

BLOCK PLAN FOR A FRACTIONAL 2^m FACTORIAL DESIGN DERIVED FROM A 2^r FACTORIAL DESIGN

TERUHIRO SHIRAKURA

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Summary

For a given fractional 2^m factorial (2^m -FF) design T , the constitution of a block plan to divide T into k ($2^{r-1} < k \leq 2^r$) blocks with r block factors each at two levels is proposed and investigated. The well-known three norms of the confounding matrix are used as measures for determining a "good" block plan. Some theorems concerning the constitution of a block plan are derived for a 2^m -FF design of odd or even resolution. Two norms which may be preferred over the other norm are slightly modified. For each value of N assemblies with $11 \leq N \leq 26$, optimum block plans for $k=2$ blocks with block sizes $[N/2]$ and $N - [N/2]$ minimizing the two norms are presented for A -optimal balanced 2^4 -FF designs of resolution V given by Srivastava and Chopra (*Technometrics*, 13, 257-269).

1. Introduction

Consider a 2^m factorial experiment with m factors. An assembly (or treatment combination) is represented by an m -rowed vector $(j_1, j_2, \dots, j_m)$, where j_i (level of i th factor) is equal to 0 or 1. As unknown effects, θ_0 , θ_i , and in general, $\theta_{i_1, \dots, i_k}$ denote the general mean, main effect of i th factor, and k -factor interaction of $i_1, \dots, i_k$ th factors, respectively. For a fixed integer l ($1 \leq l \leq m$), let θ be the $\nu \times 1$ vector composed of effects up to l -factor interactions, where $\nu = \sum_{i=0}^l \binom{m}{i}$, i.e.,

$$\theta' = (\theta_0; \theta_1, \theta_2, \dots, \theta_m; \theta_{12}, \dots, \theta_{m-1, m}; \theta_{12, \dots, l}, \dots, \theta_{m-l+1, \dots, m}).$$

Assume throughout this paper that $(l+1)$ -factor and higher order interactions are negligible and that the m factors are different from block factors. Let T be a fractional 2^m factorial (2^m -FF) design which is a

Key words: Confounding matrix, norm, balanced array.

suitable set of N assemblies. Note that the assemblies in T are not always distinct. Using a design T , we consider the estimation of a $\nu_0 \times 1$ vector of linear parametric functions $\theta_0 = C\theta$ for some $\nu_0 \times \nu$ matrix C . For an $N \times 1$ observation vector y_T of T (whose observations are assumed to be independent random variables with common variance σ^2), consider the model

$$(1.1) \quad E(y_T) = E\theta$$

where $E(\cdot)$ stands for an expected value and E is the $N \times \nu$ design matrix with elements ± 1 (see, e.g., Yamamoto, Shirakura and Kuwada [13]). Suppose that there exists a $\nu_0 \times \nu$ matrix K satisfying $KM = C$ (i.e., $\text{rank } M = \text{rank } [M: C]$), which is equivalent to the estimability of θ_0 , where $M = E'E$ is called the information matrix of T . The best linear unbiased estimate of θ_0 can then be given by

$$(1.2) \quad \hat{\theta}_0 = KE'y_T.$$

When $\theta_0 = \theta$, i.e., $C = I$ (identity matrix of appropriate order) and $\nu_0 = \nu$, T corresponds to a 2^m -FF design of resolution $2l+1$. In this case, note that the nonsingularity of M is assumed. On the other hand, when $\theta_0 = (\theta_1, \dots, \theta_m; \dots; \theta_{12\dots l-1}, \dots, \theta_{m-l+2\dots m})'$, i.e., $C = [0: I: O]$ (0 and O are respectively zero vector and zero matrix of appropriate orders) and $\nu_0 = \nu - 1 - \binom{m}{l}$, T corresponds to a design of resolution $2l$ (see Box and Hunter [1]).

In order to get $\hat{\theta}_0$ in (1.2), it is required to make the plots of N assemblies under conditions as homogeneous as possible for the m factors. After planning a design T for $\hat{\theta}_0$, however, it may occur that N observations for T can not be yielded simultaneously by physical, chemical and/or economical reasons, etc. For example, consider an experiment of a certain reaction for a mixture of m raw materials each at two levels. Then, after accomodating the N mixtures in a given T , it may occur that each reaction of them can not be observed under a homogeneous condition. Therefore we consider an arrangement of T in some blocks. The less is the number of assemblies in which we have to experiment simultaneously, the larger is the possibility that we obtain a homogeneous condition. Of course, the number of blocks (say k) should be small compared to N . The problem is to constitute the k blocks such that the estimate $\hat{\theta}_0$ is not sensitive to the block division. The present paper discusses this problem under special situations. For the k blocks, we use a 2^r design with r factors each at two levels which consists of k distinct assemblies ($2^{r-1} < k \leq 2^r$). As block factors, for example, consider experimenter, day and place. Our situation is then the case where the N observations in a given T have to be yielded

by two experimenters, in two days and/or at two places. As measures of the insensitivity to block effects, we use well-known three norms of a confounding matrix of T . In Section 2, we introduce the three norms of a confounding matrix for r block factors and present a procedure for the constitution of k ($2^{r-1} < k \leq 2^r$) blocks with some block sizes for a design T . For a 2^m -FF design of resolution $2l$ or $2l+1$, we give the constitution of the k blocks such that the norms have appropriate values and that the confounding matrix has a certain desirable pattern. In particular, it is shown that for a 2^m -FF design of even resolution, there exist the blocks for which the three norms are zero (i.e., its confounding matrix is zero). Section 3 deals with the constitution of $k=2$ blocks for a balanced fractional 2^m factorial (2^m -BFF) design of resolution $2l+1$. Balanced designs (including orthogonal designs) have various desirable properties and wide applications. A 2^m -BFF design of resolution V ($l=2$) was first discussed by Srivastava [9] and a design of resolution $2l+1$ has been generalized by Yamamoto, Shirakura and Kuwada [13], [14]. Some properties of the preferable two norms are presented for 2^m -BFF designs of resolution $2l+1$. Also, the norms are slightly modified. For A-optimal 2^4 -BFF designs of resolution V given by Srivastava and Chopra [10], the constitutions of 2 blocks with block sizes $[N/2]$ and $N-[N/2]$ minimizing the modified norms are presented for the values of N satisfying $11 \leq N \leq 26$.

