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SOME PROPERTIES OF INVARIANT POLYNOMIALS WITH MATRIX ARGUMENTS AND THEIR APPLICATIONS IN ECONOMETRICS

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Summary

Further properties are derived for a class of invariant polynomials with several matrix arguments which extend the zonal polynomials. Generalized Laguerre polynomials are defined, and used to obtain expansions of the sum of independent noncentral Wishart matrices and an associated generalized regression coefficient matrix. The latter includes the k-class estimator in econometrics.

1. Introduction

The zonal polynomials $C_\kappa(X)$ arise mathematically as zonal spherical functions on the space of real positive definite symmetric $m\times m$ matrices X, which may be regarded as the quotient space $Gl(m, R)/O(m)$, where $Gl(m, R)$ is the real linear group of real nonsingular $m\times m$ matrices and $O(m)$ is the orthogonal group of $m\times m$ orthogonal matrices. The application of the $C_\kappa(X)$ to multivariate distribution theory was recognized and developed by James [14] [15] and Constantine [3]. A class of invariant polynomials $C_\phi^{\kappa[r]}(X_{[r]})$ generalizing the $C_\kappa(X)$ to any number r of symmetric matrices $X_{[r]}=(X_1,\cdots,X_r)$ has been defined by Davis [6] in the case $r=2$, and extended to $r\geqq 2$ by Chikuse [2]. These polynomials have also been applied to multivariate distribution theory (see e.g. Chikuse [2], Davis [5], [7], [8], [9], Hayakawa [10], [11], Hillier et al. [13], Mathai and Pillai [18], Phillips [19]).

Section 2 of the present paper reviews various properties of the $C_\phi^{\kappa[r]}$ which were indicated by Chikuse [2]. In particular, equation (3.12) of the latter paper expresses a basic result from which various other properties can be obtained. This result is restated in Section 2 with a minor correction. A further generalization of Constantine's [4] Laguerre

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polynomial of matrix argument is also defined. Proofs of the results in Section 2 are indicated in Section 3. In Section 4, we shall present methods to determine the coefficients which are 'suitably' defined in the expansions obtained in Section 2.

Various estimators of the regression coefficient in a single structural equation with multiple endogenous variables have been considered in the econometric literature. Theil's [24] k-class estimators contain the ordinary least squares (OLS) and two-stage least squares (TSLS) estimators as special cases ($k=0, 1$ respectively). These have the form of the regression coefficient vector $A_{22}^{-1}a_{21}$ for a random matrix $A=\overset{\quad 1 \qquad m-1}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & A_{22} \end{bmatrix}}\begin{matrix} 1 \\ m-1 \end{matrix}$, where A is a noncentral Wishart matrix in the case of the OLS and TSLS estimators, and is the sum S_1+S_2 of independent noncentral and central Wishart matrices respectively for the general k-class estimator. In Section 5 we derive the density functions of $A=\sum_{i=1}^{r} S_i$, a sum of independent noncentral Wishart matrices with possibly distinct covariance matrices. The associated generalized regression coefficient matrix $B=A_{22}^{-1}A_{21}$, where $A=\overset{\quad m_1 \quad m_2}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}\begin{matrix} m_1 \\ m_2 \end{matrix}$, is considered in Section 6, together with particular cases of econometric interest.

2. Properties of the $C_\phi^{\kappa[r]}$

If $P_k(X)$ denotes the class of homogeneous polynomials of degree k in the elements of a symmetric matrix X, then under the representation of $Gl(m, R)$ in $P_k(X)$ generated by the congruence transformation $X \rightarrow LXL'$, $L \in Gl(m, R)$, $P_k(X)$ decomposes into the direct sum of uniquely defined irreducible invariant subspaces [15]

$$P_k(X)=\bigoplus_{\kappa} \mathcal{V}_\kappa(X) ,$$

where κ ranges over all ordered partitions of k into $\leqq m$ parts. The restriction of the representation to $\mathcal{V}_\kappa(X)$ is the irreducible representation of $Gl(m, R)$ indexed by $[2\kappa]$, and $\mathcal{V}_\kappa(X)$ has a one-dimensional subspace, invariant under the restriction to $O(m)$, which is generated by the (suitably normalized) zonal polynomial $C_\kappa(X)$.

The class $P_{k[r]}(X_{[r]})$ of homogeneous polynomials of degree $k_1, \cdots, k_r$ in the elements of $X_1, \cdots, X_r$, respectively, may be written as the Kronecker product

$$(2.1) \qquad P_{k[r]}(X_{[r]})=\bigotimes_{i=1}^{r} P_{k_i}(X_i)=\bigotimes_{\kappa[r]} \{\mathcal{V}_{\kappa_1}(X_1)\otimes\cdots\otimes\mathcal{V}_{\kappa_r}(X_r)\}$$

(Chikuse [2]), where $\kappa[r]=(\kappa_1, \cdots, \kappa_r)$. Under the representation of

$Gl(m, R)$ in this space generated by

$$X_i \to LX_iL' \qquad (i=1,\cdots,r;\ L \in Gl(m, R)),$$

the restriction to the product space in braces in (2.1) is the Kronecker product $\bigotimes_{i=1}^{r}[2\kappa_i]$, which may be decomposed into a direct sum of irreducible representations of $Gl(m, R)$ indexed by partitions Φ of $2f$ into $\leqq m$ parts, where $f=\sum_{i=1}^{r} k_i$. Thus we may write

$$P_{\kappa[r]}(X_{[r]})=\bigoplus_{\kappa[r]}\bigoplus_{\Phi}\mathcal{V}_{\Phi}^{\kappa[r]}(X_{[r]}).$$

