be regarded as error estimations on the probability measures escaped from the approximate main domain A. The lower and upper bounds are almost the same order of magnitude in V^*, which results from the estimation due to the inequalities in the following lemma.

LEMMA 2.1. *For any* $t>0$,

$$l(t)|\ln t|\leqq|t-1|\leqq u(t)|\ln t|\,, \tag{2.12}$$

where $l(t)$ *and* $u(t)$ *are the functions defined respectively by* (2.10) *and* (2.11) *and both of them are monotone increasing in* t. *The equality signs in* (2.12) *hold if and only if* $t=1$.

PROOF. Instead of (2.12) we shall show the inequality

$$\frac{30y(y+1)^3}{3y^4+68y^3+98y^2+68y+3}|\ln y| \leqq|y-1|\leqq\frac{3(y+1)(y^2+8y+1)}{11y^2+38y+11}|\ln y|\,,\qquad (y>0)\,, \tag{2.13}$$

from which we can get desired inequality (2.12) by transforming $y=t^{1/3}$.

Using the following well-known expansion

$$S\equiv\frac{1}{2x}\ln\frac{1+x}{1-x}=1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^5}{7}+\cdots\qquad(|x|<1)$$

and putting $x=(y-1)/(y+1)$ $(y>0)$ and $x=1/(2z+1)$ $(z>0$ or $z<-1)$ we have

$$S=\frac{y+1}{2(y-1)}\ln y=(z+1/2)\ln(1+1/z)=1+\sum_{i=1}^{\infty}1/\{(2i+1)(2z+1)^{2i}\}\,. \tag{2.14}$$

Then,

$$\begin{aligned} S&<1+\frac{1}{3(2z+1)^2}+\frac{1}{5(2z+1)^4}\left\{1+\frac{1}{(2z+1)^2}+\frac{1}{(2z+1)^4}+\cdots\right\}\\ &=1+[5\{(2z+1)^2-1\}+3]/[15(2z+1)^2\{(2z+1)^2-1\}]\\ &=1+x^2(5-2x^2)/\{15(1-x^2)\}\,, \end{aligned} \tag{2.15}$$

hence

$$\frac{\ln y}{y-1}<\frac{2}{y+1}\left\{1+\frac{(y-1)^2(3y^2+14y+3)}{60y(y+1)^2}\right\}=\frac{3y^4+68y^3+98y^2+68y+3}{30y(y+1)^3}$$

from which we obtain the L.H.S. inequality in (2.13).

To the contrary

$$S>1+\frac{1}{3(2z+1)^2}\left\{1+\frac{1}{(5/3)(2z+1)^2}+\frac{1}{(5/3)^2(2z+1)^4}+\cdots\right\} \tag{2.16}$$
$$=1+\frac{1}{3(2z+1)^2}\cdot\frac{1}{1-1/((5/3)(2z+1)^2)}=1+\frac{5x^2}{3(5-3x^2)}\ .$$

Therefore,

$$\frac{\ln y}{y-1}>\frac{2}{y+1}\left[1+\frac{5(y-1)^2}{3\{5(y+1)^2-3(y-1)^2\}}\right]=1+\frac{5x^2}{3(5-3x^2)}$$

from which we obtain the R.H.S. inequality in (2.13).

The monotonous property of $l(t)$ and $u(t)$ are easily proved by differentiating them. Thus, the proof of the lemma is accomplished.

Remark 2.3. The inequalities (2.15) and (2.16) are quite accurate, but they can be further refined by taking the exact leading terms of the series in (2.14). So, we can improve the inequalities (2.12) with our required accuracy. The transformation $y=t^{1/3}$ $(t>0)$ makes it possible to adopt fairly wide intervals around neighbourhood of $t=1$ as approximate main domains where we are particularly interested in discussing meaningful approximations between f^* and g^*. For comparisons we shall show numerical results to the quantity $Q\equiv|u-1|$. In the following numerical examples i_k means that the figure 'i' successibly appears k times. For $u=0.95$, $Q=0.05$, $0.049_{10}18<Q<0.050_{10}33$ by (2.13) and $0.049_{12}88<Q<0.050_{13}45$ by (2.12). For $u=1.05$, $Q=0.05$, $0.049_{10}39<Q<0.05_{10}25$ by (2.13) and $0.049_{13}17<Q<0.050_{13}34$ by (2.12). For $u=0.05$, $Q=0.95$, $0.781<Q<1.024$ by (2.13) and $0.949<Q<0.951$ by (2.12). For $u=10$, $Q=9$, $8.47<Q<9.23$ by (2.13) and $8.9985<Q<9.0_36$ by (2.12).