For the case where one knows in advance that block factors are explicitly needed for the estimation of θ_0 , a design should be planned such that the effect θ_0 is unconfounded with block effects. For the case of $\theta_0 = \theta$, however, it is in general difficult to plan such a design and even if we do it, a larger number of assemblies are required. If the effects of block factors may not be so large as effects in θ and not be so neglected as higher order interactions, the methods given in this paper would be also useful.

2. Constitution of a confounding block plan

Let T be a 2^m -FF design with N assemblies whose α th assembly is given by $(j_1^{(\alpha)}, j_2^{(\alpha)}, \dots, j_m^{(\alpha)})$ for $\alpha=1, \dots, N$. For $2^r < N$, consider another 2^r design D with r factors and N assemblies whose α th assembly is given by $(d_1^{(\alpha)}, d_2^{(\alpha)}, \dots, d_r^{(\alpha)})$ for $\alpha=1, \dots, N$. Suppose k ($2^{r-1} < k \leq 2^r$) distinct assemblies are exactly included in D . Further let $[T; D]$ be a set of assemblies obtained by juxtaposing T and D in a way such that its α th assembly is $(j_1^{(\alpha)}, \dots, j_m^{(\alpha)}; d_1^{(\alpha)}, \dots, d_r^{(\alpha)})$ for $\alpha=1, \dots, N$. The set $[T; D]$ can be considered as a 2^{m+r} -FF design with $m+r$ factors. However, the r factors play here the role of block factors and D gives a block plan for T . That is, k blocks $B_{(d_1, \dots, d_r)}$ for T are constituted in a

way such that the α th assembly of $[T; D]$ is $(j_1^{(\alpha)}, \dots, j_m^{(\alpha)}; d_1^{(\alpha)}, \dots, d_r^{(\alpha)})$ if and only if the α th assembly of T belongs to $B_{(d_1^{(\alpha)}, \dots, d_r^{(\alpha)})}$. Then the number of times an assembly $(d_1, \dots, d_r)$ occurs in D , $k(d_1 \dots d_r)$, is called a block size for $B(d_1 \dots d_r)$. Consider now the $N \times 1$ observation vector $\mathbf{y}_{[T; D]}$ of $[T; D]$. According to model (1.1), we then have the following model:

$$(2.1) \quad E(\mathbf{y}_{[T; D]}) = E\boldsymbol{\theta} + X(D)\boldsymbol{\eta},$$

where $\boldsymbol{\eta}' = (\eta_1, \dots, \eta_r)$, (η_i is the main effects of i th block factor), and $X(D)$ is the $N \times r$ design matrix for $\boldsymbol{\eta}$ of D with elements ± 1 . Assume that there are no interactions between the m factors and r block factors, and between the r block factors. Note that the level of j th factor for the α th assembly in D is 0 and 1 if and only if the (α, j) element of $X(D)$ is -1 and 1 , respectively. We still insist on using $\hat{\boldsymbol{\theta}}_0$ in (1.2) for the estimation of $\boldsymbol{\theta}_0$, because T is a design which is in advance planned to obtain $\hat{\boldsymbol{\theta}}_0$. Under model (2.1), the expected value of $\hat{\boldsymbol{\theta}}_0$, ($\mathbf{y}_T$ in $\hat{\boldsymbol{\theta}}_0$ is replaced with $\mathbf{y}_{[T; D]}$), becomes

$$(2.2) \quad E(\hat{\boldsymbol{\theta}}_0) = \boldsymbol{\theta}_0 + A(D)\boldsymbol{\eta},$$

where $A(D) = KE'X(D)$ is said to be a confounding matrix of D . As measures how $E(\hat{\boldsymbol{\theta}}_0)$ in (2.2) can be close to $\boldsymbol{\theta}_0$, the following three norms of $A(D)$ may be considered:

$$(2.3) \quad \begin{aligned} \text{(i)} \quad & \|A(D)\|_1 = \max_{1 \leq i \leq v_0} \left\{ \sum_{j=1}^r |a_{ij}| \right\}, \\ \text{(ii)} \quad & \|A(D)\|_2 = \{\text{tr}(A(D)'A(D))\}^{1/2} = \left\{ \sum_{i=1}^{v_0} \sum_{j=1}^r a_{ij}^2 \right\}^{1/2}, \\ \text{(iii)} \quad & \|A(D)\|_3 = \max_{1 \leq j \leq r} \left\{ \sum_{i=1}^{v_0} |a_{ij}| \right\}, \end{aligned}$$

where a_{ij} are the (i, j) elements of $A(D)$.