Subspaces such that $\Phi=2\phi$, where ϕ is a partition of f into $\leqq m$ parts, also contain a one-dimensional subspace generated by a polynomial invariant under the simultaneous transformations

$$X_i \to HX_iH' \qquad (i=1,\cdots,r;\ H \in O(m)).$$

A complication which arises at this point is that $[2\phi]$ may occur with a multiplicity greater than one for a given $\kappa[r]$, and the corresponding subspaces $\mathcal{V}_{2\phi}^{\kappa[r]}$ and their invariant polynomials are then not uniquely defined. However, the direct sum of equivalent subspaces

$$\mathcal{U}_{\phi}^{\kappa[r]}(X_{[r]})=\bigoplus_{\phi'\equiv\phi}\mathcal{V}_{2\phi'}^{\kappa[r]}(X_{[r]})$$

is uniquely defined, and it has so far proved sufficient for applications (Davis [6]) to construct a basis $\{C_{\phi'}^{\kappa[r]}(X_{[r]}),\ \phi'\equiv\phi\}$ for the subspace of invariant polynomials in $\mathcal{U}_{\phi}^{\kappa[r]}$ which is "orthogonal" in a certain sense. Any such basis (with a suitable normalization) then has the fundamental property

$$\text{(2.2)} \qquad \int_{O(m)}\prod_{i=1}^{r} C_{\kappa_i}(A_iHX_iH')dH=\sum_{\phi\in\kappa_1\cdots\kappa_r} C_{\phi}^{\kappa[r]}(A_{[r]})C_{\phi}^{\kappa[r]}(X_{[r]})/C_{\phi}(I).$$

Here the A_i are $m\times m$ symmetric matrices, I is the $m\times m$ unit matrix, dH is the invariant Haar measure over $O(m)$, and the sum on the right-hand side extends over the representations $[2\phi]$ in the decomposition of $\bigotimes_{i=1}^{r}[2\kappa_i]$. The appropriate multiplicity of ϕ is implied by the notation (2.2), but in other situations may need to be indicated explicitly, e.g. by $\bigoplus_{\phi'\equiv\phi}$. Although the basis is only required to be orthogonal, it is convenient to take the uniquely-defined projection of $\prod_{i=1}^{r}\{\operatorname{tr}(X_i)\}^{k_i}$ onto $\mathcal{U}_{\phi}^{\kappa[r]}$ as the first member of the set, and to choose the remaining polynomials to be orthogonal to it. No attempt is made to specify individual $\mathcal{V}_{2\phi}^{\kappa[r]}$, or to relate the $C_{\phi}^{\kappa[r]}$ to them. Coefficients in formulae based on the $C_{\phi}^{\kappa[r]}$ must therefore in general be regarded as relative to a particular tabulation of the polynomials.

With regard to the effect on $C_{\phi}^{\kappa[r]}(X_{[r]})$ of simultaneously permuting X_i and the corresponding κ_i, it is convenient for tabulation to *define* the polynomials to be invariant under transpositions

$$(X_i, \kappa_i) \longleftrightarrow (X_j, \kappa_j) , \qquad (i \neq j) , \tag{2.3}$$

provided that $\kappa_i \neq \kappa_j$. However, reference to the tables in [8] shows that invariance cannot be assumed when $\kappa_i = \kappa_j$. On the other hand, since the class of polynomials defined by $\mathcal{U}_{\phi}^{\kappa[r]}$ is necessarily invariant under (2.3) when $\kappa_i = \kappa_j$, it follows that the result of simultaneously permuting (X_i, κ_i) in $C_{\phi}^{\kappa[r]}(X_{[r]})$ can in general be expressed as a linear combination of the original $C_{\phi'}^{\kappa[r]}(X_{[r]})$, $\phi' \equiv \phi$.

Clarification of these points has necessitated some adjustment of various formulae in Chikuse [2]. The equation numbers of the latter paper are cited for convenience under the present equation numbers. Proofs of the results are indicated in Section 3. We shall use the notation $\alpha[s, r] = (\alpha_s, \cdots, \alpha_r)$.

LEMMA 2.1.

(2.4) [(3.12)]
$$\int_{O(m)} C_{\phi}^{\kappa[r]}(A'H'X_{[q]}HA, A_{[q+1,r]})dH = \sum_{\substack{\sigma \in \kappa_1 \cdots \kappa_q \\ (\phi \in \sigma^* \cdot \kappa_{q+1} \cdots \kappa_r)}} \sum_{\phi' \equiv \phi} \gamma_{\sigma;\phi'}^{\kappa[r];\phi} C_{\sigma}^{\kappa[q]}(X_{[q]}) C_{\phi'}^{\sigma^*, \kappa[q+1,r]}(A'A, A_{[q+1,r]}) / C_{\sigma}(I) ,$$

for suitably defined coefficients γ, where σ^* denotes the partition σ ignoring multiplicity. No general formula is given for the γ's, but in particular

$$\gamma_{\sigma;\phi'}^{\kappa[r];\phi} = \begin{cases} \delta_{\kappa_1,\sigma} \delta_{\phi,\phi'} & (q=1) \\ \delta_{\sigma,\phi} \delta_{\phi,\phi'} & (q=r) \end{cases}$$

([2] (3.15), (3.14) *respectively*). *A number of further properties may be derived from Lemma* 2.1.

LEMMA 2.2. (i)

(2.5) [(3.1)]
$$C_{\phi}^{\kappa[r]}(X_{[q]}, I, \cdots, I) = C_{\phi}(I) \sum_{\substack{\sigma \in \kappa_1 \cdots \kappa_q \\ (\phi \in \sigma^* \cdot \kappa_{q+1} \cdots \kappa_r)}} \alpha_{\sigma}^{\kappa[r];\phi} C_{\sigma}^{\kappa[q]}(X_{[q]}) / C_{\sigma}(I)$$

where

$$\alpha_{\sigma}^{\kappa[r];\phi} = \sum_{\phi' \equiv \phi} \gamma_{\sigma;\phi'}^{\kappa[r];\phi} \theta_{\phi'}^{\sigma^*, \kappa[q+1,r]} , \qquad \theta_{\phi}^{\kappa[r]} = C_{\phi}^{\kappa[r]}(I, \cdots, I) / C_{\phi}(I) .$$

(ii)