LEMMA 2.2. *Let A be any measurable set in $\boldsymbol{B}$, then it holds that*

$$D(X, Y; \boldsymbol{B})\geqq V^*(X, Y; A)/2+|P^X(A)-P^Y(A)|/2\ , \tag{2.17}$$

and

$$D(X, Y; \boldsymbol{B})\leqq V^*(X, Y; A)/2+1-(P^X(A)+P^Y(A))/2\ . \tag{2.18}$$

PROOF. Consider a decomposition of R such that

$$R^+=\{E;\ P^X(E)\geqq P^Y(E),\ E\in \boldsymbol{B}\}\qquad (\in R)$$

and

$$R^-=\{E;\ P^X(E)<P^Y(E),\ E\in \boldsymbol{B}\}\qquad (=R-R^+)\ .$$

Then,

$$\begin{aligned} D(X, Y; \boldsymbol{B})&=P^X(R^+)-P^Y(R^+)\\ &=P^Y(R^-)-P^X(R^-)\end{aligned} \tag{2.19}$$

$$=P^X(A^+\cup(R^+-A^+))-P^Y(A^+\cup(R^+-A^+))$$
$$=(P^X(A^+)-P^Y(A^+))+(P^X(R^+-A^+)-P^Y(R^+-A^+))$$
$$\geqq\int_{A^+}(f^*-g^*)d\mu$$
$$=\left\{\int_A|f^*-g^*|d\mu+\int_A(f^*-g^*)d\mu\right\}\Big/2$$
$$=V^*(X, Y; A)/2+(P^X(A)-P^Y(A))/2\ .$$

Similarly, it follows that

$$D(X, Y; \boldsymbol{B})\geqq V^*(X, Y; A)/2+(P^Y(A)-P^X(A))/2\ .$$

Therefore, we get the inequality (2.17).

On the other hand, from (2.19)

$$\begin{aligned}D(X, Y; \boldsymbol{B})=&V^*(X, Y; A)/2-(P^X(A)-P^Y(A))/2\\&+\{(P^X(A)-P^Y(A))+(P^X(R^+-A^+)-P^Y(R^+-A^+))\}\ .\end{aligned}$$

The terms in the curly brackets can be estimated as follows:

$$\begin{aligned}\{\qquad\}=&P^X(A^+)+P^X(A^-)-P^Y(A^+)-P^Y(A^-)\\&+P^X(R^+)-P^X(A^+)-P^Y(R^+)+P^Y(A^+)\\=&(P^X(R^+)+P^X(A^-))-(P^Y(R^+)+P^Y(A^-))\\=&P^X(R^+\cup A^-)-P^Y(R^+\cup A^-)\\<&P^X(R^+\cup R^-)-P^Y(A^+\cup A^-)\\=&P^X(R)-P^Y(A)=1-P^Y(A)\ .\end{aligned}$$

Thus, the inequality (2.18) immediately follows and the proof of the lemma is completed.

Now, let us try to give the lower and upper bounds of the quantity $V^*(X, Y; A)$. We can prove the following

LEMMA 2.3. *It holds that*

(2.20) $$\underline{V}^*\leqq V^*(X, Y; A)\leqq\bar{V}^*\ ,$$

where $\underline{V}^*$ *and* $\bar{V}^*$ *are the quantities in* (2.8) *and* (2.9), *respectively,*

PROOF.

$$\begin{aligned}V^*(X, Y; A)=&\int_A|f^*-g^*|d\mu\geqq-\int_A(f^*-g^*)d\mu=-\int_A f^*(1-g^*/f^*)d\mu\\&\geqq-\int_A f^*\ln(f^*/g^*)d\mu\qquad(\because\ \ln t\geqq1-1/t\ \text{for}\ t>0)\ .\end{aligned}$$

Namely,

$$V^* \geqq -I^*(X, Y; A) . \tag{2.21}$$

Similarly, we have the dual inequality:

$$V^* \geqq -I^*(Y, X; A) . \tag{2.22}$$

Since

$$\begin{aligned}
V^* &= \int_A \{2(f^*-g^*)^+ - (f^*-g^*)\} d\mu \\
&= 2\int_{A^+} (f^*-g^*)d\mu - \int_A (f^*-g^*)d\mu \\
&= 2\int_{A^+} f^*(1-g^*/f^*)d\mu - (P^X(A)-P^Y(A)) \\
&\geqq 2\int_{A^+} f^* l(g^*/f^*) \ln (f^*/g^*)d\mu - (P^X(A)-P^Y(A)) \\
& \qquad\qquad\qquad (\because \text{ Lemma 2.1})
\end{aligned}$$

$$\geqq 2l(a^+)\cdot I^*(X, Y; A^+) - (P^X(A)-P^Y(A)) . \tag{2.23}$$

Similarly,

$$\begin{aligned}
V^* &= \int_A \{2(f^*-g^*)^- + (f^*-g^*)\} d\mu \\
&= 2\int_{A^-} (g^*-f^*)d\mu + \int_A (f^*-g^*)d\mu \\
&= 2\int_{A^-} f^*(g^*/f^*-1)d\mu + (P^X(A)-P^Y(A)) \\
&\geqq 2\int_{A^-} f^* l(g^*/f^*) \ln (g^*/f^*)d\mu + (P^X(A)-P^Y(A))
\end{aligned}$$

$$> -2l(a^-)\cdot I^*(X, Y; A^-) + (P^X(A)-P^Y(A)) . \tag{2.24}$$

We have also

$$\begin{aligned}
V^* &= 2\int_{A^+} g^*(f^*/g^*-1)d\mu - (P^X(A)-P^Y(A)) \\
&\geqq 2\int_{A^+} g^* l(f^*/g^*) \ln (f^*/g^*)d\mu - (P^X(A)-P^Y(A))
\end{aligned}$$