Under (2.1), the mean square error of $\hat{\boldsymbol{\theta}}_0$ for $\boldsymbol{\theta}_0$ is given by

$$\text{MSE}(\hat{\boldsymbol{\theta}}_0) = KMK'\sigma^2 + A(D)\boldsymbol{\eta}\boldsymbol{\eta}'A(D)'$$

It is seen that for a given design T , $\text{MSE}(\hat{\boldsymbol{\theta}}_0)$ is dependent on $A(D)$ only through $A(D)\boldsymbol{\eta}\boldsymbol{\eta}'A(D)'$, because σ^2 and $\boldsymbol{\eta}$ are constant. Therefore it is reasonable to be considered that minimizing each norm in (2.3) may also reflect decreasing a magnitude of $\text{MSE}(\hat{\boldsymbol{\theta}}_0)$. In particular, this is more obvious for the norm $\|A(D)\|_2$ for the case $r=1$ which will be treated later, since

$$(2.4) \quad \text{tr}(\text{MSE}(\hat{\boldsymbol{\theta}}_0)) = \sigma^2 \text{tr}(KMK') + \eta_1^2 \|A(D)\|_2^2.$$

Let $D_{\{k(d_1 \dots d_r)\}}^N$ denote the collection of all possible D with N assemblies in which the assembly $(d_1, \dots, d_r)$ occurs $k(d_1 \dots d_r)$ times. For a given T , D in $D_{\{k(d_1 \dots d_r)\}}^N$ is said to be a confounding block (CB) plan when D is selected in a consideration of the norm $\|A(D)\|_i$, ($i=1, 2, 3$). A plan D is said to be an optimum confounding block (OCB) plan (with block sizes $k(d_1 \dots d_r)$) for T with respect to the norm $\|A(D)\|_i$ if $\|A(D)\|_i$ is a minimum in the class $D_{\{k(d_1 \dots d_r)\}}^N$ for each $i=1, 2, 3$. Also the set $B_{T, \{k(d_1 \dots d_r)\}}$ of the corresponding blocks $B_{(d_1 \dots d_r)}$ is said to be an optimum block for T . This idea is due to Hedayat, Raktue and Federer [4]. However they have proposed the norm $\|A\|=(\text{tr}(A'A))^{1/2}$ of alias matrix A as a measure in selecting a design T .

2^m -FF designs of resolution $2l$ or $2l+1$ are particularly important for our practical uses. Therefore we shall discuss some properties of the norms in (2.3) for these designs.

THEOREM 2.1. *Let T be a 2^m -FF design of resolution $2l$. Suppose an $N \times r$ submatrix of E whose r columns correspond to r l -factor interactions has k ($2^{r-1} < k \leq 2^r$) distinct rows. Then there exists an OCB plan $D \in D_{\{k(d_1 \dots d_r)\}}^N$ for T such that $\|A(D)\|_i=0$, ($i=1, 2, 3$), i.e., $A(D)$ is a zero matrix, where $k(d_1 \dots d_r)$ is the number of row vectors $(2d_1-1, \dots, 2d_r-1)$ occurring in the submatrix.*

PROOF. Denote the submatrix by Y . Assume that the r l -factor interactions are at the s_1 th, s_2 th, $\dots$, s_r th positions of θ and let Z be the $\nu \times r$ matrix in which the i th element of s_i th row is 1 and 0 elsewhere. Then $Y=EZ$ holds. Consider D satisfying $X(D)=Y$. Then, from the assumption, $D \in D_{\{k(d_1 \dots d_r)\}}^N$. Also, $A(D)=KE'X(D)=KMZ=CZ=O$. This completes the proof.

THEOREM 2.2. *Let T be a 2^m -FF design of resolution $2l+1$. Consider the same submatrix and $D_{\{k(d_1 \dots d_r)\}}^N$ as in Theorem 2.1. Then, there exists a CB plan $D \in D_{\{k(d_1 \dots d_r)\}}^N$ for T such that $\|A(D)\|_i=1$, ($i=1, 3$), and $\|A(D)\|_2=\sqrt{r}$. Moreover, in the plan D , every effect up to $(l-1)$ -factor interactions is unconfounded with block effects η .*

PROOF. Consider the same matrices Y and Z , and the plan $D \in D_{\{k(d_1 \dots d_r)\}}^N$ as in Proof of Theorem 2.1. Then $A(D)=Z$, since $C=I$. From the structure of Z , $\|Z\|_i=1$, ($i=1, 3$), $\|Z\|_2=\sqrt{r}$, and furthermore from (2.2), $E(\hat{\theta}_0)=\theta+Z\eta$, which completes the proof.

In what follows, (optimum) CB plans with two blocks (i.e., $r=1$) are constituted for designs. Note that $A(D)$ and $X(D)$ become column vectors. In view of Theorem 2.1, we can easily obtain an OCB plan with two blocks for some 2^m -FF design of even resolution. The following theorem implies that for a 2^m -FF design of odd resolution, there

exists a CB plan better than one in Theorem 2.2 w.r.t. the norm $\|A(D)\|_1$:

THEOREM 2.3. *Let $2 \leq l \leq m-2$. Then there exists a CB plan D in $D_{(k(d_l))}^N$, ($k(d_l)$ are certain block sizes), for a 2^m -FF design of resolution $2l+1$ such that $\|A(D)\|_1=1/2$ holds. Moreover, in the plan D , any effects up to $(l-1)$ -factor interactions are unconfounded with block effect η .*

PROOF. Since $2 \leq l \leq m-2$, there exist two distinct main effects θ_i and $\theta_{i'}$ such that $t, t' \notin \{t_1, \dots, t_{l-1}, t'_1, \dots, t'_{l-1}\}$ for some two distinct $(l-1)$ -factor interactions $\theta_{t_1 \dots t_{l-1}}$ and $\theta_{t'_1 \dots t'_{l-1}}$. Denote a column vector of E corresponding to an effect θ_z in θ by e_z . Consider the $N \times 1$ vector $\mathbf{x}$ given as follows:

$$\mathbf{x} = \frac{1}{2} \{e_i * (e_{t_1 \dots t_{l-1}} + e_{t'_1 \dots t'_{l-1}}) + e_{i'} * (e_{t_1 \dots t_{l-1}} - e_{t'_1 \dots t'_{l-1}})\}.$$

where $*$ denotes the product operation defined by $(a_1, a_2, \dots, a_n)' * (b_1, b_2, \dots, b_n)' = (a_1 b_1, a_2 b_2, \dots, a_n b_n)'$. Then it is easy to verify that $\mathbf{x}$ is a vector with elements 1 or -1 . From the property of design matrix E , $\mathbf{x}$ can also be written by

$$\mathbf{x} = \frac{1}{2} (e_{t_1 \dots t_{l-1}} + e_{t'_1 \dots t'_{l-1}} + e_{i' t_1 \dots t_{l-1}} - e_{i' t'_1 \dots t'_{l-1}}).$$