(2.6) [(3.7)]
$$C_{\phi}^{\kappa[r]}(X, \cdots, X, A_{[q+1,r]}) = \sum_{\sigma^* \in \kappa_1 \cdots \kappa_q} \sum_{\phi' \equiv \phi} \beta_{\sigma^*;\phi'}^{\kappa[r];\phi} C_{\phi'}^{\sigma^*, \kappa[q+1,r]}(X, A_{[q+1,r]})$$

where

$$\beta_{\sigma^*;\phi'}^{\kappa[r];\phi}=\sum_{\sigma\equiv\sigma^*}\theta_\sigma^{\kappa[q]}\gamma_{\sigma;\phi'}^{\kappa[r];\phi}$$

and the summation in (2.6) *is over distinct partitions* σ^* *ignoring multiplicity.*

If the arguments are permuted in the left-hand sides of (2.4), (2.5) *and* (2.6), *then as noted earlier the result is a linear combination of polynomials with arguments as shown in these equations. Thus the formulae continue to hold, except that the coefficients* γ, α, β *must be replaced by the corresponding linear combinations. In particular, if the last* q *arguments are involved, we shall denote the resulting coefficients* $\bar{\gamma}$, $\bar{\alpha}$, $\bar{\beta}$.

(iii) *Product of polynomials*

(2.7) [(3.11)]
$$C_\sigma^{\kappa[q]}(X_{[q]})C_\tau^{\kappa[q+1,r]}(X_{[q+1,r]})=\sum_{\phi\in\sigma^*\cdot\tau^*}\pi_{\sigma,\tau}^{\kappa[r];\phi}C_\phi^{\kappa[r]}(X_{[r]}) \ ,$$

where

$$\pi_{\sigma,\tau}^{\kappa[r];\phi}=\sum_{\phi'\equiv\phi}\gamma_{\sigma;\phi'}^{\kappa[r];\phi}\bar{\alpha}_\tau^{\sigma^*,\kappa[q+1,r];\phi'} \ .$$

(iv) (*Extension of* (ii)). *If* $r=\sum_{i=1}^t q_i$, *then*

(2.8)
$$\begin{aligned}&C_\phi^{\kappa[r]}(\underbrace{X_1,\cdots,}_{q_1} X_1,\cdots\underbrace{, X_t,\cdots,}_{q_t} X_t)\\ &\quad=\sum_{\sigma_1^*\in\kappa_1\cdots\kappa_q}\cdots\sum_{\sigma_t^*\in\kappa_{r-q_t+1}\cdots\kappa_r}\sum_{\phi'\equiv\phi}\tilde{\beta}_{\sigma^*[t];\phi'}^{\kappa[r];\phi}C_{\phi'}^{\sigma^*[t]}(X_{[t]}) \ ,\end{aligned}$$

for suitable coefficients $\tilde{\beta}$ *which reduce to the* β*'s of* (2.6) *when* $q_2=\cdots=q_t=1$.

(v) *Multinomial expansion*

(2.9)
$$\begin{aligned}&C_\phi^{\sigma[t]}\left(\sum_{i=1}^{q_1}X_{1i},\cdots,\sum_{i=1}^{q_t}X_{ti}\right)\\ &\quad=\sum_{\substack{\kappa[r]\\(\sigma_i\in\kappa_{r_{i-1}+1}\cdots\kappa_{r_i})}}\sum_{\phi'\equiv\phi}\prod_{i=1}^t\begin{bmatrix}s_i\\k_{r_{i-1}+1},\cdots,k_{r_i}\end{bmatrix}\tilde{\beta}_{\sigma[t];\phi}^{\kappa[r];\phi'}\\ &\qquad\cdot C_{\phi'}^{\kappa[r]}(X_{11},\cdots,X_{1q_1},\cdots,X_{tq_t})\end{aligned}$$

where $r_i=\sum_{j=1}^i q_j$ $(i=1,\cdots,t)$, $r=r_t$.

LEMMA 2.3.

(2.10)
$$C_\phi^{\kappa[r]}(X_{[r]}A)=\sum_{\sigma\in\phi^*\cdot\phi^*}\zeta_\sigma^{\kappa[r];\phi}C_\sigma^{\kappa[r],\phi^*}(X_{[r]}, A) \ ,$$

for suitable coefficients ζ. *If* A *is placed first on the right-hand side,* *the coefficient will be written* $\bar{\zeta}$.

Generalized Laguerre polynomial. Extending Constantine's [4] definition of a Laguerre polynomial of matrix argument (see also [2], [5], [16]), let

(2.11)
$$L^{u[q]}_{\kappa[r];\phi}(X_{[q]}; B_{[r]}) = \operatorname{etr}\left(\sum_{i=1}^{q} X_i\right) \int_{R_1>0} \cdots \int_{R_q>0} \prod_{i=1}^{q} \{\operatorname{etr}(-R_i)|R_i|^{u_i} A_{u_i}(X_i R_i)\} \cdot C^{\kappa[r]}_{\phi}((BRB')_{[q]}, (BB')_{[q+1,r]}) \prod_{i=1}^{q} dR_i ,$$

where A_u is Herz's [12] Bessel function of matrix argument. The Laplace transform with respect to $X_{[q]}$ is

(2.12)
$$\int_{X_1>0} \cdots \int_{X_q>0} \operatorname{etr}\left(-\sum_{i=1}^{q} X_i Z_i\right) \prod_{i=1}^{q} |X_i|^{u_i} L^{u[q]}_{\kappa[r];\phi}(X_{[q]}; B_{[r]}) \prod_{i=1}^{q} dX_i = \prod_{i=1}^{q} \{\Gamma_m(u_i+p, \kappa_i)|Z_i|^{-u_i-p}\} C^{\kappa[r]}_{\phi}((B(I-Z^{-1})B')_{[q]}, (BB')_{[q+1,r]}) ,$$

where $p=(m+1)/2$. Applying (2.9) and [2] (3.23), we obtain the serial expression