$$= -2l(b^+)\cdot I^*(Y, X; A^+) - (P^X(A)-P^Y(A)) , \tag{2.25}$$

and

$$\begin{aligned}
V^* &= 2\int_{A^-} g^*(1-f^*/g^*)d\mu + (P^X(A)-P^Y(A)) \\
&\geqq 2\int_{A^-} g^* l(f^*/g^*) \ln (g^*/f^*)d\mu + (P^X(A)-P^Y(A))
\end{aligned}$$

$$\geqq 2l(b^-)\cdot I^*(Y, X; A^-) + (P^X(A)-P^Y(A)) . \tag{2.26}$$

Further,

$$V^*=\int_A f^*|1-g^*/f^*|d\mu \geqq \int_A f^* l(g^*/f^*)|\ln(f^*/g^*)|d\mu$$
$$\geqq l(a)\cdot\int_A f^*|\ln(f^*/g^*)|d\mu = l(a)\cdot I_a^*(X, Y; A)$$
$$\geqq l(a)\left|\int_A f^*\ln(f^*/g^*)d\mu\right| = l(a)|I^*(X, Y; A)| , \tag{2.27}$$

and similarly

$$V^*\geqq l(b)\cdot I_a^*(Y, X; A)\geqq l(b)|I^*(Y, X; A)| . \tag{2.28}$$

Thus, we have proved the L.H.S. inequality in (2.20). Analogously, the R.H.S. inequality (2.20) can be proved by showing the corresponding inequalities in parallel with (2.23)-(2.28), which is easy and will be ommitted.

PROOF OF THEOREM 2.1. The theorem is now straight-forwardly obtained by Lemma 2.2 and Lemma 2.3.

Now, we shall proceed to consider the two-sided ϕ-equivalence of probability distributions. Let us begin with a practically interesting case. Let ε be any given non-negative number and consider the set

$$B_\varepsilon=\{x ; |\ln f^*(x)/g^*(x)|\leqq\zeta(\varepsilon),\ x\in A\subset R\} , \tag{2.29}$$

where $\zeta(\varepsilon)\to 0$ as $\varepsilon\to 0$ and A is an approximate main domain for X and Y. Assume that

$$\min(P^X(B_\varepsilon), P^Y(B_\varepsilon))\geqq 1-\xi(\varepsilon) , \tag{2.30}$$

here $\xi(\varepsilon)$ is a non-negative and monotone decreasing function of ε as ε decreases. Then there exist decreasing functions $\xi_1(\varepsilon)$, $\xi_2(\varepsilon)$ and $\xi_3(\varepsilon)$ such that $0\leqq\xi_1(\varepsilon)\leqq\xi_2(\varepsilon)\leqq\xi_3(\varepsilon)<\xi(\varepsilon)<1$ and that

$$\begin{aligned}\max(P^X(A), P^Y(A))&\geqq 1-\xi_1(\varepsilon)\geqq 1-\xi_2(\varepsilon)\geqq\min(P^X(A), P^Y(A))\\ &\geqq 1-\xi_3(\varepsilon)\geqq\min(P^X(B_\varepsilon), P^Y(B_\varepsilon))\\ &\geqq 1-\xi(\varepsilon) .\end{aligned} \tag{2.31}$$

It should be noted that the functions ξ, ξ_1, ξ_2 and ξ_3 are determined according to the set A and B_ε for individual problems. Further, we have the following inequality by scrutinizing the proof of Lemma 2.2 in Matsunawa [5].

$$\min(P^X(B_\varepsilon), P^Y(B_\varepsilon))\geqq\max(P^X(A), P^Y(A))-\frac{1}{c(\varepsilon)}V^*(X, Y; A) , \tag{2.32}$$

where $c(\varepsilon)=\min(\zeta(\varepsilon), 1)$.

This inequality and (2.17) yield

$$\min(P^X(B_\varepsilon), P^Y(B_\varepsilon))$$

$$\geqq \max(P^X(A), P^Y(A)) - \frac{2}{c(\varepsilon)} D(X, Y; \boldsymbol{B})$$

$$+\frac{1}{c(\varepsilon)} |P^X(A) - P^Y(A)|$$

$$= \left(1+\frac{1}{c(\varepsilon)}\right) \max(P^X(A), P^Y(A)) - \frac{1}{c(\varepsilon)} \min(P^X(A), P^Y(A)) \tag{2.33}$$

$$-\frac{2}{c(\varepsilon)} D(X, Y; \boldsymbol{B}) .$$

Hence,

$$D(X, Y; \boldsymbol{B}) \geqq \frac{c(\varepsilon)}{2}\Big[\left(1+\frac{1}{c(\varepsilon)}\right) \max(P^X(A), P^Y(A))$$

$$-\frac{1}{c(\varepsilon)} \min(P^X(A), P^Y(A)) - \min(P^X(B_\varepsilon), P^Y(B_\varepsilon))\Big]$$

$$\geqq \frac{c(\varepsilon)}{2}(\xi_3(\varepsilon) - \xi_1(\varepsilon)) + \frac{1}{2}(\xi_2(\varepsilon) - \xi_1(\varepsilon)) \geqq 0 \qquad (\because\ (2.31)) . \tag{2.34}$$