This means that $\mathbf{x}$ can be expressed as a linear combination of columns of E corresponding to the four l -factor interactions $\theta_{t_1 \dots t_{l-1}}$, $\theta_{t'_1 \dots t'_{l-1}}$, $\theta_{i' t_1 \dots t_{l-1}}$ and $\theta_{i' t'_1 \dots t'_{l-1}}$. Suppose now $\mathbf{z}$ is a $\nu \times 1$ vector with elements 0, 1 or -1 obtained from θ by replacing the above first three l -factor interactions with 1, the last one with -1 , and the other effects with 0. Further consider D given by $X(D) = \mathbf{x}$. Then we have $X(D) = E\mathbf{z}/2$. Therefore, $A(D) = M^{-1}E'X(D) = \mathbf{z}/2$, since $K = M^{-1}$. Hence $\|A(D)\|_1 = 1/2$. This completes the proof.

It is remarked that the block sizes $k(0)$ and $k(1)$ of the plan D in the above theorem are equal to the numbers of -1 and 1 in $\mathbf{x}$, respectively.

Example 1. Consider

$$T = \{\Omega(5, 0), \Omega(5, 2), \Omega(5, 4)\},$$

where $\Omega(m, h)$, ($0 \leq h \leq m$), is the set of $\binom{m}{h}$ distinct assemblies with weight h (number of 1's). Then it is easy to see that T is a 2^5 -FF design of resolution V ($l=2$) with $N=16$ assemblies such that $M=16I$. This design is called an orthogonal design which is the most popular design. T can be expressed by the 16×5 matrix

$$T' = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

whose α th row denotes the α th assembly. Suppose two main effects are θ_2 and θ_3 , and the other main effects are θ_4, θ_5 (i.e., $t=2, t'=3, t_1=4$ and $t'_1=5$) in Proof of Theorem 2.3. Therefore $\mathbf{x} = (\mathbf{e}_{24} + \mathbf{e}_{25} + \mathbf{e}_{34} - \mathbf{e}_{35})/2 = (1; -1, 1, -1, 1, -1, -1, 1, 1, -1, -1; 1, -1, 1, -1, 1)'$. Thus we have the plan $D = \{\mathbf{x} + (1, 1, \dots, 1)'\}/2 \in D_{\{k(0)=8, k(1)=8\}}^6$ with $\|A(D)\|_1 = 1/2$. The corresponding blocks $B_{(0)}$ and $B_{(1)}$ are given by

$$B_{(0)} = \{(1, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 1, 1, 0, 0), (0, 1, 0, 1, 0), \\ (0, 0, 1, 0, 1), (0, 0, 0, 1, 1), (1, 1, 1, 0, 1), (1, 0, 1, 1, 1)\},$$

$$B_{(1)} = T - B_{(0)}.$$

In fact, the above plan D turns out to be an OCB plan over $D_{\{k(0)=8, k(1)=8\}}^6$ w.r.t. $\|A(D)\|_1$.

As compared with the norms $\|A(D)\|_2$ and $\|A(D)\|_3$, the decrease of $\|A(D)\|_1$ directly acts to reduce the bias for each estimate in $\hat{\theta}_0$. In the sense, $\|A(D)\|_1$ is preferred over $\|A(D)\|_2$ and $\|A(D)\|_3$. Also, for the present case $r=1$, $\|A(D)\|_2$ is mathematically preferred over $\|A(D)\|_1$ and $\|A(D)\|_3$ in view of (2.4). For reference, note that $\|A(D)\|_2 = 1$ and $\|A(D)\|_3 = 2$ for the plan D in Theorem 2.3. We are henceforth interested in an OCB plan with two blocks w.r.t. $\|A(D)\|_i$, ($i=1, 2$).

3. OCB plans for balanced designs

CB plans in $D_{\{k(a_1)\}}^N$, ($N=k(0)+k(1)$), are developed for 2^m -BFF designs of resolution $2l+1$. We first define a balanced design. A 2^m -FF design of resolution $2l+1$ is called balanced if the covariance matrix $\text{Var}(\hat{\theta}) = M^{-1}\sigma^2$ of $\hat{\theta} = \hat{\theta}_0$ in (1.2) is invariant under any permutation of m factors, i.e., if for two estimates $\hat{\theta}_{t_1 \dots t_u}$ and $\hat{\theta}_{t'_1 \dots t'_u}$ in $\hat{\theta}$,

$$(3.1) \quad \begin{aligned} \text{Var}(\hat{\theta}_{t_1 \dots t_u}) &= \text{Var}(\hat{\theta}_{\tau(t_1 \dots t_u)}), \\ \text{Cov}(\hat{\theta}_{t_1 \dots t_u}, \hat{\theta}_{t'_1 \dots t'_u}) &= \text{Cov}(\hat{\theta}_{\tau(t_1 \dots t_u)}, \hat{\theta}_{\tau(t'_1 \dots t'_u)}) \end{aligned}$$

where $\text{Var}(\cdot)$ and $\text{Cov}(\cdot, \cdot)$ stand for variance and covariance of estimates, respectively, and τ is an element of the symmetric group $\mathcal{C} = \left\{ \tau; \tau = \begin{pmatrix} 1 & 2 & \dots & m \\ \tau(1) & \tau(2) & \dots & \tau(m) \end{pmatrix} \right\}$ of order m . In practice, for a given m , u' -factor interactions ($m/2 < u' \leq m$) are often assumed to be negligible.