(2.13)
$$L^{u[q]}_{\kappa[r];\phi}(X_{[q]}; B_{[r]}) = \left\{\prod_{i=1}^{q} (u_i+p)_{\kappa_i}\right\} \sum_{\substack{(\rho,\sigma)[q] \\ (\kappa_i \in \rho_i \cdot \sigma_i)}} \sum_{\phi' \equiv \phi} \left\{\prod_{i=1}^{q} \binom{k_i}{r_i} \Big/ (u_i+p)_{\sigma_i}\right\} \cdot \tilde{\beta}^{(\rho,\sigma)[q],\kappa[q+1,r];\phi'}_{\kappa[r];\phi} C^{(\rho,\sigma)[q],\kappa[q+1,r]}_{\phi'}((BB', -BXB')_{[q]}, (BB')_{[q+1,r]}) .$$

$L^{u[q]}_{\kappa[r];\phi}$ is the coefficient of $C^{\kappa[r]}_{\phi}(U_{[r]}) \Big/ \prod_{i=1}^{r} k_i! C_{\phi}(I)$ in the expansion of

(2.14)
$$\int_{O(m)} \prod_{i=1}^{q} |I - B_i'H'U_iHB_i|^{-u_i-p} \operatorname{etr}\left[-\sum_{i=1}^{q} X_i B_i'H'U_iHB_i(I-B_i'H'U_iHB_i)^{-1} + \sum_{i=q+1}^{r} B_i'H'U_iHB_i\right] dH .$$

3. Proofs of the results in Section 2

PROOF OF LEMMA 2.1. The integral in (2.4) is the coefficient of $C^{\kappa[r]}_{\phi}(U_{[r]})/C_{\phi}(I)$ in

$$\int_{O(m)} dH \int_{O(m)} \prod_{i=1}^{q} C_{\kappa_i}(A'H'X_iHAK'U_iK) \prod_{j=q+1}^{r} C_{\kappa_j}(A_jK'U_jK) dK .$$

Interchanging the order of integration, this becomes

$$\sum_{\sigma \in \kappa_1 \cdots \kappa_q} \{C^{\kappa[q]}_{\sigma}(X_{[q]})/C_{\sigma}(I)\} \int_{O(m)} C^{\kappa[q]}_{\sigma}(BK'U_{[q]}K) \prod_{i=q+1}^{r} C_{\kappa_i}(A_iK'U_iK) dK$$

where $B=A'A$. Following James's [14] argument, we obtain

$$C_\sigma^{\kappa[q]}(BU_{[q]}) \in \mathcal{U}_{\sigma^*}^{\kappa[q]}(U_{[q]}) \otimes \mathcal{V}_{\sigma^*}(B)$$

$$C_{\kappa_i}(A_iU_i) \in \mathcal{V}_{\kappa_i}(U_i) \otimes \mathcal{V}_{\kappa_i}(A_i)\,, \qquad (i=q+1,\cdots,r)\,,$$

and from invariance under

$$B,\, A_i \to LBL',\, LA_iL' \qquad (i=q+1,\cdots,r)\,,$$

$$U_i \to L'^{-1}U_iL^{-1} \qquad (i=1,\cdots,r\colon\ L \in Gl(m, R))$$

we have

$$C_\sigma^{\kappa[q]}(BU_{[q]}) \prod_{i=q+1}^{r} C_{\kappa_i}(A_iU_i) \in \bigoplus_{\phi \in \sigma^* \cdot \kappa_{q+1} \cdots \kappa_r} \mathcal{U}_{\phi^*}^{\kappa[r]}(U_{[r]}) \otimes \mathcal{U}_{\phi^*}^{\sigma^*, \kappa[q+1,r]}(B, A_{[q+1,r]})\,.$$

Hence the integral in (2.4) has the required form.

Proof of Lemma 2.2. (i) Set $A=A_{q+1}=\cdots=A_r=I$ in Lemma 2.1.
(ii) Set $X_1=\cdots=X_q=I$ in Lemma 2.1.
(iii) The left-hand side in (2.7) is the coefficient of $C_\sigma^{\kappa[q]}(A_{[q]})C_\tau^{\kappa[q+1,r]}(A_{[q+1,r]})\Big/\prod_{i=1}^{r} k_i!C_\sigma(I)C_\tau(I)$ in

$$P=\int_{O(m)}\int_{O(m)} \operatorname{etr}\left(\sum_{i=1}^{q} X_iH'A_iH\right) \operatorname{etr}\left(\sum_{j=q+1}^{r} X_jK'A_jK\right)dHdK\,.$$

Let $H \to HK$. By invariance of Haar measure

$$P=\sum_{\kappa[r];\phi}^{\infty} C_\phi^{\kappa[r]}(X_{[r]})\left[\prod_{i=1}^{r} k_i!C_\phi(I)\right]^{-1} \int_{O(m)} C_\phi^{\kappa[r]}(H'A_{[q]}H, A_{[q+1,r]})dH\,.$$

From Lemma 2.1 and (2.5) the integral may be expressed as

$$C_\phi(I) \sum_{\sigma \in \kappa_1 \cdots \kappa_q} \sum_{\substack{\tau \in \kappa_{q+1} \cdots \kappa_r \\ (\phi \in \sigma^* \cdot \tau^*)}} \sum_{\phi' \equiv \phi} \gamma_{\sigma;\phi'}^{\kappa[r];\phi} \overline{\alpha}_\tau^{\sigma^*, \kappa[q+1,r];\phi'} C_\sigma^{\kappa[q]}(A_{[q]})$$

$$\cdot\, C_\tau^{\kappa[q+1,r]}(A_{[q+1,r]})/C_\sigma(I)C_\tau(I)\,.$$

The result follows.
(iv) Apply (ii) successively, with appropriate interchange of arguments.
(v) The left-hand side of (2.9) is the coefficient of $C_\phi^{\sigma[t]}(U_{[t]})\Big/\prod_{i=1}^{t} s_i!C_\phi(I)$ in

$$\int_{O(m)} \operatorname{etr}\left\{\sum_{i=1}^{t}\left[\sum_{j=1}^{q_i} X_{ij}\right]HU_iH'\right\}dH$$

$$=\sum_{\kappa[r];\phi}^{\infty} C_\phi^{\kappa[r]}(X_{11},\cdots,X_{1q_1},\cdots,X_{tq_t})$$

$$\cdot\, C_\phi^{\kappa[r]}(\underbrace{U_1,\cdots,U_1}_{q_1},\cdots,\underbrace{U_t,\cdots,U_t}_{q_t})\Big/\prod_{i=1}^{r} k_i!C_\phi(I)\,.$$