On the other hand, since

$$V^*(X, Y; A) \leqq \int_{A-B_\varepsilon} |f^* - g^*| d\mu + \int_{B_\varepsilon} \frac{f^*+g^*}{2} \left|\ln \frac{f^*}{g^*}\right| d\mu$$

$$\leqq \max(P^X(A), P^Y(A)) - \min(P^X(B_\varepsilon), P^Y(B_\varepsilon))$$

$$+\zeta(\varepsilon)\cdot \max(P^X(A), P^Y(A))$$

$$\leqq \xi(\varepsilon) + \zeta(\varepsilon) ,$$

we have from (2.18), (2.30) and (2.31)

$$D(X, Y; \boldsymbol{B}) \leqq V^*(X, Y; A)/2 + 1 - (P^X(A) + P^Y(A))/2$$

$$\leqq \frac{1}{2}\xi(\varepsilon) + \frac{1}{2}\zeta(\varepsilon) + \xi_3(\varepsilon) . \tag{2.35}$$

Thus, we have proved the following two-sided ϕ-equivalence for X and Y:

THEOREM 2.2. *If the condition* (2.30) *is satisfied, then*

$$X \sim Y \qquad [(\boldsymbol{B})_d ; \underline{\phi}, \bar{\phi}] , \tag{2.36}$$

with

$$\underline{\phi} = \underline{\phi}(\underline{\eta}(\varepsilon); V^*) = \frac{c(\varepsilon)}{2}(\xi_3(\varepsilon) - \xi_1(\varepsilon)) + \frac{1}{2}(\xi_2(\varepsilon) - \xi_1(\varepsilon)) , \tag{2.37}$$

$$\bar{\phi} = \bar{\phi}(\bar{\eta}(\varepsilon); V^*) = \frac{1}{2}\xi(\varepsilon) + \frac{1}{2}\zeta(\varepsilon) + \xi_3(\varepsilon) , \tag{2.38}$$

where $\underline{\eta}(\varepsilon) \geqq c(\varepsilon)(\xi_3(\varepsilon) - \xi_1(\varepsilon))$, $\bar{\eta}(\varepsilon) \leqq \xi(\varepsilon) + \zeta(\varepsilon)$ *and* $c(\varepsilon) = \min(\zeta(\varepsilon), 1)$. $\xi(\varepsilon)$, $\xi_i(\varepsilon)$ $(i=1, 2, 3)$ *and* $\zeta(\varepsilon)$ *are the functions in* (2.31) *and* (2.29), *respectively.*

Remark 2.4. In practical situations concrete forms for $\underline{\eta}(\varepsilon)$ and $\bar{\eta}(\varepsilon)$ are not necessarily needed, though these functions are conceptually significant in the definition and the qualitative development of underlying ϕ-equivalence theory. In fact, the most important point of the ϕ-equivalence is to find sharp lower and upper bounds to $D(X, Y; \boldsymbol{B})$, and we can very often realize it without giving the forms of the functions in advance.

From Theorem 2.1, Theorem 2.2 and Remark 2.1 we can state the following extended result:

COROLLARY 2.1. *If there exist two decreasing functions* $\underline{\eta}_l(\varepsilon)$ *and* $\bar{\eta}_u(\varepsilon)$ *such that*

$$(2.39) \qquad (0 \leqq)\, \underline{\eta}_l(\varepsilon) \leqq \underline{V}^* \,(\leqq V^* \leqq)\, \bar{V}^* \leqq \bar{\eta}_u(\varepsilon)\,; \qquad \bar{\eta}_u(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\,,$$

are satisfied, then under the condition (2.30) X *and* Y *are two-sided* ϕ-*equivalent with respect to* $(\underline{V}^*, \bar{V}^*)$ *with*

$$(2.37)' \qquad \underline{\phi} = \underline{\phi}(\underline{\eta}_l(\varepsilon)\,; \underline{V}^*) \geqq \underline{\eta}_l(\varepsilon)/2 + (\xi_2(\varepsilon) - \xi_1(\varepsilon))/2\,,$$

$$(2.38)' \qquad \bar{\phi} = \bar{\phi}(\bar{\eta}_u(\varepsilon)\,; \bar{V}^*) \leqq \bar{\eta}_u(\varepsilon)/2 + \xi_3(\varepsilon)\,,$$

where $\xi_i(\varepsilon)$ $(i=1, 2, 3)$ *are the same functions as those in Theorem* 2.2, $\underline{V}^*$ *and* $\bar{V}^*$ *are the bounds of the quantity* $V^*(X, Y; A)$ *given in the preceding theorem.*

A partial inverse of Theorem 2.2 holds as follows:

THEOREM 2.3. *If for any positive number* ε *there exist functions* $\underline{\tau}(\varepsilon)$ *and* $\bar{\tau}(\varepsilon)$ *such that*

$$(0 \leqq)\, \underline{\tau}(\varepsilon) \leqq D(X, Y; \boldsymbol{B}) \leqq \bar{\tau}(\varepsilon)\,; \qquad \bar{\tau}(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\,,$$

then there exists a measurable set A *in* $\boldsymbol{B}$ *such that*

$$(2.40) \qquad \min(P^X(A), P^Y(A)) \geqq 1 - \underline{\tau}(\varepsilon)$$

and that

$$(2.41) \qquad (0 \leqq)\, 2(\underline{\tau}(\varepsilon) - \xi_3^*(\varepsilon)) \leqq V^*(X, Y; A) \leqq 2\bar{\tau}(\varepsilon) + \xi_1^*(\varepsilon) - \xi_2^*(\varepsilon)\,,$$

where $\xi_i^*(\varepsilon)$ $(i=1, 2, 3)$ *are non-negative functions such that* $\xi_1^*(\varepsilon) \leqq \xi_2^*(\varepsilon) \leqq \xi_3^*(\varepsilon) \leqq \underline{\tau}(\varepsilon)$.