Therefore, we here give the restriction $2l \leq m$. It is known from Srivastava [9] and Yamamoto, Shirakura and Kuwada [13] that a 2^m -BFF design of resolution $2l+1$ with N assemblies is equivalent to a balanced array of size N , m constraints, strength $2l$ and index set $\mathcal{U} = \{\mu_0, \mu_1, \dots, \mu_{2l}\}$, (simply, B -array $[N, m, 2l, \mathcal{U}]$), provided M is non-singular. For the save of space, see the above papers for the definition of a balanced array. In particular, a B -array $[N, m, 2l, \mathcal{U}]$ is called an orthogonal array with index λ if $\lambda = \mu_0 = \mu_1 = \dots = \mu_{2l}$, which is equivalent to an orthogonal 2^m -FF design of resolution $2l+1$ with $M = 2^{2l} \lambda I$ (see Example 1).

Now, for a $\nu \times 1$ vector $\mathbf{c}$, attach the same subindices as the effects of $\boldsymbol{\theta}$ to the elements of $\mathbf{c}$, i.e.,

$$\mathbf{c}' = (c_0; c_1, \dots, c_m; c_{12}, \dots, c_{m-1m}; \dots; c_{12\dots l}, \dots, c_{m-l+1\dots m}).$$

Further consider a $\nu \times 1$ vector $\tau(\mathbf{c})$ for $\tau \in \mathcal{C}$ defined by

$$\tau(\mathbf{c})' = (c_0; c_{\tau(1)}, \dots, c_{\tau(m)}; c_{\tau(12)}, \dots, c_{\tau(m-1m)}; \dots; c_{\tau(12\dots l)}, \dots, c_{\tau(m-l+1\dots m)}).$$

Then, we have the following lemma which can easily be proved:

LEMMA 3.1. For any $\tau \in \mathcal{C}$,

$$\mathbf{c}_1' \tau(\mathbf{c}_2) = \tau^{-1}(\mathbf{c}_1)' \mathbf{c}_2$$

holds where $\mathbf{c}_1$ and $\mathbf{c}_2$ are $\nu \times 1$ vectors.

THEOREM 3.2. Let T be a 2^m -BFF design of resolution $2l+1$ with N assemblies and D be a plan in $\mathbf{D}_{k(d_1)}^N$. Then, for any $\tau \in \mathcal{C}$,

$$\|A(D)\|_i = \|A^\tau(D)\|_i, \quad i=1, 2$$

hold where $A^\tau(D) = M^{-1} \tau(E'X(D))$.

PROOF. For $\tau_1, \tau_2 \in \mathcal{C}$, suppose $V(\tau_1(\boldsymbol{\theta}), \tau_2(\boldsymbol{\theta}))\sigma^2$ is the covariance matrix for $\boldsymbol{\theta}$ in which the row and column orders correspond to $\tau_1(\boldsymbol{\theta})$ and $\tau_2(\boldsymbol{\theta})$, respectively. In particular, note that $V(\boldsymbol{\theta}, \boldsymbol{\theta}) = M^{-1}$ since $\text{Var}(\hat{\boldsymbol{\theta}}) = M^{-1}\sigma^2$. It follows from Lemma 3.1 that

$$A^\tau(D) = M^{-1} \tau(E'X(D)) = V(\boldsymbol{\theta}, \boldsymbol{\theta}) \tau(E'X(D)) = V(\boldsymbol{\theta}, \tau^{-1}(\boldsymbol{\theta}))(E'X(D)).$$

Therefore, we have

$$\tau^{-1}(A^\tau(D)) = V(\tau^{-1}(\boldsymbol{\theta}), \tau^{-1}(\boldsymbol{\theta}))(E'X(D)).$$

From (3.1), $V(\boldsymbol{\theta}, \boldsymbol{\theta}) = V(\tau^{-1}(\boldsymbol{\theta}), \tau^{-1}(\boldsymbol{\theta}))$ holds. Hence we have

$$\|A^\tau(D)\|_i = \|\tau^{-1}(A^\tau(D))\|_i = \|M^{-1}E'X(D)\|_i = \|A(D)\|_i.$$

A design T is called a simple array with parameters $\lambda_0, \lambda_1, \dots, \lambda_m$,

(simply, S -array $[m; \lambda_0, \lambda_1, \dots, \lambda_m]$), if the assemblies in T can be obtained by λ_h repetitions of the set $\mathcal{Q}(m, h)$ for each $h=0, 1, \dots, m$, where $\mathcal{Q}(m, h)$ are explained in Example 1 in Section 2. It is easily seen that an S -array $[m; \lambda_0, \dots, \lambda_m]$ is a B -array $[N, m, 2l, \mathcal{U}]$, where

$$N = \sum_{h=0}^m \lambda_h \binom{m}{h},$$

$$\mu_i = \sum_{h=0}^m \lambda_h \binom{m-2l}{h-i}, \quad i=0, 1, \dots, 2l.$$

As will be seen from Chopra [2], Chopra and Srivastava [3], Shirakura [6], [7], Srivastava and Chopra [10], [11], etc., most of balanced arrays are of simple types for practical values of m and N . Therefore, it is desirable to study CB plans for 2^m -BFF designs derivable from simple arrays.