Proof of Lemma 2.3.

$$C_{\phi}^{\kappa[r]}(X_{[r]}A) \in \mathcal{U}_{\phi^*}^{\kappa[r]}(X_{[r]}) \otimes \mathcal{V}_{\phi^*}(A) \subseteq \bigoplus_{\sigma^* \in \phi^* \cdot \phi^*} \mathcal{U}_{\sigma^*}^{\kappa[r],\phi^*}(X_{[r]}, A) .$$

Since the left-hand side is invariant under the simultaneous transformations $X_i \to HX_iH'$, $A \to HAH'$ ($H \in O(m)$), it must be a linear combination of the $C_{\sigma}^{\kappa[r],\phi^*}(X_{[r]}, A)$.

4. Construction of the coefficients

In Section 2 we have shown the existence of the coefficients γ, $\tilde{\beta}$ and ζ in the expansions (2.4), (2.8) (or (2.9)) and (2.10) respectively in terms of the invariant polynomials. In this section, we indicate methods of constructing them using matrix differential operators. The tabulation is lengthy and hence, due to limited space, it will be presented in a subsequent paper.

We utilize the matrices of differential operators

$$\partial U = \begin{cases} ((1/2)(1+\delta_{ij})\partial/\partial u_{ij}) , & \text{for an } m\times m \text{ symmetric matrix } U=(u_{ij}) , \\ (\partial/\partial u_{ij}) , & \text{for an } m\times m \text{ asymmetric matrix } U , \end{cases} \tag{4.1}$$

with δ being Kronecker's delta. Applications of the operators ∂U in multivariate analysis are indicated in Richards [21], [22] and Phillips [20]. The ∂U have useful properties

$$f(R)\,\mathrm{etr}\,(UR) = f(\partial U)\,\mathrm{etr}\,(UR) , \tag{4.2}$$

and hence

$$f(R) = f(\partial U)\,\mathrm{etr}\,(UR)|_{U=0} . \tag{4.3}$$

We note here an alternative representation of the multinomial expansion (2.9),

$$\begin{aligned} C_{\phi}^{\sigma[t]}\left(\sum_{i=1}^{q_1} X_{1i}, \cdots, \sum_{i=1}^{q_t} X_{ti}\right) \\ = \prod_{k=1}^{t} \left[\mathrm{etr}\left(\sum_{j=2}^{q_k} X_{kj}\right)\partial X_{k1}\right] C_{\phi}^{\sigma[t]}(X_{11}, \cdots, X_{k1}, \cdots, X_{t1}) , \end{aligned} \tag{4.4}$$

which is the formal representation of Taylor's expansion

$$f(U+R) = \mathrm{etr}\,(R\partial U) f(U) . \tag{4.5}$$

(4.4) may provide a mechanism for the generation of the coefficients $\tilde{\beta}$ in the expansion (2.9).

In similar fashion to that in Richards [21], [22], we define differential operators having a certain orthogonality property. Let $\tilde{t}_1, \tilde{t}_2, \cdots$

be the distinct traces of products of $X_1, \cdots, X_r$. Then, the orthonormal polynomials $\tilde{\Gamma}_\phi^{\kappa[r]}(X_{[r]}) = z_\phi^{-1/2} C_\phi^{\kappa[r]}(X_{[r]})$, with $z_\phi = C_\phi(I_m)/2^f (m/2)_\phi$, are expressed as

$$\tag{4.6} \tilde{\Gamma}_\phi^{\kappa[r]}(X_{[r]}) = \sum_\nu \tilde{\gamma}_{\phi;\nu}^{\kappa[r]} \tilde{t}_1^{\nu_1} \tilde{t}_2^{\nu_2} \cdots ,$$

for suitable sets $\nu = (\nu_1, \nu_2, \cdots)$ of nonnegative integers $\nu_1, \nu_2, \cdots$. The coefficients $\tilde{\gamma}_{\phi;\nu}^{\kappa[r]}$ are tabulated up to $r=3$ and $f=5$ in Davis [5], [8]. Let σ_i be a partition of k_i, $i=1, \cdots, r$, and $\tau \in \sigma[r]$. Based on the orthogonality property of the invariant polynomials, associated with each invariant polynomial $C_\phi^{\kappa[r]}(X_{[r]})$, we define a differential operator $\partial_\phi^{\kappa[r]}[X_{[r]}]$ having the basic property that

$$\tag{4.7} \partial_\phi^{\kappa[r]}[X_{[r]}] C_\tau^{\sigma[r]}(X_{[r]}) = \delta_{\kappa[r],\phi;\sigma[r],\tau} ,$$

as

$$\tag{4.8} \partial_\phi^{\kappa[r]}[X_{[r]}] = \sum_\nu \frac{z_\phi^{-1/2} \tilde{\gamma}_{\phi;\nu}^{\kappa[r]}}{(\nu_1!\, \nu_2! \cdots) z_{(f)}^{-1/2} \tilde{\gamma}_{f;\nu}^{k_1 \cdots k_r}} \frac{\partial^{\nu_1+\nu_2+\cdots}}{\partial \tilde{t}_1^{\nu_1} \partial \tilde{t}_2^{\nu_2} \cdots} .$$

Now, (4.3) yields

$$\tag{4.9} f(\partial U)\, {}_0F_1\left(\frac{1}{2}m ; \frac{1}{4} BUU'B'\right)\bigg|_{U=0} = \int_{O(m)} f(HB) dH .$$