PROOF. From the assumption of the theorem it is obvious that we can take some measurable set $A \in \boldsymbol{B}$ and there exist non-negative functions $\xi_i^*(\varepsilon)$ $(i=1, 2, 3)$ such that $\xi_1^*(\varepsilon) \leqq \xi_2^*(\varepsilon) \leqq \xi_3^*(\varepsilon) \leqq \underline{\tau}(\varepsilon)$ and that

$$\begin{aligned} \max(P^X(A), P^Y(A)) &\geqq 1-\xi_1^*(\varepsilon) \geqq 1-\xi_2^*(\varepsilon) \\ &\geqq \min(P^X(A), P^Y(A)) \geqq 1-\xi_3^*(\varepsilon) \geqq 1-\underline{\tau}(\varepsilon) . \end{aligned}$$

Then, from (2.18)

$$\begin{aligned} V^*(X, Y; A) &\geqq 2D(X, Y; \boldsymbol{B}) - 2\{1-\min(P^X(A), P^Y(A))\} \\ &\geqq 2\{\underline{\tau}(\varepsilon) - \xi_3^*(\varepsilon)\} \geqq 0 , \end{aligned}$$

which shows the L.H.S. inequality of (2.41).

On the other hand, from (2.17) and (2.40).

$$\begin{aligned} V^*(X, Y; A) &\leqq 2D(X, Y; \boldsymbol{B}) - |P^X(A) - P^Y(A)| \\ &\leqq 2\bar{\tau}(\varepsilon) - \{\max(P^X(A), P^Y(A)) - \min(P^X(A), P^Y(A)\} \\ &\leqq 2\bar{\tau}(\varepsilon) - \{(1-\xi_1^*(\varepsilon)) - (1-\xi_2^*(\varepsilon))\} \\ &= 2\bar{\tau}(\varepsilon) + \xi_1^*(\varepsilon) - \xi_2^*(\varepsilon) , \end{aligned}$$

which proves the R.H.S. inequality of (2.41).

Application 1. Now, as an example of the two-sided ϕ-equivalence we shall consider the difference between sampling with and without replacement from a finite population. Let $\Omega_N = \{x_1^N, x_2^N, \cdots, x_N^N\}$ be a set of N distinct elements, and let $X_{(n)}^N = (X_1^N, X_2^N, \cdots, X_n^N)$ be a random variable to a random sampling *without replacement* of size n with probability

$$P(X_{(n)}^N = x_{(n)}) = 1/N_n \qquad (= p_N(x_{(n)}), \text{ say}) ,$$

where $x_{(n)} \in D_{(n)}^N$, $D_{(n)}^N$ being the set of all non-repeated n-permutations out of Ω_N and $N_n = N(n-1)\cdots(N-n+1)$. On the other hand, let $Y_{(n)}^N = (Y_1^N, Y_2^N, \cdots, Y_n^N)$ be a random variable corresponding to a random sampling *with replacement* of size n with probability

$$P(Y_{(n)}^N = y_{(n)}) = 1/N^n \qquad (= q_N(y_{(n)}), \text{ say}) ,$$

where $y_{(n)} \in E_{(n)}^N$, the set of all repeated n-permutations out of Ω_N. The restricted K-L information over $D_{(n)}^N$ is given by

$$\begin{aligned} I^*(X_{(n)}^N, Y_{(n)}^N; D_{(n)}^N) &= \sum_{x_{(n)} \in D_{(n)}^N} p_N(x_{(n)}) \ln \frac{p_N(x_{(n)})}{q_N(x_{(n)})} \\ &= \ln N^n/N_n = -\ln \prod_{i=1}^{n-1} (1-i/N) . \end{aligned} \tag{2.42}$$

Since $P^{X_{(n)}^N}(D_{(n)}^N) = 1$ and $P^{Y_{(n)}^N}(D_{(n)}^N) = N_n/N^n \geqq 1 - n(n-1)/(2N)$, then putting $A = A^+ = D_{(n)}^N$ in Theorem 2.1 we have

(2.43) $\underline{V}^*/2+(1-N_n/N^n)/2 \leqq D(X^N_{(n)}, Y^N_{(n)}; \boldsymbol{B}) \leqq \bar{V}^*/2+(1-N_n/N^n)/2$.

Now, as $\underline{V}^*$ and $\bar{V}^*$ let us adopt $2l(a^+)I^*(X, Y; A^+)-(P^X(A)-P^Y(A))$ and $2u(c^+)I^*(X, Y; A^+)-(P^X(A)-P^Y(A))$, respectively. Then, since $a^+=N_n/N^n=c^+$, from (2.42) and (2.43) we get

(2.44) $l(N_n/N^n)\ln (N^n/N_n) \leqq D(X^N_{(n)}, Y^N_{(n)}; \boldsymbol{B}) \leqq u(N_n/N^n)\ln (N^n/N_n)$,

where $l(\cdot)$ and $u(\cdot)$ are the functions defined by (2.10) and (2.11), respectively. In view of Lemma 2.1 and the numerical examples the lower and upper bounds in (2.44) are very close except for the cases where the quantity $N_n/N^n=\prod_{i=1}^{n-1}(1-i/N)$ take considerably small values near to zero.