THEOREM 3.3. *Let T be an S -array $[m, \lambda_0, \dots, \lambda_m]$ with $N = \sum_{h=0}^m \lambda_h \binom{m}{h}$. Then, for any $D \in \mathcal{D}_{[k(d_1)]}^N$ and $\tau \in \mathcal{C}$, there exists D_0 in $\mathcal{D}_{[k(d_1)]}^N$ such that*

$$\tau(E'X(D)) = E'X(D_0).$$

PROOF. The design matrix E is represented by

$$E = (e_0; e_1, \dots, e_m; e_{12}, \dots, e_{m-1m}; \dots; e_{12\dots l}, \dots, e_{m-l+1\dots m}),$$

where e_x is defined in Proof of Theorem 2.3. Suppose $\tau(E)$ is the $N \times \nu$ matrix obtained from E by exchanging e_x with $e_{\tau(x)}$. (In particular, $e_{\tau(0)} = e_0$.) Further define the $N \times m$ matrices

$$E_1 = (e_1, \dots, e_m) \quad \text{and} \quad \tau(E_1) = (e_{\tau(1)}, \dots, e_{\tau(m)}).$$

Since T is a simple array, both E_1 and $\tau(E_1)$ then include just λ_h matrices of size $\binom{m}{h} \times m$ with elements ± 1 composed of $\binom{m}{h}$ distinct m -rowed vectors in which the numbers of 1's are h for each $h=0, 1, \dots, m$. Therefore, there exists a permutation matrix P_τ of order N such that

$$(3.2) \quad \tau(E_1) = P_\tau E_1.$$

On the other hand, we have

$$e_{\tau(t_1 t_2 \dots t_u)} = e_{\tau(t_1)} * e_{\tau(t_2)} * \dots * e_{\tau(t_u)},$$

where $*$ denotes the product operator in Theorem 2.3. By (3.2), $e_{\tau(t_i)} = P_\tau e_{t_i}$ hold for $i=1, \dots, u$. Thus it can be shown that

$$e_{\tau(t_1 t_2 \dots t_u)} = (P_{\tau} e_{t_1}) * \dots * (P_{\tau} e_{t_u}) = P_{\tau}(e_{t_1} * \dots * e_{t_u}) = P_{\tau} e_{t_1 \dots t_u}.$$

Also, $e_0 = P_{\tau} e_0$, since the elements of e_0 are all one. Therefore, the matrix P_{τ} satisfies $\tau(E) = P_{\tau} E$. This means that

$$\tau(E'X(D)) = \tau(E)'X(D) = E'P_{\tau}'X(D).$$

From the one-to-one correspondence of $X(D)$ and D , there exists D_0 in $D_{[k(d_1)]}^N$ satisfying $X(D_0) = P_{\tau}'X(D)$. This completes the proof.

Theorems 3.2 and 3.3 are useful in determining an OCB plan D for a design T in which $\|A(D)\|_i$, ($i=1, 2$), has a minimum over $D_{[k(d_1)]}^N$. Suppose $\mathbf{c} = (c_0; c_1, \dots, c_m; \dots, c_{m-l+1} \dots m)' = E'X(D)$ and consider $D_{[k(d_1)]}^{*N} = \{D \in D_{[k(d_1)]}^N; c_1 \leq c_2 \leq \dots \leq c_m\}$. Now let D^* be an OCB plan over $D_{[k(d_1)]}^{*N}$. Further consider a permutation $\tau \in \mathcal{C}$ such that $(c_1^*, \dots, c_m^*)$ transforms $(c_{\tau(1)}^*, \dots, c_{\tau(m)}^*)$, where $\mathbf{c}^* = E'X(D^*) = (c_0^*; c_1^*, \dots, c_m^*; \dots, c_{m-l+1}^* \dots m)'$ and $c_{\tau(1)}^* \leq c_{\tau(2)}^* \leq \dots \leq c_{\tau(m)}^*$. Then, by Theorems 3.2 and 3.3, there exists $D_0^* \in D_{[k(d_1)]}^{*N}$ satisfying $\|A(D^*)\|_i = \|A^{\tau}(D^*)\|_i = \|A(D_0^*)\|_i$. This means that D_0^* is also an OCB plan over $D_{[k(d_1)]}^N$. That is, an OCB plan over $D_{[k(d_1)]}^N$ can be selected in $D_{[k(d_1)]}^{*N}$.

Consider the subclass of $D_{[k(d_1)]}^N$,

$$S^N = \{D \in D_{[k(d_1)]}^N; k(1) = [N/2]\}.$$

where $[x]$ denotes the greatest integer less than or equal to x . Also we take no interest in the confounding of the general mean and block effect. Therefore, the following slightly modified norms may be introduced:

- (i) $\|A^*(D)\|_1 = \max_{1 \leq i \leq \nu-1} |a_i^*|$,
- (ii) $\|A^*(D)\|_2 = \{\text{tr}(A^*(D)'A^*(D))\}^{1/2}$,

where $A^*(D)$ is the $(\nu-1) \times 1$ vector obtained by removing the first element of $A(D)$ and a_i^* 's are the elements of $A^*(D)$. Our interest now lies in a CB plan such that $\|A^*(D)\|_i$ is a minimum over S^N . The vector $A^*(D)$ can be written by $A^*(D) = M^*E'X(D)$, where M^* is the $(\nu-1) \times \nu$ matrix obtained by removing the first row of M^{-1} . Arguments similar to Proofs of Theorems 3.2 and 3.4 give the following two theorems:

THEOREM 3.4. *Consider the design T and plan D of Theorem 3.2. Then, for any $\tau \in \mathcal{C}$,*

$$\|A^*(D)\|_i = \|A^{*\tau}(D)\|_i, \quad i=1, 2,$$

where $A^{*\tau}(D) = M^*\tau(E'X(D))$.

THEOREM 3.5. *Consider the simple array T of Theorem 3.3. Then, for any $D \in S^N$ and $\tau \in \mathcal{C}$, there exists $D_0 \in S^N$ such that*

$$\tau(E'X(D)) = E'X(D_0) .$$

From the above theorems, we observe similarly that an OCB plan (w.r.t. $\|A^*(D)\|_i$) over S^N can also be selected in the subclass

$$S^{*N} = \{D \in S^N; D \in D_{\{k(a_1)\}}^{*N}\} .$$

On the other hand, the calculation of $\|A(D)\|_i$ or $\|A^*(D)\|_i$ requires the inverse of $\nu \times \nu$ matrix M . However, Yamamoto, Shirakura and Kuwada [13] have shown that for a 2^m -BFF design of resolution $2l+1$, M^{-1} has at most $\binom{l+3}{3}$ distinct elements. Furthermore, Shirakura and Kuwada [14] have given an explicit expression of these elements.