This gives a representation of the left-hand side of (2.4),

$$\tag{4.10} \begin{aligned} &C_\phi^{\kappa[r]}(A'\partial U' X_{[q]} \partial U A, A_{[q+1,r]})\, {}_0F_1\left(\frac{1}{2}m ; \frac{1}{4} UU'\right)\bigg|_{U=0} \\ &= C_\phi^{\kappa[r]}(A'\partial U' X_{[q]} \partial U A, A_{[q+1,r]}) \sum_\tau \left(t!\, 4^t \left(\frac{1}{2}m\right)_\tau\right)^{-1} C_\tau(UU') , \end{aligned}$$

where τ runs through the partitions of $t = \sum_{i=1}^q k_i$. The representation makes the evaluation of the left-hand side of (2.4) much easier, involving only the differential computation. Applying the differential operators $\partial_\sigma^{\kappa[q]}[X_{[q]}]$ and then $\partial_{\phi'}^{\sigma^*, \kappa[q+1,r]}[A'A, A_{[q+1,r]}]$ to both sides of (2.4) with the left-hand side replaced by (4.10), we obtain

$$\tag{4.11} \begin{aligned} \gamma_{\tau;\phi'}^{\kappa[r];\phi} / C_\sigma(I) = \sum_\tau &\left(t!\, 4^t \left(\frac{1}{2}m\right)_\tau\right)^{-1} \partial_{\phi'}^{\sigma^*, \kappa[q+1,r]}[A'A, A_{[q+1,r]}] \\ &\cdot \partial_\sigma^{\kappa[q]}[X_{[q]}] C_\phi^{\kappa[r]}(A'\partial U' X_{[q]} \partial U A, A_{[q+1,r]}) C_\tau(UU') . \end{aligned}$$

Similarly, the coefficients $\tilde{\beta}$ and ζ are determined. Applying the differential operators $\partial_{\phi'}^{\sigma^*[t]}[X_{[t]}]$, or $\partial_{\phi'}^{\kappa[r]}[X_{11}, \cdots, X_{1q_1}, \cdots, X_{tq_t}]$, and $\partial_\sigma^{\kappa[r],\phi^*}$ $[X_{[r]}, A]$ to both sides of (2.8), or (2.9), and (2.10) respectively, we obtain

$$\partial_{\phi'}^{\sigma^*[t]}[X_{[t]}]C_{\phi}^{\kappa[r]}(X_1,\cdots,X_1,\cdots,X_t,\cdots,X_t)=\tilde{\beta}_{\sigma^*[t];\phi'}^{\kappa[r];\phi}, \tag{4.12}$$

or

$$\partial_{\phi'}^{\kappa[r]}[X_{11},\cdots,X_{1q_1},\cdots,X_{tq_t}]C_{\phi}^{\sigma[t]}\left(\sum_{i=1}^{q_1}X_{1i},\cdots\sum_{i=1}^{q_t}X_{ti}\right)$$
$$=\prod_{i=1}^{t}\begin{bmatrix} s_i \\ k_{r_{i-1}+1},\cdots,k_{r_i}\end{bmatrix}\tilde{\beta}_{\sigma[t];\phi}^{\kappa[r];\phi'}, \tag{4.13}$$

and

$$\partial_{\sigma}^{\kappa[r],\phi^*}[X_{[r]},A]C_{\phi}^{\kappa[r]}(X_{[r]}A)=\zeta_{\sigma}^{\kappa[r];\phi}. \tag{4.14}$$

5. Sum of noncentral Wishart matrices

Let $A=\sum_{i=1}^{r}S_i$, where the S_i are independently distributed as $W_m(n_i, \Sigma_i, \Omega_i)$ $(i=1,\cdots,r)$. We shall also use the notation

$$\Gamma_i=(1/2)T_i\Omega_iT_i', \qquad \text{where } \Sigma_i^{-1}=T_iT_i' \ (i=1,\cdots,r),$$
$$\Lambda_i=\Sigma_r^{-1}-\Sigma_i^{-1}=G_iG_i' \qquad (i=1,\cdots,r).$$

Then following Chikuse's [2] derivation, we obtain for the density of A

$$f(A)=c_1\,\mathrm{etr}\left(-\frac{1}{2}\Sigma_r^{-1}A\right)|A|^{\Sigma n_i/2-p}\sum_{\kappa[r];\phi}^{\infty}\left\{\theta_{\phi}^{\kappa[r]}\prod_{i=1}^{r-1}\left(\frac{1}{2}n_i\right)_{\kappa_i}\Big/\right.$$
$$\left.\prod_{i=1}^{r}k_i!\left(\frac{1}{2}\sum_{i=1}^{r}n_i\right)_{\phi}\right\}\sum_{\substack{(\rho,\sigma)[r-1]\\(\kappa_i\in\rho_i\cdot\sigma_i)}}\sum_{\phi'\equiv\phi}\left\{\prod_{i=1}^{r-1}\binom{k_i}{r_i}\Big/\left(\frac{1}{2}n_i\right)_{\sigma_i}\right\}$$
$$\cdot\,\tilde{\beta}_{\kappa[r];\phi}^{(\rho,\sigma)[r-1],\kappa_r;\phi'}C_{\phi'}^{(\rho,\sigma)[r-1],\kappa_r}\left(\frac{1}{2}A[(\Lambda,\Gamma)_{[r-1]},\Gamma_r]\right), \tag{5.1}$$

where

$$c_1=\mathrm{etr}\left(-\frac{1}{2}\sum_{i=1}^{r}\Omega_i\right)\Big/\Gamma_m\left(\frac{1}{2}\sum_{i=1}^{r}n_i\right)\prod_{i=1}^{r}|2\Sigma_i|^{n_i/2}.$$

From (2.13), (5.1) may be written as

$$f(A)=c_1\,\mathrm{etr}\left(-\frac{1}{2}\Sigma_r^{-1}A\right)|A|^{\Sigma n_i/2-p}\sum_{\kappa[r];\phi}^{\infty}\theta_{\phi}^{\kappa[r]}$$
$$\cdot\,L_{\kappa[r];\phi}^{(n/2-p)[r-1]}\left(-(G^{-1}\Gamma G'^{-1})_{[r-1]};\left(\frac{1}{2}A\right)^{1/2}(G_{[r-1]},\Gamma_r^{1/2})\right)\Big/$$
$$\prod_{i=1}^{r}k_i!\left(\frac{1}{2}\sum_{i=1}^{r}n_i\right)_{\phi}, \tag{5.2}$$

which does not involve unknown coefficients.