For $N>n\geqq 2$ rather rough evaluations than (2.44) can be given as

(2.45) $$D(X^N_{(n)}, Y^N_{(n)}; \boldsymbol{B}) \geqq l\left(\left(1-\frac{n}{N}\right)^{n-1}\right)\cdot\left[\frac{n(n-1)}{2N}+\frac{n^2(2n-3)}{12N(N-n)}\right] ,$$

(2.46) $$D(X^N_{(n)}, Y^N_{(n)}; \boldsymbol{B}) \leqq u\left(\left(1-\frac{1}{N}\right)^{n-2}\right)\cdot\left[\frac{n(n-1)}{2N}+\frac{n^3}{6N^2} - \frac{n\{N(n-1)-n\}}{4N^2(N-n)}\right] .$$

In view of the above inequalities we can state the following two-sided uniform ϕ-equivalence with respect to $I^*(X, Y; A)$: Let $\varepsilon=\varepsilon_N=n^2/N$, $n=n(N)$, for each N. Then $\min(P^X(A), P^Y(A))=P^Y(A)\geqq 1-n(n-1)/2N\geqq 1-\varepsilon/2$ and

(2.47) $$X^N_{(n)}\sim Y^N_{(n)} \qquad [(\boldsymbol{B})_d; \underline{\phi}, \bar{\phi}] ,$$

where

$$\underline{\phi}=\underline{\phi}(\underline{\eta}(\varepsilon); I^*)=l((1-\sqrt{\varepsilon/N})^{\sqrt{N\varepsilon}-1})\cdot\underline{\eta}(\varepsilon) ,$$

$$\bar{\phi}=\bar{\phi}(\bar{\eta}(\varepsilon); I^*)=u((1-1/N)^{\sqrt{N\varepsilon}-1})\cdot\bar{\eta}(\varepsilon) ,$$

$$\underline{\eta}(\varepsilon)=\frac{\varepsilon}{2}\left(1-\frac{1}{\sqrt{N\varepsilon}}\right)+\frac{\varepsilon(2\sqrt{\varepsilon}-3/\sqrt{N})}{12\sqrt{N}(1-1/\sqrt{N\varepsilon})} ,$$

$$\bar{\eta}(\varepsilon)=\frac{\varepsilon}{2}\left(1-\frac{1}{\sqrt{N\varepsilon}}\right)+\frac{\varepsilon\sqrt{\varepsilon}}{6\sqrt{N}}-\frac{\varepsilon\{(1-1/\sqrt{N\varepsilon})-1/N\}}{4N(1-\sqrt{\varepsilon/N})}$$

and $l(\cdot)$ and $u(\cdot)$ are the functions defined by (2.10) and (2.11), respectively. Clearly, $\underline{\eta}(\varepsilon)$, $\bar{\eta}(\varepsilon)$, $\underline{\phi}$ and $\bar{\phi}$ decreases to zero, as $\varepsilon\to 0$.

Concerning the above topic Freedman [1] gave the following exact result

(2.48) $$D(X^N_{(n)}, Y^N_{(n)}; \boldsymbol{B})=1-N_n/N^n ,$$

from which he also gave the simple bounds

$$(2.49) \qquad 1-\exp(-n(n-1)/N)<D(X^N_{(n)}, Y^N_{(n)}; \boldsymbol{B})<n(n-1)/2N .$$

Therefore, making use of the inequalities, we can get the different forms of the two-sided ϕ-equivalence. Further, from the above results we can easily see that

$$(2.50) \qquad X^N_{(n)} \sim Y^N_{(n)} , \qquad (\boldsymbol{B})_d \ (N\to\infty) \ \text{ iff } \ n^2/N\to 0, \text{ as } N\to\infty ,$$

which is the result obtained through a different approach in Ikeda and Matsunawa [2] without notice of Freedman's above contribution.

3. Criteria for the type $(\boldsymbol{B})_d$ uniform asymptotic equivalence

In the last part of the previous section we referred to an asymptotic result (2.50) on sampling distributions from a finite population. For more broad class of asymptotic problems some modified information criteria given below are practically useful. Let $\{X_s\}$ $(s=1, 2, \cdots)$ and $\{Y_s\}$ $(s=1, 2, \cdots)$ be two sequences of random variables defined on the sequence of abstract spaces $(R_s, \boldsymbol{B}_s)$ $(s=1, 2, \cdots)$. Corresponding to (1.1), the two sequences are said to be asymptotic equivalent in the sence of type $(\boldsymbol{B})_d$, if

$$(3.1) \qquad D(X_s, Y_s; \boldsymbol{B}_s)=\sup_{E\in\boldsymbol{B}_s}|P^{X_s}(E)-P^{Y_s}(E)|\to 0 , \qquad (s\to\infty) ,$$

and, according to Ikeda and Matsunawa [2], is symbolically denoted by

$$(3.2) \qquad X_s \sim Y_s , \qquad (\boldsymbol{B})_d, \ (s\to\infty) .$$

Let, as previous sections, both distributions $P^{X_s}(E)$ and $P^{Y_s}(E)$ be absolutely continuous with respect to a σ-finite measure μ_s over a non-empty sub-σ-field of $\boldsymbol{B}_s$ generated by a domain of our interest A_s $(\in \boldsymbol{B}_s)$, and let f^*_s (>0) and g^*_s (>0) be the respective Radon-Nikodym derivatives of P^{X_s} and P^{Y_s} on the set A_s, for each s. In what follows we also assume that related supremums and infimums corresponding to the quantities a, b, c, d and so on in Theorem 2.1 exist with finite values. Under these set-up we have the following theorem whose proof can be easily accomplished along the line of the proofs of Theorem 2.2 and Theorem 2.3. Thus, it will be omitted here.