In Table, optimum blocks $B_T (=B_{T, \{k(a_1)\}})$ corresponding to OCB plans w.r.t. $\|A^*(D)\|_1$ and/or $\|A^*(D)\|_2$ over S^N are listed for A -optimal 2^4 -BFF designs T of resolution V (minimizing $\text{tr } M^{-1}$) for the values of N with $11 \leq N \leq 26$. The A -optimal balanced designs have been already given by Srivastava and Chopra [10]. Note that a B -array $[N, m=4, 4, \mathcal{U} = \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}]$ is equivalent to an S -array $[m=4; \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4]$, where $N = \lambda_0 + 4\lambda_1 + 6\lambda_2 + 4\lambda_3 + \lambda_4$ and $\mu_i = \lambda_i$, ($i=0, 1, 2, 3, 4$). Therefore, 2^4 -BFF designs of resolution V can be represented by the sets $\Omega(m=4, h)$, ($h=0, 1, 2, 3, 4$), of assemblies. Briefly, we write $T = \{\lambda_0\Omega(0), \lambda_1\Omega(1), \lambda_2\Omega(2), \lambda_3\Omega(3), \lambda_4\Omega(4)\}$, where $\Omega(h) = \Omega(4, h)$ for $h=0, \dots, 4$. In Table, note that the other set $B_{(1)}$ can be obtained by $B_{(1)} = T - B_{(0)}$ for each T . Fortunately, it turns out that relatively many optimum blocks B_T are simply expressed by $\Omega(h)$ and that the optimum blocks w.r.t. $\|A^*(D)\|_{i=1,2}$ are identical except for $N=12, 14, 15$ and 21 . The values of $\|A^*(D)\|_{i=1,2}$ are also given in the table for reference. From the above theorems, it may be remarked that an OCB plan is not unique for a given design. Furthermore, it is easily seen that if D is an OCB plan for T , then it is also so for $\bar{T}$, where $\bar{T}$ is the design obtained by an interchange of 0 and 1 in T . By the interchange, the most of properties for T are preserved (e.g., balanced property or A -, D -, E -optimalities). The following example is helpful in referring to Table.

Example 2.

(i) Consider a B -array $[N=12, m=4, 4, \mathcal{U} = \{1, 0, 1, 1, 1\}]$ given by

$$T = \{\Omega(0), \Omega(2), \Omega(3), \Omega(4)\} ,$$

which is a 2^4 -BFF design of resolution V with 12 assemblies. Then, T can be rewritten by the 12×4 matrix

Table. Optimum blocks for 2⁴-BFF designs of resolution V

N	μ_0	μ_1	μ_2	μ_3	μ_4	T	$B_{(0)}$	$\ A^*(D)\ _1$	$\ A^*(D)\ _2$
11	1	0	1	1	0	$\{\mathcal{Q}(0), \mathcal{Q}(2), \mathcal{Q}(3)\}$	$\{\mathcal{Q}(0), (1, 0, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}$	0.3331	0.8333
12	1	0	1	1	1	$\{\mathcal{Q}(0), \mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\dagger\{\mathcal{Q}(2)\}$ $\dagger\dagger\{(1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 1)\}$	0.25 0.5	0.7181 0.7071
13	2	0	1	1	1	$\{2\mathcal{Q}(0), \mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\{\mathcal{Q}(2), \mathcal{Q}(4)\}$	0.1429	0.3499
14	0	1	1	1	0	$\{\mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3)\}$	$\dagger\{(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 0, 1), (0, 1, 1, 1)\}$	0.25	0.5
15	1	1	1	1	0	$\{\mathcal{Q}(0), \mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3)\}$	$\dagger\dagger\{(0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}$	0.375	0.4330
16	1	1	1	1	1	$\{\mathcal{Q}(0), \mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\dagger\{\mathcal{Q}(1), \mathcal{Q}(3)\}$	0.2	0.6325
17	2	1	1	1	1	$\{2\mathcal{Q}(0), \mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\{\mathcal{Q}(1), \mathcal{Q}(3)\}$	0.25	0.5
18	2	1	1	1	2	$\{2\mathcal{Q}(0), \mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), 2\mathcal{Q}(4)\}$	$\{\mathcal{Q}(1), \mathcal{Q}(3), \mathcal{Q}(4)\}$	0	0
19	3	1	1	1	2	$\{3\mathcal{Q}(0), \mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), 2\mathcal{Q}(4)\}$	$\{\mathcal{Q}(0), \mathcal{Q}(1), \mathcal{Q}(3), \mathcal{Q}(4)\}$	0.0370	0.1171
20	1	2	1	1	1	$\{\mathcal{Q}(0), 2\mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\{(1, 0, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (0, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1), \mathcal{Q}(3)\}$	0.0833	0.1667
21	1	2	1	1	2	$\{\mathcal{Q}(0), 2\mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), 2\mathcal{Q}(4)\}$	$\dagger\{\mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(4)\}$	0.0437	0.1169
22	1	1	2	1	1	$\{\mathcal{Q}(0), \mathcal{Q}(1), 2\mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\dagger\dagger\{(0, 0, 1, 0), (0, 0, 0, 1), \mathcal{Q}(1), \mathcal{Q}(3), \mathcal{Q}(4)\}$	0.1429	0.2427
23	2	1	2	1	1	$\{2\mathcal{Q}(0), \mathcal{Q}(1), 2\mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\}$	$\{\mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(4)\}$	0.1398	0.3424
24	2	1	2	1	2	$\{2\mathcal{Q}(0), \mathcal{Q}(1), 2\mathcal{Q}(2), \mathcal{Q}(3), 2\mathcal{Q}(4)\}$	$\{2\mathcal{Q}(0), \mathcal{Q}(2), \mathcal{Q}(3)\}$	0.1801	0.2724
24	1	2	1	2	1	$\{\mathcal{Q}(0), 2\mathcal{Q}(1), \mathcal{Q}(2), 2\mathcal{Q}(3), \mathcal{Q}(4)\}$	$\{2\mathcal{Q}(0), \mathcal{Q}(2), \mathcal{Q}(3)\}$	0.125	0.25
25	1	2	1	2	2	$\{\mathcal{Q}(0), 2\mathcal{Q}(1), \mathcal{Q}(2), 2\mathcal{Q}(3), 2\mathcal{Q}(4)\}$	$\{(1, 0, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), \mathcal{Q}(1), \mathcal{Q}(4)\}$	0.0690	0.1873
26	2	2	1	2	2	$\{2\mathcal{Q}(0), 2\mathcal{Q}(1), \mathcal{Q}(2), 2\mathcal{Q}(3), 2\mathcal{Q}(4)\}$	$\{\mathcal{Q}(0), \mathcal{Q}(1), 2\mathcal{Q}(3)\}$	0	0
							$\{2\mathcal{Q}(1), \mathcal{Q}(3), \mathcal{Q}(4)\}$	0.0395	0.1101
								0.0625	0.125