In view of (2.11), the Laguerre polynomial is expressible in the form

$$\text{(5.3)} \quad \operatorname{etr}\left(-\sum_{i=1}^{r-1} \Gamma_i \Lambda_i^{-1}\right) \int_{R_1>0} \cdots \int_{R_{r-1}>0} \prod_{i=1}^{r-1} \{\operatorname{etr}(-R_i)|R_i|^{n_i/2-p}$$

$$\cdot A_{n_i/2-p}(-G_i^{-1}\Gamma_i G_i'^{-1} R_i)\} C_\phi^{\kappa[r]}\left(\frac{1}{2} A[(GRG')_{[r-1]}, \Gamma_r]\right) \prod_{i=1}^{r-1} dR_i \,.$$

6. Distribution of the generalized regression coefficient

(a) *General case.* Using the notation in the Introduction, we make the transformation $(A_{11}, A_{12}, A_{22}) \to (A_{11.2}, B, A_{22})$ where $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$. The Jacobian is $|A_{22}|^{m_1}$; also $|A| = |A_{11.2}||A_{22}|$ and

$$A = \begin{bmatrix} A_{11.2} & 0 \\ 0 & 0 \end{bmatrix} + [B, I_{m_2}]' A_{22} [B, I_{m_2}] = \mathcal{A} + \mathcal{B} \,, \qquad \text{say} \,.$$

Writing $\Sigma_r^{-1} = \begin{bmatrix} \Sigma_r^{11} & \Sigma_r^{12} \\ \Sigma_r^{21} & \Sigma_r^{22} \end{bmatrix}$, we have

$$\operatorname{tr}(\Sigma_r^{-1}A) = \operatorname{tr}(\Sigma_r^{11}A_{11.2} + QA_{22})$$

$$Q = [B, I_{m_2}]\Sigma_r^{-1}[B, I_{m_2}]' = B\Sigma_r^{11}B' + B\Sigma_r^{12} + \Sigma_r^{21}B' + \Sigma_r^{22} \,.$$

The invariant polynomial in (5.3) may be expanded by Lemma 2.3 as

$$\text{(6.1)} \qquad \sum_{\rho \in \phi\cdot\phi} \zeta_\rho^{\kappa[r];\phi} C_\rho^{\kappa[r],\phi*}((GRG')_{[r-1]}, \Gamma_r, A/2) \,.$$

Substituting $A = \mathcal{A} + \mathcal{B}$, applying the multinomial expansion, and integrating with respect to $A_{11.2}$ and A_{22}, we obtain the density of B in the form

$$\text{(6.2)} \quad f(B) = c_2|Q|^{-(\Sigma n_i + m_1)/2} \sum_{\kappa_1\lambda;\phi}^{\infty} \left\{\left(\frac{1}{2}\left(\sum_{i=1}^r n_i - m_2\right)\right)_\kappa \left(\frac{1}{2}\left(\sum_{i=1}^r n_i + m_1\right)\right)_\lambda \Big/\right.$$

$$\left. k!l!\left(\frac{1}{2}\sum_{i=1}^r n_i\right)_\phi\right\} \sum_{\substack{\sigma[r] \\ (\phi \in \sigma_1 \cdots \sigma_r)}} \binom{f}{s_1, \cdots, s_r} \theta_\phi^{\sigma[r]} \sum_{\tau \in \phi\cdot\phi} \sum_{\tau' \equiv \tau} \zeta_\tau^{\sigma[r];\phi} \bar{\beta}_{\phi;\tau}^{\sigma[r],\kappa,\lambda;\tau'}$$

$$\cdot L_{\sigma[r],\kappa,\lambda;\tau'}^{(n/2-p)[r-1]}(-(G^{-1}\Gamma G'^{-1})_{[r-1]}; G_{[r-1]}, \Gamma_r^{1/2}, [(\Sigma_r^{11})^{-1/2}, 0]',$$

$$[B, I_{m_2}]' Q^{-1/2})$$

where

$$c_2 = \operatorname{etr}\left(-\frac{1}{2}\sum_{i=1}^r \Omega_i\right) \Gamma_{m_1}\left(\frac{1}{2}\left(\sum_{i=1}^r n_i - m_2\right)\right) \Gamma_{m_2}\left(\frac{1}{2}\left(\sum_{i=1}^r n_i + m_1\right)\right) \Big/$$

$$\left\{\Gamma_m\left(\frac{1}{2}\sum_{i=1}^r n_i\right) \prod_{i=1}^r |\Sigma_i|^{n_i/2} |\Sigma_r^{11}|^{(\Sigma n_i - m_2)/2}\right\} \,.$$