THEOREM 3.1. *The asymptotic equivalence* (3.2) *holds if and only if the following conditions are satisfied: There exists a sequence of measurable subsets* $\{B_s\,(\subset A_s \in \boldsymbol{B}_s)\}$ $(s=1, 2, \cdots)$ *such that*

$$(3.3) \qquad P^{X_s}(B_s)\to 1 , \qquad (s\to\infty)$$

and that simultaneously each one of the following conditions holds:

(3.4) $$I^*(X_s, Y_s; B_s^+)\to 0\,, \qquad (s\to\infty)\,,$$

(3.5) $$I_a^*(X_s, Y_s; B_s)\to 0\,, \qquad (s\to\infty)\,,$$

(3.6) $$I^*(X_s, Y_s; B_s)\to 0\,, \qquad (s\to\infty)\,.$$

Remark 3.1. The above theorem is a reinforcement of Theorem 3.1 in Matsunawa [5]. As the set B_s in the 'only if' part we may take, for instance, the corresponding set to (2.29) of the form

$$B_s=\{x;\ |\ln f_s^*(x)/g_s^*(x)|\leqq\zeta_s(\varepsilon_s),\ x\in A_s\subset R_s\}\ ,$$

where ε_s $(s=1, 2,\cdots)$ is an any given sequence of the non-negative numbers such that $\varepsilon_s\to 0$ as $s\to\infty$, and where $\zeta_s(\varepsilon_s)$ is a decreasing function such that $\zeta_s(\cdot)\to 0$ as $\varepsilon_s\to 0$ with $s\to\infty$.

Application 2. Finally we shall apply the criteria in Theorem 3.1 to investigate the asymptotic theory of quasi-extreme order statistics. Let $X_1, X_2,\cdots, X_N$ be a random sample of size N drawn from a continuous distributions with cdf. $F(x)$ and pdf. $f(x)$. Denote the corresponding order statistics by $X_{N1}<X_{N2}<\cdots<X_{NN}$. Assume that the support of the underlying distribution is one open interval

$$(\alpha, \beta)=\{x;\ f(x)>0\}\ ,$$

where α and β may be extended real numbers. Assume that $N-n\geqq 2$ and define the lower $n=n(N)$ extremes by

$$X_{N(n)}=(X_{N1}, X_{N2},\cdots, X_{Nn})\ .$$

Corresponding to this, consider the following $n=n(N)$ dimensional random variable

$$\tilde{X}_{N(n)}=(\tilde{X}_{N1}, \tilde{X}_{N2},\cdots, \tilde{X}_{Nn})$$

whose distribution has the Radon-Nikodym derivative with respect to Lebesgue measure given by

(3.7) $$\tilde{p}_N(x_{(n)})=N^n\cdot\exp\,[-N\cdot F(x_n)]\prod_{i=1}^{n} f(x_i)\ ,$$

on the domain

(3.8) $$A_{N,n}=\{x_{(n)}=(x_1,\cdots, x_n)\,|\,\alpha<x_1<x_2<\cdots<x_n<\beta\}\ .$$

Note that $\tilde{p}_N(x_{(n)})$ is not necessarily a pdf. If we multiply a normalizing constant to the R.H.S. of (3.7), we get formally the corresponding pdf., which is the case considered in Matsunawa and Ikeda [3].

Concerning to the distributions of the above two random variables we get

THEOREM 3.2.

$$X_{N(n)} \sim \tilde{X}_{N(n)}, \qquad (\boldsymbol{B})_d \ (N \to \infty), \tag{3.9}$$

if and only if the condition

$$n/N \to 0, \qquad (N \to \infty) \tag{3.10}$$

is fulfilled.

PROOF. The 'if' part is closely related to Theorem 2.2 in Matsunawa and Ikeda [3]. This part, however, needs to be proved by the modified information criteria presented in Theorem 3.1. The pdf. of $X_{N(n)}$ is clearly given by

$$p_N(x_{(n)}) = \frac{N!}{(N-n)!}[1-F(x_n)]^{N-n} \prod_{i=1}^{n} f(x_i), \qquad (\alpha < x_1 < \cdots < x_n < \beta). \tag{3.11}$$