$\dagger$ These are optimum blocks w.r.t. $\|A^*(D)\|_1$; $\dagger\dagger$ these are optimum blocks w.r.t. $\|A^*(D)\|_2$; the others are optimum blocks w.r.t. the two norms.

$$T' = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

whose α th row denotes the α th assembly in T . Let $D_1 = (1; 0, 0, 0, 0, 0, 0; 1, 1, 1, 1; 1)$ and $D_2 = (1; 1, 0, 0, 0, 0, 0; 1, 1, 1, 1; 0)$ be CB plans in S^N , whose α th elements denote the α th assemblies in D_1 and D_2 , respectively. Then $\|A^*(D_1)\|_1 = 0.25$, $\|A^*(D_1)\|_2 = 0.7181$, $\|A^*(D_2)\|_1 = 0.5$ and $\|A^*(D_2)\|_2 = 0.7071$. This means that D_1 is a CB plan better than D_2 w.r.t. $\|A^*(D)\|_1$ and D_2 is one better than D_1 w.r.t. $\|A^*(D)\|_2$ for T . In fact, $\|A^*(D_1)\|_1$ and $\|A^*(D_2)\|_2$ have minimum values over S^N . Hence D_1 and D_2 are OCB plans w.r.t. $\|A^*(D)\|_1$ and $\|A^*(D)\|_2$, respectively. An optimum block corresponding to D_1 is

$$B_T = \{B_{(0)} = \{\mathcal{Q}(2)\}, B_{(1)} = \{\mathcal{Q}(0), \mathcal{Q}(3), \mathcal{Q}(4)\}\}.$$

An optimum block to D_2 is $B_T = \{B_{(0)}, B_{(1)}\}$, where

$$B_{(0)} = \{(1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), \\ (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 1)\},$$

$$B_{(1)} = \{(0, 0, 0, 0), (1, 1, 0, 0), \mathcal{Q}(3)\}.$$

This is the indication of Table for $N=12$.

(ii) Consider a B -array $[20, 4, 4, \{1, 2, 1, 1, 1\}]$ given by

$$T = \{\mathcal{Q}(0), 2\mathcal{Q}(1), \mathcal{Q}(2), \mathcal{Q}(3), \mathcal{Q}(4)\},$$

which is a 2^4 -BFF design of resolution V with 20 assemblies. Similarly, T can be rewritten by the following

$$T' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $D = (1; 0, 1, 0, 0; 1, 0, 0, 0; 1, 1, 1, 1, 1, 1; 0, 0, 0, 0; 1) \in S^N$. Then, $\|A^*(D)\|_1 = 0.1429$ and $\|A^*(D)\|_2 = 0.2474$, which have minimum values over S^N . Hence D is an OCB plan for T w.r.t. the two norms and the corresponding optimum block $B_T = \{B_{(0)}, B_{(1)}\}$, where

$$B_{(0)} = \{(1, 0, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), \\ (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), \mathcal{Q}(3)\},$$

$$B_{(1)} = \{\mathcal{Q}(0), (0, 1, 0, 0), (1, 0, 0, 0), \mathcal{Q}(2), \mathcal{Q}(4)\}.$$

Remark. For $r \geq 2$, in general, it is an enormous amount of work

to find an OCB plan D in $D_{\{k(d_1, \dots, d_r)\}}^N$ w.r.t. each norm for a design T . In the case where block factors rise in turn for the complete experiment of a design T , we can consider a successive block division. For simplicity, consider the case of $r=2$ and $k=4$. This is the case where after the division of two blocks for one factor (experimenter), both of the two blocks must be divided into two blocks for a new factor (day). First we find an optimum block $B_{T, \{k(0), k(1)\}} = \{B_{(0)}, B_{(1)}\}$ w.r.t. each norm for the block effect η_1 . Next we find the set $B_{T, \{k(0,0), k(0,1), k(1,0), k(1,1)\}} = \{B_{(0,0)}, B_{(0,1)}, B_{(1,0)}, B_{(1,1)}\}$ of blocks minimizing the norm for the effect η_2 , where $B_{(0,s)}$ and $B_{(1,s)}$, ($s=0, 1$), are two divisions of $B_{(0)}$ and $B_{(1)}$, respectively, and $k(q) = k(q, 0) + k(q, 1)$ for $q=0, 1$. If $N=16$ and $k(0, 0) = \dots = k(1, 1) = 4$, and if the assemblies in T are all distinct, then the total number of plans in $D_{\{k(d_1, d_2)\}}^N$ is $16!/(4!)^4 = 63,063,000$, whereas the plan in the above procedure is chosen among $\binom{16}{8} + \binom{8}{4} \binom{8}{4} = 17,770$ plans.

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