(b) *Theil's k-class estimators.* These correspond to the special case $r=2$, $m_1=1$,

$$\Sigma_1 = \Sigma \,, \quad \Sigma_2 = (1-k)\Sigma \,, \qquad (0 < k < 1)$$

$$\Gamma_1=(1/2)\Sigma^{-1}[\beta, I_{m_2}]'\Delta[\beta, I_{m_2}]\Sigma^{-1},$$
$$\Gamma_2=0,$$

where β is the population value of B, and Δ is an $m_2\times m_2$ positive definite matrix. Unfortunately, the most convenient form of $f(B)$ in this case does not follow immediately from (6.2), although it can be obtained by a minor manipulation. However, it is preferable to return to (5.3), in which the invariant polynomial reduces to $\delta_{\kappa_2,0}C_\phi((1/2)R_1G_1'AG_1)$, i.e. L reduces to Khatri's [16] Laguerre polynomial $\delta_{\kappa_2,0}L_\phi^{n_1/2-p}(-G_1^{-1}\Gamma_1G_1'^{-1}, (1/2)G_1'AG_1)$. In place of (6.1), we now expand

$$C_\phi\left(\frac{1}{2}R_1G_1'AG_1\right)=\sum_{\substack{\kappa,\lambda\\(\phi\in\kappa\cdot\lambda)}}\sum_{\phi'\equiv\phi}\binom{f}{k}\theta_{\phi'}^{\kappa,\lambda}\sum_{\sigma\in\phi^*\cdot\phi^*}\bar{\zeta}_\sigma^{\kappa,\lambda;\phi'}$$
$$\cdot\, C_\sigma^{\phi^*,\kappa,\lambda}\left(R_1, \frac{1}{2}G_1'\mathcal{A}G_1, \frac{1}{2}G_1'\mathcal{B}G_1\right).$$

Averaging over $A_{11.2}$ and A_{22} as before, and taking

$$G_1=(\boldsymbol{k}^{-1}-1)^{-1/2}T$$
$$T=\begin{bmatrix}\Sigma_{11.2}^{-1/2} & 0\\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1/2} & \Sigma_{22}^{-1/2}\end{bmatrix}, \qquad (\Sigma^{-1}=TT'),$$
$$\Sigma_{11.2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21},$$

the density of the $\boldsymbol{k}$-class estimator may be expressed in terms of the standardized quantities (Anderson and Sawa [1])

$$B^*=\Sigma_{22}^{1/2}(B-\Sigma_{22}^{-1}\Sigma_{21})\Sigma_{11.2}^{-1/2}$$
$$\beta^*=\Sigma_{22}^{1/2}(\beta-\Sigma_{22}^{-1}\Sigma_{21})\Sigma_{11.2}^{-1/2}$$
$$\Delta^*=\Sigma_{22}^{-1/2}\Delta\Sigma_{22}^{-1/2}$$

as

$$\text{(6.3)}\quad f(B)=c_3(1-\boldsymbol{k})^{mn_1/2}|I_{m_2}+B^*B^{*\prime}|^{-(\Sigma n_i+m_1)/2}$$
$$\cdot\sum_{\kappa,\lambda;\phi}^{\infty}\left\{\left(\frac{1}{2}\left(\sum_{i=1}^{2}n_i-m_2\right)\right)_\kappa\left(\frac{1}{2}\left(\sum_{i=1}^{2}n_i+m_1\right)\right)_\lambda\theta_\phi^{\kappa,\lambda}\boldsymbol{k}^{k+1}/k!l!\left(\frac{1}{2}\sum_{i=1}^{2}n_i\right)_\phi\right\}$$
$$\cdot\sum_{\sigma\in\phi\cdot\phi}\bar{\zeta}_\sigma^{\kappa,\lambda;\phi}L_{\phi^*,\kappa,\lambda;\sigma}^{n_1/2-p}\left(-\frac{1}{2}(\boldsymbol{k}^{-1}-1)[\beta^*, I_{m_2}]'\Delta^*[\beta^*, I_{m_2}];\right.$$
$$\left.\begin{bmatrix}I_{m_1} & 0\\ 0 & 0\end{bmatrix}, [B^*, I_{m_2}]'(I_{m_2}+B^*B^{*\prime})^{-1}[B^*, I_{m_2}]\right)$$

where L here denotes Chikuse's [2] Laguerre polynomial, and

$$c_3=\text{etr}\left(-\frac{1}{2}\Delta^*(I_{m_2}+\beta^*\beta^{*\prime})\right)\Gamma_{m_1}\left(\frac{1}{2}\left(\sum_{i=1}^{2}n_i-m_2\right)\right)\Gamma_{m_2}\left(\frac{1}{2}\left(\sum_{i=1}^{2}n_i+m_1\right)\right)$$
$$\cdot\,|\Sigma_{22}|^{m_1/2}/\Gamma_m\left(\frac{1}{2}\sum_{i=1}^{2}n_i\right)|\Sigma_{11.2}|^{m_2/2}\,.$$

It is useful to note that

$$[B,\ I_{m_2}]T=\Sigma_{22}^{-1/2}[B^*,\ I_{m_2}]\ ,$$

with a corresponding result for β. Anderson and Sawa [1] have given a series expansion of the cumulative distribution function of B in the case $m_1=m_2=1$. An asymptotic expansion for the general k-class estimator was given by Kunitomo et al. [17].

The density of the OLS estimator may be deduced by letting $k\to 0$ in (6.3). From (2.13), the Laguerre polynomial, multiplied by k^{k+1}, approaches a $C_\sigma^{\phi,\kappa,\lambda}$ polynomial, and applying Lemma 2.3 once again we obtain

$$f(B)=c_3|I_{m_2}+B^*B^{*\prime}|^{-(n+m_1)/2}\sum_{\kappa,\lambda;\phi}^{\infty}\left\{\left(\frac{1}{2}(n-m_2)\right)_\kappa\left(\frac{1}{2}(n+m_1)\right)_\lambda\theta_\phi^{\kappa,\lambda}/k!l!\left(\frac{1}{2}n\right)_\phi\right\}$$
$$\cdot\,C_\phi^{\kappa,\lambda}\left(\frac{1}{2}\Delta^*\beta^*\beta^{*\prime},\ \frac{1}{2}\Delta^*(I_{m_2}+\beta^*B^{*\prime})(I_{m_2}+B^*B^{*\prime})^{-1}(I_{m_2}+B^*\beta^{*\prime})\right)\ ,$$

where $n=n_1+n_2$. The TSLS case follows by replacing n by n_1. Both the OLS and the TSLS cases correspond to the result in Phillips [19], and we refer to [1] and [19] for interpretations of the quantities involved. Taking $m_1=m_2=1$ we immediately obtain Richardson's [23] result.

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