Then, after some calculations we have

$$\begin{aligned} I^*(X_{N(n)}, \tilde{X}_{N(n)}; A_{N,n}) &= \int_{A_{N,n}} p_N(x_{(n)}) \ln [p_N(x_{(n)})/\tilde{p}_N(x_{(n)})] dx_{(n)} \\ &= -\frac{1}{2} \ln\left(1-\frac{n}{N}\right) - \frac{n}{2N} + \frac{1}{12}\left(\frac{1}{N} - \frac{1}{N-n}\right) \\ &\quad + R(N-n) - R(N) + \left(1-\frac{n}{N}\right)T(N) - T(N-n) + \frac{n}{N(N+1)}, \end{aligned} \tag{3.12}$$

where

$$R(s) = \sum_{i=1}^{\infty} \frac{a_{i+1}}{s(s+1)\cdots(s+i)}, \qquad (s \geqq 2),$$

$$T(s) = \sum_{i=1}^{\infty} \frac{c_{i+1}}{(s+1)\cdots(s+i)}, \qquad (s > 1),$$

with for any integer $r \geqq 2$

$$a_r = \frac{1}{r} \int_0^1 t(1-t)(2-t)\cdots(r-1-t)\left(\frac{1}{2}-t\right) dt,$$

$$c_r = \frac{1}{r} \int_0^1 t(1-t)(2-t)\cdots(r-1-t) dt.$$

Using the inequalities on $R(s)$ and $T(s)$ discussed in Matsunawa [4] we see that

(3.13) $I^*(X_{N(n)}, \tilde{X}_{N(n)}; A_{N,n})$

$$>-\frac{1}{2}\ln\left(1-\frac{n}{N}\right)-\frac{n}{2N}+\frac{n}{N(N+1)}-\frac{n}{12N^2}+\frac{1}{6N+0.5}$$
$$-\frac{1}{6(N-n)}-\frac{N-n}{12}\left\{\frac{1}{N^3}+\frac{1}{(N-n)^2(N-n-1)}-\frac{1}{N^3(N+1)}\right.$$
$$\left.+\frac{2}{N(N-1)(N+1)(N+2)}\right\} \quad (\equiv \underline{\eta}_N(\varepsilon_N) \text{ with } \varepsilon_N=n/N),$$

and that

(3.14) $I^*(X_{N(n)}, \tilde{X}_{N(n)}; A_{N,n})$

$$<-\frac{1}{2}\ln\left(1-\frac{n}{N}\right)-\frac{n}{2N}+\frac{n}{N(N+1)}-\frac{n}{12N^2}+\frac{1}{6N}$$
$$-\frac{1}{6(N-n)+0.5}+\frac{N-n}{12}\left\{\frac{1}{(N-n)^3}+\frac{1}{N^2(N-1)}\right.$$
$$-\frac{1}{(N-n)^3(N-n+1)}$$
$$\left.+\frac{2}{(N-n)(N-n-1)(N-n+1)(N-n+2)}\right\}$$
$$(\equiv \bar{\eta}_N(\varepsilon_N) \text{ with } \varepsilon_N=n/N).$$

Clearly $P^{Y_{N(n)}}(A_{N,n})=1$, and from (3.13) and (3.14) the condition (3.10) implies

$$|I^*(X_{N(n)}, \tilde{X}_{N(n)}; A_{N,n})|\to 0, \quad (N\to\infty), \tag{3.15}$$

which proves (3.9).

On the other hand, in order to prove the 'only if part' of the theorem, suppose that (3.10) does not hold. Namely, assume that there exists a positive constant p $(0<p\leqq 1)$ such that $n/N\to p$ as $N\to\infty$. It is enough for us to disprove (3.15) under the assumption. When $p=1$, two cases may happen, where (i) $N-n\to m$ (= a positive integer larger than 2) as $N\to\infty$ and (ii) $N-n\to\infty$ as $N\to\infty$. It is obvious from (3.13) that both cases lead to $I^*(X_{N(n)}, \tilde{X}_{N(n)}; A_{N,n})\to\infty$ as $N\to\infty$. If $0<p<1$, we have from (3.14) $I^*(X_{N(n)}, \tilde{X}_{N(n)}; A_{N,n})\geqq -(1/2)\ln(1-p)-p/2 \geqq p^2/4>0$ as $N\to\infty$. This also contradicts (3.15). Therefore we have completed the proof of the theorem.

From the theorem we can derive the asymptotic distribution of the n-th order statistic $X_{N,n}$ by considering the marginal distribution of $X_{N(n)}$. The pdf. of $X_{N,n}$ is given by

$$q_N(x)=\frac{\Gamma(N+1)}{\Gamma(n)\Gamma(N-n+1)}[F(x)]^{n-1}[1-F(x)]^{N-n}f(x), \tag{3.16}$$
$$(\alpha<x<\beta).$$

Let $\tilde{X}_{N,n}$ be the random variable whose distribution has the Radon-Nikodym derivative with respect to Lebesgue measure given by

$$(3.17) \quad \tilde{q}_N(x)=\frac{N}{\Gamma(n)}[F(x)]^{n-1}\exp[-N\cdot F(x)]f(x)\,, \qquad (\alpha<x<\beta)\,.$$

We obtain the following

COROLLARY 3.1.

$$(3.17) \qquad X_{N,n}\sim\tilde{X}_{N,n} \qquad (\boldsymbol{B})_d,\ (N\to\infty)\,,$$

if and only if (3.10) *is satisfied.*

Remark 3.2. If we set $\varepsilon=\varepsilon_N=n/N$ and make use of the inequalities (3.13) and (3.14), we have the corresponding two-sided uniformly ϕ-equivalent statements to (3.9) and (3.17). As for the upper $m=m(N)$ extremes $Y_{N(m)}=(X_{N,N-m+1},\cdots,X_{N,N})$ and the $(N-m+1)$-th order statistic, we obviously have parallel results to Theorem 3.2 and Corollary 3.1.

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